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Certain Results involving Fractional q-integral Operators

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1. Introduction

In 1966, Alsalam, W. A. defined a fractional q-integral operator through the q-integral

$$K_q^{\eta,\alpha} f(x) = \frac{q^{-\eta} x^{\eta}}{(1-q)^{1-\alpha}} \Pi_q \begin{bmatrix} \alpha ; \\ 1 \end{bmatrix} \int_x^{\infty} [y-x]_{\alpha-1} \, y^{-\eta-\alpha} . f(yq^{1-\alpha}) \, d(y, q) \quad (1)$$

where $\alpha \neq 0, -1, -2, \ldots\ldots\ldots$

Later, Agarwal, R. P., in 1967, defined the operators :

$$I_q^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{(1-q)^{1-\alpha}} \Pi_q \begin{bmatrix} \alpha ; \\ 1 \end{bmatrix} \int_0^x (x-tq)_{\alpha-1} \, t^{\eta} f(t) \, d(t; q). \quad (2)$$

In 1970, Upadhyay, M. defined two operators which are the extensions of Al-salam's and Agarwal's operators. The operators defined by Upadhyay, M. are as follows :

$$I_q [(a); (b); z, \eta : f(x)] = \frac{x^{-\eta-1}}{(1-q)} \int_0^x {}_A\phi_B \begin{bmatrix} (a) ; & \frac{zt}{x} \\ (b) ; & \end{bmatrix} f(t) \, d(t; q) \quad (3)$$

$$K_q [(a); (b); z, \eta : f(x)] = \frac{x^{\eta} q^{-\eta}}{(1-q)} \int_x^{\infty} t^{-\eta-1} \, {}_A\phi_B \begin{bmatrix} (a) ; & \frac{zt}{x} \\ (b) ; & \end{bmatrix} f(t) \, d(t; q). \quad (4)$$

In 1978, Sharma, S. defined the following operators :

$$I_q \begin{bmatrix} (a) : (d) ; (f) ; \\ (b) : (e) ; (g) ; \end{bmatrix} z_1, z_2, \eta_1, \eta_2 : f(x, y) \Bigg] = \frac{x^{-\eta_1 - 1} y^{-\eta_2 - 1}}{(1-q)^2} .$$

$$\int_0^x \int_0^y t_1^{\eta_1} t_2^{\eta_2} \phi \left[\begin{matrix} (a) : (d) ; (f) ; \\ (b) : (e) ; (g) ; \end{matrix} \frac{z_1 t_1}{x}, \frac{z_2 t_2}{y} \right] f(t_1, t_2) . d(t_1; q) d(t_2; q)$$

$$= \sum_{k, j = 0}^{\infty} q^{k(\eta_1 + 1) + j(\eta_2 + 1)} \phi \left[\begin{matrix} (a) : (d) ; (f) ; \\ (b) : (e) ; (g) ; \end{matrix} z_1 q^k, z_2 q^j \right] f(xq^k, yq^j) \qquad (5)$$

and

$$K_q \left[\begin{matrix} (a) : (d) ; (f) ; \\ (b) : (e) ; (g) ; \end{matrix} z_1, z_2, \eta_1, \eta_2 : f(x, y) \right] = \frac{x^{\eta_1} y^{\eta_2} q^{-\eta_1 - \eta_2}}{(1 - q)^2} .$$

$$\int_x^{\infty} \int_y^{\infty} t_1^{-\eta_1 - 1} t_2^{-\eta_2 - 1} \phi \left[\begin{matrix} (a) : (d) ; (f) ; \\ (b) : (e) ; (g) ; \end{matrix} \frac{z_1 x}{t_1}, \frac{z_2 x}{t_2} \right] f(t_1, t_2) d(t_1; q) d(t_2; q)$$

$$= \sum_{k, j = 0}^{\infty} q^{k\eta_1 + j\eta_2} \phi \left[\begin{matrix} (a) : (d) ; (f) ; \\ (b) : (e) ; (g) ; \end{matrix} z_1 q^{k+1}, z_2 q^{j+1} \right] f(xq^{-k-1}, yq^{-j-1}). \qquad (6)$$

The operators defined by Upadhyay, M. and also by Alasalam, W. A. and Agarwal, R. P. are limiting cases of (5) and (6).

