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Finslerian Hypersurface and its Generalized Matsumoto changed Space

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Abstract

The present paper is devoted to the study of three kinds of hyperplane and generalized Matsumoto β-change of Finsler metric. Here $\beta = b_i(x, y)\, y^i$, $b_i(x, y)$ is h-vector in (M^n, L). The h-vector b_i is v covariantly constant with respect to Cartan's connection CΓ and satisfies the relation $LC^h_{ij}\, b_h = \rho\, h_{ij}$ and not only a function of coordinate but it is also a function of directional arguments.

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1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space where M^n is an n-dimensional differential manifold and $L(x, y)$ is the fundamental function. In 1984 C. Shibata introduced the transformation of Finsler metric [5], which is defined as :

$$L^*(x, y) = f(L, \beta), \tag{1.1}$$

where $\beta = b_i(x)\, y^i$, $b_i(x)$ are components of a covariant vector in (M^n, L) and f is positively homogeneous of degree one in L and β. This change of metric is called a β-change. In this paper we shall study a generalized Matsumoto β-change metric, which is defined as :

$$L^*(x, y) = \frac{\beta^2}{\beta - L} = f(L, \beta), \tag{1.2}$$

where $\beta = b_i(x, y)\, y^i$, $b_i(x, y)$ is an h-vector. As before Kropina and various geometers has taken b_i a covariant vector. Now we have taken b_i as h-vector, which is v-covariantly constant with respect to Cartan's connection CΓ and satisfies the

$LC^h_{ij}\, b_h = \rho\, h_{ij}$. Thus the h-vector b_i is not only a function of coordinate but it is also a function of directional argument and b_i satisfies all the condition of h-vector which is given by Izumi [6] in 1980. M. Matsumoto [8] studied the theory of Finslerian hypersurfaces and defined three types of hyperplane, which were later on studied by various geometers ([1], [2], [7], [9]). In the present paper using the field of linear frame ([2], [7], [9]) we shall consider Finslerian hypersurfaces given by a generalized Matsumoto change of Finsler metric with h-vector. The purpose of the present paper is to obtain the relation between original Finslerian hypersurfaces of (M^n, L) and another Finslerian hypersurfaces given by the generalized Matsumoto change of Finsler metric (M^n, L^*) with h-vector.

2. Preliminaries

The vector field b_i (x, y) in the Finsler space (M^n, L), is called h-vector if b_i (x, y) satisfy the following conditions :

(i) $b_{i\,|\,j} = 0$, (b) $LC^h_{ij}\, b_h = \rho\, h_{ij}$. (2.1)

Here | j denotes the v-covariant derivative with respect to Cartan's connection $C\Gamma$, C^h_{ij} is the Cartan's tensor, h_{ij} is the angular metric tensor and ρ is a function given by :

$$\rho = \frac{1}{(n-1)} LC^i\, b_i, \tag{2.2}$$

where $C^i = C^i_{jk}\, g^{jk}$. From (2.1), we get

$$\dot{\partial}_j\, b_i = L^{-1}\, \rho\, h_{ij}\,. \tag{2.3}$$

For an h-vector the function r and the magnitude of h-vector are independent of y [6]. Let $F^n = (M^n, L)$, be an n-dimensional Finsler space whose metric function is L(x, y) on M^n. The metric tensor g_{ij} (x, y) and Cartan's C-tensor C_{ijk} (x, y) of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial L^2}{\partial y^i\, \partial y^j} \quad \text{and} \quad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k},$$

respectively and we can introduce introduced the Cartan's connection $C\Gamma = (F^i_{jk}$,

N^i_j, C^i_{jk}) along F^n. A hypersurface M^{n-1} represented by the equation $x^i = x^i(u^\alpha)$, where u^α is Gaussian coordinates on M^{n-1} and greek indices vary from 1 to n −1. As for the matrix we assumed that the projection factor $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank n − 1 and also employed the notations $B^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$ and $B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}$. At a point u^α, the supporting element y^i is tangential to M^{n-1}. We may then write $y^i = B^i_\alpha(u)\, v^\alpha$, where v^α is the supporting element of M^{n-1} at the point u^α. We get a Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$ of n −1 dimensional, where $\underline{L}(u, v) = L(x(u), y(u, v))$ along M^{n-1}. The unit normal vector $N^i(u, v)$ at each point u^α of F^{n-1} is given by

$$g_{ij} B^i_\alpha N^j = 0 \quad \text{and} \quad g_{ij} N^i N^j = 1. \tag{2.4}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), then we get