Now, we shall introduce the following two more generalized fractional q-integral operators

$$I_q \left[\begin{matrix} (a): (b) ; (c) ; (d) :: (e); (f) ; (g) ; z_1, z_2, z_3 \\ (a'): (b') ; (c'); (d') :: (e') ; (f') ; (g') ; \eta_1, \eta_2 \eta_3 \end{matrix} q : f(x, y, z) \right]$$

$$= \frac{x^{-\eta_1 - 1} y^{-\eta_2 - 1} z^{-\eta_3 - 1}}{(1 - q)^3} \int_0^x \int_0^y \int_0^z t_1^{\eta_1} t_2^{\eta_2} t_3^{\eta_3} \times$$

$$\phi \left[\begin{matrix} (a): (b) ; (c) ; (d) :: (e); (f) ; (g) ; \\ (a'): (b') ; (c'); (d') :: (e') ; (f') ; (g') ; \end{matrix} q : \frac{z_1 t_1}{x}, \frac{z_2 t_2}{y}, \frac{z_3 t_3}{z} \right] \times$$

$$f(t_1, t_2, t_3) d(t_1; q) d(t_2; q) d(t_3; q)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q^{i(\eta_1 + 1) + j(\eta_2 + 1) + k(\eta_3 + 1)} .$$

$$\phi\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\; q: z_1q^i, z_2q^j, z_3q^k\right] f(xq^i, yq^j, zq^k) \quad (7)$$

and

$$K_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\; q: f(x,y,z)\right]$$

$$= \frac{x^{\eta_1} y^{\eta_2} z^{\eta_3} q^{-\eta_1-\eta_2-\eta_3}}{(1-q)^3} \int_x^\infty \int_y^\infty \int_z^\infty t_1^{-\eta_1-1}\, t_2^{-\eta_2-1}\, t_3^{-\eta_3-1} \times$$

$$\phi\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\; q: \frac{z_1x}{t_1}, \frac{z_2y}{t_2}, \frac{z_3z}{t_3}\right] \times$$

$$f(t_1, t_2, t_3)\, d(t_1;q)\, d(t_2;q)\, d(t_3;q)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q^{i\eta_1 + j\eta_2 + k\eta_3}$$

$$\phi\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\; q: z_1q^{i+1}, z_2q^{j+1}, z_3q^{k+1}\right]$$

$$f(xq^{-i-1}, yq^{-j-1}, zq^{-k-1}). \quad (8)$$

The operators I_q, K_q defined by Sharma, S. and Upadhyay, M. and also by Alasalam, W. A. and Agarwal, R. P. are limiting cases of (7) and (8). The basic integrals are defined by means of the relations :

$$\int_x^\infty f(t)\, d(t;q) = x(1-q) \sum_{k=1}^{\infty} q^{-k} f(xq^{-k}), \quad (9)$$

$$\int_0^x f(t)\, d(t;q) = x(1-q) \sum_{k=0}^{\infty} q^{k} f(xq^{k}), \quad (10)$$

$$\int_0^\infty f(t)\, d(t;q) = (1-q) \sum_{k=-\infty}^{\infty} q^{k} f(q^{k}). \quad (11)$$

The basic analogue of Mellin Transform of f(x, y, z) is defined as :

$$M_q\,[f(x,y,z)] = \int_0^\infty \int_0^\infty \int_0^\infty x^{s_1-1}\, y^{s_2-1}\, z^{s_3-1}\, f(x,y,z)\, d(x;q)\, d(y;q)\, d(z;q). \quad (12)$$

The main results to be established are :

Theorem 1. If $\sum_{\lambda_1,\lambda_2,\lambda_3=-\infty}^{\infty} |q^{\lambda_1(1+\eta_1)+\lambda_2(1+\eta_2)+\lambda_3(1+\eta_3)}\,.\, f(q^{\lambda_1}, q^{\lambda_2}, q^{\lambda_3})|$

and $\sum_{\lambda_1,\lambda_2,\lambda_3=-\infty}^{\infty} |q^{-\eta_1\lambda_1-\lambda_2\eta_2-\lambda_3\eta_3}\,.\, g(q^{\lambda_1}, q^{\lambda_2}, q^{\lambda_3})|$ converge, where $|q| < 1$, $|z_1| < 1$, $|z_2| < 1$, $|z_3| < 1$, $Rl\,(\eta_1, \eta_2, \eta_3) > -1$, then, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z).\, K_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\ (a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\; q : g(x,y,z)\right] d(x;q)\, d(y;q)\, d(z;q)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(xq^{-1}, yq^{-1}, zq^{-1}).$$

$$I_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\ (a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\; q : f(x,y,z)\right] d(x;q)\, d(y;q)\, d(z;q). \quad (13)$$

This is an interconnection between I_q and K_q operators.

Theorem 2. If $Rl\,(\eta_1, \eta_2, \eta_3) > -1$, $|q| < 1$ and $|z_1| < 1$, $|z_2| < 1$, $|z_3| < 1$; then we have the following relation between I_q and K_q operators :

$$I_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\, z_1, z_2, z_3\\ (a'):(b');(c');(d')::(e');(f');(g');\, \eta_1, \eta_2\, \eta_3\end{matrix}\; :\right.$$

$$\left. : K_q\left[\begin{matrix}(l):(m);(n);(p)::(u);(v);(w);\, z_1, z_2, z_3\\ (l'):(m');(n');(p')::(u');(v');(w');\, \eta_1, \eta_2, \eta_3\end{matrix}\; q : f(x,y,z)\right]\right]$$

$$= K_q\left[\begin{matrix}(l):(m);(n);(p)::(u);(v);(w);\, z_1; z_2, z_3;\\ (l'):(m');(n');(p')::(u');(v');(w');\, \eta_1, \eta_2, \eta_3;\end{matrix}\; :\right.$$

$$I_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);z_1,z_2,z_3\\(a'):(b');(c');(d')::(e');(f');(g');\eta_1,\eta_2\,\eta_3\end{matrix}\;q:f(x,y,z)\right]\right] \qquad (14)$$

Theorem 3. If $\sum_{\lambda_1,\lambda_2,\lambda_3=-\infty}^{\infty} |q^{\lambda_1 s_1+\lambda_2 s_2+\lambda_3 s_3}.\, f(q^{\lambda_1}, q^{\lambda_2}, q^{\lambda_3})|$ converges, where $|q|<1, |z_1|<1, |z_2|<1, |z_3|<1$, $Rl\,(\eta_i+s_i)>0$, $(i=1,2,3)$, then

$$M_q\left[K_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);z_1,z_2,z_3;\\(a'):(b');(c');(d')::(e');(f');(g');\eta_1,\eta_2\,\eta_3;\end{matrix}\;q:f(x,y,z)\right]\right]$$

$$= q^{s_1+s_2+s_3}\,(1-q^{\eta_1+s_1})^{-1}(1-q^{\eta_2+s_2})^{-1}(1-q^{\eta_3+s_3})^{-1}\times$$

$$\phi\left[\begin{matrix}(a):(b);(c);(d)::(e),\eta_1+s_1;(f),\eta_2+s_2;(g),\eta_3+s_3;\\(a'):(b');(c');(d')::(e'),\eta_1+s_1+1;(f'),\eta_2+s_2+1;(g'),\eta_3+s_3+1;\end{matrix}\right.$$

$$\left. q:\; qz_1,qz_2,qz_3\right].\,M_q\,[f(x,y,z)]. \qquad (15)$$

Theorem 4. If $\sum_{\lambda_1,\lambda_2,\lambda_3=-\infty}^{\infty} |q^{\lambda_1 s_1+\lambda_2 s_2+\lambda_3 s_3}.\, f(q^{\lambda_1}, q^{\lambda_2}, q^{\lambda_3})|$ converges, where $|q|<1, |z_i|<1$, $Rl\,(\eta_i-S_i)>-1$, $(i=1,2,3)$, then

$$M_q\left[I_q\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);z_1,z_2,z_3;\\(a'):(b');(c');(d')::(e');(f');(g');\eta_1,\eta_2\,\eta_3;\end{matrix}\;q:f(x,y,z)\right]\right]$$

$$= (1-q^{1+\eta_1-s_1})^{-1}(1-q^{1+\eta_2-s_2})^{-1}(1-q^{1+\eta_3-s_3})^{-1}\times$$

$$\phi\left[\begin{matrix}(a):(b);(c);(d)::(e),1+\eta_1-s_1;(f),1+\eta_2-s_2;(g),1+\eta_3-s_3;\\(a'):(b');(c');(d')::(e'),2+\eta_1-s_1;(f'),2+\eta_2-s_2;(g'),2+\eta_3-s_3+1;\end{matrix}\right.$$

$$\left. q:\; z_1,z_2,z_3\right].\,M_q\,[f(x,y,z)]. \qquad (16)$$

Proof of Results (13) to (16)

To prove result (13), the left hand side can be written as :

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} f(x, y, z).\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q^{i\eta_1 + j\eta_2 + k\eta_3} \times$$

$$\phi\left[\begin{matrix}(a): (b) ; (c) ; (d) :: (e); (f) ; (g) ; \\ (a'): (b') ; (c'); (d') :: (e') ; (f') ; (g') ;\end{matrix} q : z_1 q^{i+1}, z_2 q^{j+1}, z_3 q^{k+1}\right] \times$$

$$g(xq^{-i-1}, yq^{-j-1}, zq^{-k-1})\, d(x, q)\, d(y; q)\, d(z, q).$$

Now, replacing x, y and z by xq^i, yq^j and zq^k in the above expression, we have