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N_i = 0, \qquad N^i N_j = 1 \tag{2.5}$$

and

$$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

From the reciprocal tensor $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we have the following relations

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \qquad N_i = g_{ij} N^j. \tag{2.6}$$

The second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α of the induced Cartan's connection $C\Gamma = (F^\alpha_{\beta\gamma}, N^\beta_\alpha, C^\alpha_{\beta\gamma})$ on F^{n-1} are respectively given by [8]

$$H_{\alpha\beta} = N_i (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \tag{2.7}$$

and

$$H_\alpha = N_i (B^i_{0\alpha} + N^i_j B^j_\alpha),$$

where

$$M_\alpha = C_{ijk} B^i_\alpha N^j N^k. \tag{2.8}$$

The contraction of $H_{\alpha\beta}$ by v^α is defined as $H_{\alpha\beta} v^\alpha = H_\beta$. Furthermore the second fundamental v-tensor $M_{\alpha\beta}$ is given by [10]

$$M_{\alpha\beta} = C_{ijk}\, B^i_\alpha\ B^j_\beta\ N^k. \tag{2.10}$$

3. Generalized Matsumoto change of Finsler metric with h-vector

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space. We shall define a function $L^*(x, y) > 0$ on M^n by the equation (1.2). To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij} and the Cartan's C-tensor C^*_{ijk} of $F^{*n} = (M^n, L^*)$, we used the following results :

$$\partial\beta/\partial y^i = b_i, \qquad \partial L/\partial y^i = l_i, \qquad \partial\, l_j/\partial y^i = L^{-1}\, h_{ij}, \tag{3.1}$$

where h_{ij} are components of angular metric tensor of F^n given by :

$$h_{ij} = g_{ij} - l_i\, l_j = L\,(\partial^2 L/\partial y^i\, \partial y^j).$$

Differentiating (1.2) with respect to y^i, we get

$$l^*_i = A_1[(\beta - 2L)\, b_i + \beta l_i], \tag{3.2}$$

where $$A_1 = \frac{\beta}{(\beta - L)^2}.$$

To obtain angular metric tensor, we differentiate (3.2) with respect to y^j which as follows :

$$h^*_{ij} = Q_0[\beta\,(1 + \rho) - 2\rho L]\, h_{ij} + Q_1\, b_i\, b_j - Q_2\ (l_i\, b_j + l_j\, b_i) + (L^2/\beta^2) l_i\, l_j, \tag{3.3}$$

where

$$Q_0 = \frac{\beta^3}{L\,(\beta - L)^3},$$

$$Q_1 = \frac{2L^2\,\beta^2}{(\beta - L)^4},$$

$$Q_2 = \frac{2\beta^3\, L}{(\beta - L)^4},$$

$$Q_3 = \frac{2\beta^4}{(\beta - L)^4}.$$

From (3.2) and (3.3), we get the following relation :

$$g^*_{ij} = Q_0\,[\beta\,(1+\rho) - 2\rho L]\,g_{ij} + S_1\,b_i\,b_j + S_2\,(l_i\,b_j + l_j\,b_i) + S_3\,[4\beta L - \beta^2\,(1+\rho) + \rho L\,(3\beta - 2L)]\,l_i\,l_j, \tag{3.4}$$

where

$$S_1 = \frac{\beta^2\,(\beta^2 + 6L^2 - 4\beta L)}{(\beta - L)^4},$$

$$S_2 = \frac{\beta^3\,(\beta - 4L)}{(\beta - L)^4},$$

$$S_3 = \frac{\beta^3}{L\,(\beta - L)^4}.$$

Now to obtain the contravariant metric tensor g^{*ij} we may here assume the tensor B_{ij} as :

$$B_{ij} = Q_0\,Cg_{ij} + C_i\,C_j, \tag{3.5}$$

where Q_0 is defined in (3.3).

$$C = \beta\,(1+\rho) - 2\rho L,$$

$$C_i = \pi b_i.$$

In view of (3.4) the unknown quantities π_{-1}, π_0 and π are obtained using the following relations :

(a) $$\pi^2_{-1} = S_3\,A,$$

(b) $$\pi_0 = \frac{S_2}{\pi_{-1}} = \frac{\beta^{3/2}\,(\beta - 4L)\,L^{1/2}}{(\beta - L)^2\,A^{1/2}},$$

(c) $$\pi^2 = S_1 - \pi_0^2 = \frac{\beta^2\,D}{(\beta - L)^4\,A},$$

where

$$A = [4\beta L - \beta^2\,(1+\rho) + \rho L(3\beta - 2L)],$$

$$D = A(\beta^2 + 6L^2 - 4\beta L) - \beta(\beta - 4L)^2 L.$$

Using the relation $B_{ij} B^{jk} = \delta_i^k$, we get

$$B^{ij} = \frac{1}{Q_0 C}\left[g^{ij} - \frac{b^i b^j LD}{\beta C(\beta - L)A + b^2 LD}\right]. \tag{3.6}$$

On account of (3.4) and (3.5) g^*_{ij} may be written as :

$$g^*_{ij} = B_{ij} + d_i d_j,$$

where $d_i = \pi_0 b_i - \pi_{-1} l_i = \frac{\beta^{3/2}}{(\beta - L)^2}[(\beta - 4L) L^{1/2} A^{-1/2} b_i - A^{1/2} L^{-1/2} l_i].$

The g^*_{ij} is defined as

$$g^*_{ij} = B_{ij} - \frac{d^i d^j}{1 + d^2},$$

where $d^i = B^{ij} d_j = \frac{(\beta - L) L^{1/2} A^{1/2}}{\beta^{3/2} C}(L^{1/2} E b^i - l^i),$

$$E = \frac{(\beta - 4L) L^{1/2}}{A} - \frac{LD(\beta - 4L) L^{1/2} A^{-1} b^2}{\beta C(\beta - L)A + b^2 LD} + \frac{\beta D L^{-1/2}}{\beta C(\beta - L)A + b^2 LD}$$

and $d^2 = d^i d_i = \frac{1}{C(\beta - L)}[L^{3/2} E(\beta - 4L) b^2 - AEL^{-1/2}\beta - \beta(\beta - 4L) + A].$

Hence g^{*ij} is given as :

$$g^{*ij} = \frac{1}{Q_0[\beta(1 + \rho) - 2\rho L]} g^{ij} - K_1 b^i b^j + K_2(l^i b^j + l^j b^i) - K_3 l^i l^j, \tag{3.7}$$

where $K_1 = \frac{(\beta - L)^3 L^2 D}{\beta^3 C[\beta C(\beta - L)A + b^2 LD]} + \frac{(\beta - L)^3 L^2 AE^2}{\beta^3 CF},$

$$K_2 = \frac{(\beta - L)^3 L^{3/2} AE}{\beta^3 CF},$$

$$K_3 = \frac{(\beta - L)^3 LA}{\beta^3 CF},$$

$$F = L^{3/2} E(\beta - 4L) b^2 - AEL^{-1/2} \beta - \beta(\beta - 4L) + A + C(\beta - L).$$

Differentiating (3.4) with respect to y^k and using (3.1), we get the following results :

$$C^*_{ijk} = Q_0 [\beta(1+\rho) - 2\rho L] C_{ijk} + T_1 (\beta^2 - 4\beta L + \rho T_2)(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + T_3 m_i m_j m_k \tag{3.8}$$

where $T_1 = \frac{\beta^2}{2L(\beta - L)^4}$,

$$T_2 = (\beta^2 - 4\beta L + 6L^2),$$

$$T_3 = \frac{6\beta L^3}{(L - \beta)^5},$$

$$m_i = b_i - \frac{\beta}{L} l_i.$$

The following important results are to be noted

$$m_i l^i = 0, \quad m_i b^i = b^2 - \frac{\beta^2}{L^2}, \quad h_{ij} m^j = h_{ij} b^j = m_i,$$

$$h_{ij} l^j = 0 \quad \text{and} \quad m^i = g^{ij} m_j = b^i - \frac{\beta}{L} l^i. \tag{3.9}$$

4. Hypersurfaces due to generalized Matsumoto change with h-vector

Let $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$ be a Finslerian hypersurface along the F^n and $F^{*n-1} = (M^{n-1}, \underline{L}^*(u, v))$ be another Finslerian hypersurface along the F^{*n} given by generalized Matsumoto change with h-vector. Let (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i) and N^i be the unit normal vector at each point of F^{n-1}. The function B^i_α may be considered as component of $n-1$ linearly independent tangent vectors of F^{n-1} and B^i_α are invariant function under generalized

Matsumoto change with an h-vector. Thus we shall show that a unit normal vector $N^{*i}(u, v)$ of F^{*n-1} is uniquely determined by :

$$g^*_{ij}\, B^i_\alpha\, N^{*j} = 0 \qquad \text{and} \qquad g^*_{ij}\, N^{*i} N^{*j} = 1. \tag{4.1}$$

Multiplication of (3.4) by $N^i N^j$ and paying attention to (2.4) and $l_i N^i = 0$, we have

$$g^*_{ij}\, N^i N^j = Q_0[\beta(1+\rho) - 2\rho L] + S_1 (b_i N^i)^2,$$

where Q_0, S_1 and C has been defined in (3.3), (3.4) and (3.5). Therefore we obtain

$$g^*_{ij}\left(\frac{\pm\sqrt{L}\,(\beta-L)^2 N^i}{\beta\sqrt{\rho R + R_0\,(b_i N^i)^2}}\right)\left(\frac{\pm\sqrt{L}\,(\beta-L)^2 N^i}{\beta\sqrt{\rho R + R_0\,(b_i N^i)^2}}\right) = 1,$$

where $$R = \beta(\beta-L)\left(\frac{\beta}{\rho} + \beta - 2L\right),$$

$$R_0 = L(\beta^2 + 6L^2 - 4\beta L).$$

Therefore, we can put

$$N^{*i} = \frac{\sqrt{L}\,(\beta-L)^2 N^i}{\beta\sqrt{\rho R + R_0\,(b_i N^i)^2}}, \tag{4.2}$$

where we have choosen only positive sign.

Using equation (3.1), (3.4), (4.2) and from (4.1), we have

$$[(\beta^2 + 6L^2 - 4\beta L)\, b_i\, \beta^i_\alpha + \beta(\beta - 4L)\, l_i\, B^i_\alpha\,]\,\frac{\sqrt{L}\,(\beta-L)^2\, b_i N^i}{\beta\sqrt{\rho R + R_0\,(b_i N^i)^2}} = 0. \tag{4.3}$$

If $[(\beta^2 + 6L^2 - 4\beta L)\, b_i\, \beta^i_\alpha + \beta(\beta - 4L)\, l_i\, B^i_\alpha\,] = 0$, then contracting it by v^α and using $y^i = B^i_\alpha\, v^\alpha$, we get $L = 0$, which is a contradiction with assumption that $L > 0$. Hence $b_i N^i = 0$. Therefore (4.2) is written as

$$N^{*i} = \frac{\sqrt{L}\,(\beta-L)^2 N^i}{\beta\sqrt{\rho R}}. \tag{4.4}$$

Proposition 4.1. There exists a field of linear frame $(B^i_1, B^i_2, \ldots\ldots\ldots, B^i_{n-1}, N^{*i} = \frac{\sqrt{L}\,(\beta - L)^2 N^i}{\beta\sqrt{\rho R}})$ of F^{*n} for a field of linear frame $(B^i_1, B^i_2, \ldots\ldots, B^i_{n-1}, N^i)$ of F^n such that (4.1) is satisfied along F^{*n-1} and then b_i is tangential to both the hypersurfaces F^{n-1} and F^{*n-1}.

We may write the quantities $B^{*\alpha}_i$ of F^{n-1} by

$$B^{*\alpha}_i = g^{*\alpha\beta} g^*_{ij} B^j_\beta,$$

where $g^{*\alpha\beta}$ is the inverse matrix of $g^*_{\alpha\beta}$. If $(B^{*\alpha}_i, N^*_i)$ be the inverse matrix of (B^i_α, N^i), then we have $B^i_\alpha B^{*\beta}_i = \delta^\beta_\alpha$, $B^i_\alpha N^*_i = 0$, $N^{*i} N^*_i = 1$ and $B^i_\alpha B^{*\alpha}_j + N^{*i} N^*_j = \delta^i_j$. We also get $N^*_i = g^*_{ij} N^{*j}$, which is on account of (3.2), (3.4) and (4.4) gives

$$N^*_i = \frac{\beta^2\,[\beta\,(1+\rho) - 2\rho L]}{(\beta - L)^2\sqrt{\rho L R}} N_i. \tag{4.5}$$

We define the Cartan's connection of F^n by $(F^i_{jk}, N^i_j, C^i_{jk})$ and Cartan's connection of F^{*n} by $(F^{*i}_{jk}, N^{*i}_j, C^{*i}_{jk})$. Let D^i_{jk} be the difference tensor which is defined as :

$$D^i_{jk} = F^{*i}_{jk} - F^i_{jk}.$$

Let b_i is the vector field in F^n such that

$$D^i_{jk} = A_{jk}\, b^i - B_{jk}\, l^i,$$

where A_{jk} and B_{jk} are components of a symmetric covariant tensor of second order. Since $N_i b^i = 0$ and $N_i l^i = 0$, contracting (4.6) by N_i, we get

$$N_i D^i_{jk} = 0 \quad \text{and} \quad N_i D^i_{0k} = 0.$$

From (3.3) and (4.5), we get

$$H^*_\alpha = \frac{\beta^2\,[\beta\,(1+\rho) - 2\rho L]}{(\beta - L)\sqrt{\rho L R}} H_\alpha. \tag{4.7}$$

If each path of a hypersurface F^{n-1} with respect to induced connection is also a path of enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind [8]. A hyperplane of the first kind is characterized by $H_\alpha = 0$. Hence from (4.7), we have

Theorem 4.1. The hypersurface F^{*n-1} is a hyperplane of the first kind if and only if the hypersurface F^{n-1} is a hyperplane of the first kind, where $b_i(x)$ is a vector field satisfying equation (4.6).

Paying attention to (2.8), (3.8) and (4.4) and by using $m_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get

$$M^*_\alpha = M_\alpha + \frac{R_1}{2\rho R} m_i B^i_\alpha , \tag{4.8}$$

where $R_1 = (\beta^2 - 4\beta L + \rho T_2)$.

Using the equations (3.3), (4.5), (4.6), (4.7) and (4.8), we get

$$H^*_{\alpha\beta} = \frac{\beta}{(\beta - L)\sqrt{\rho L R}} \left[(\beta^2 + \rho\beta^2 - 2\rho L\beta) H_{\alpha\beta} + \frac{R_1}{2(\beta - L)} H_\beta m_i B^i_\alpha \right]. \tag{4.9}$$

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind [8]. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$. From (4.7) and (4.8), we have the following :

Theorem 4.2. The hypersurface F^{*n-1} is a hyperplane of the second kind if and only if the hypersurface F^{n-1} is a hyperplane of the second kind where $b_i(x)$ is a vector field satisfying equation (4.6).

Using equations (2.9), (3.8) and (4.4), we get

$$M^*_{\alpha\beta} = \frac{\beta^2 [\beta(1+\rho) - 2\rho L]}{(\beta - L)\sqrt{L \rho R}} M_{\alpha\beta} . \tag{4.10}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of the third kind [8]. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0$, $M_{\alpha\beta} = 0$. From (4.7), (4.9) and (4.10), we have

Theorem 4.3. The hypersurface F^{*n-1} is a hyperplane of the third kind if and only if the hypersurface F^{n-1} is a hyperplane of the third kind. where $b_i(x)$ is a vector field satisfying equation (4.6).

Finally we have shown that a generalized Matsumoto change with h-vector makes three types of hyperplanes invariant under certain conditions.

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