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} g(xq^{-1}, yq^{-1}, zq^{-1}).\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q^{i(1+\eta_1) + j(1+\eta_2) + k(1+\eta_3)} \times$$

$$\phi\left[\begin{matrix}(a): (b) ; (c) ; (d) :: (e); (f) ; (g) ; \\ (a'): (b') ; (c'); (d') :: (e') ; (f') ; (g') ;\end{matrix} q : z_1 q^{i+1}, z_2 q^{j+1}, z_3 q^{k+1}\right] \times$$

$$f(xq^{i}, yq^{j}, zq^{k})\, d(x, q)\, d(y; q)\, d(z, q)$$

which is equal to right hand side of result (13).

To prove result (14), we write its left side as

$$\sum_{i, j, k=0}^{\infty} q^{i\eta_1 + j\eta_2 + k\eta_3} \times$$

$$\phi\left[\begin{matrix}(l): (m) ; (n) ; (p) :: (u), (v) , (w) ; \\ (l'): (m') ; (n'); (p') :: (u') , (v') , (w') ;\end{matrix} q : z_1 q^{i+1}, z_2 q^{j+1}, z_3 q^{k+1}\right]$$

$$I_q\left[\begin{matrix}(a): (b) ; (c) ; (d) :: (e); (f) ; (g) ; z_1, z_2, z_3; \\ (a'): (b') ; (c'); (d') :: (e') ; (f') ; (g') ; \eta_1, \eta_2, \eta_3;\end{matrix} q : f(xq^{-i-1}, y^{-j-1}, z^{-k-1})\right]$$

$$= \sum_{i, j, k=0}^{\infty} q^{i\eta_1 + j\eta_2 + k\eta_3} \times$$

$$\phi\left[\begin{matrix}(l): (m) ; (n) ; (p) :: (u), (v) , (w) ; \\ (l'): (m') ; (n'); (p') :: (u') , (v') , (w') ;\end{matrix} q : z_1 q^{i+1}, z_2 q^{j+1}, z_3 q^{k+1}\right]$$

$$\sum_{r, s, t=0}^{\infty} q^{r(1+\eta_1) + s(1+\eta_2) + t(1+\eta_3)} \times$$

$$\phi\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\, q: z_1q^r, z_2q^s, z_3q^t\right] \times$$

$$f(xq^{r-i-1}, yq^{s-j-1}, zq^{t-k-1})$$

which is equal to right hand side of result (14). To prove result (15), we write its left side as :

$$M_q\left[\sum_{i,j,k=0}^{\infty} q^{i\eta_1+j\eta_2+k\eta_3} \times \phi\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\, q: z_1q^{i+1}, z_2q^{j+1}, z_3q^{k+1}\right] . f(xq^{-i-1}, yq^{-j-1}, zq^{-k-1})\right]$$

$$= \sum_{i,j,k=0}^{\infty} q^{i\eta_1+j\eta_2+k\eta_3} \times \phi\left[\begin{matrix}(a):(b);(c);(d)::(e);(f);(g);\\(a'):(b');(c');(d')::(e');(f');(g');\end{matrix}\, q: z_1q^{i+1}, z_2q^{j+1}, z_3q^{k+1}\right]$$

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^{s_1-1}y^{s_2-1}z^{s_3-1}.f(xq^{-i-1}, yq^{-j-1}, zq^{-k-1}) . d(x;q)\, d(y;q)\, d(z;q)$$

which on replacing x, y, z by $xq^{i+1}, yq^{j+1}, zq^{k+1}$ respectively, becomes

$$= \sum_{i,j,k=0}^{\infty} q^{i\eta_1+j\eta_2+k\eta_3} \times \sum_{r,s,t=0}^{\infty} \frac{((a))_{r+s+t}((b))_{r+s}((c))_{s+t}((d))_{r+t}}{((a'))_{r+s+t}((b'))_{r+s}((c'))_{s+t}((d'))_{r+t}} \times$$

$$\frac{((e))_r((f))_s((g))_t(z_1q)^r(z_2q)^s(z_3q)^t . q^{ir+js+kt}}{((e'))_r((f'))_s((g'))_t(q)_s(q)_t(q)_r}(q^{s_1+s_2+s_3+is_1+js_2+ks_3})$$

$$\times \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^{s_1-1}y^{s_2-1}z^{s_3-1}.f(x, y, g). d(x;q)\, d(y;q)\, d(z;q)$$

$$= q^{s_1+s_2+s_3}\sum_{r,s,t=0}^{\infty} \frac{((a))_{r+s+t}((b))_{r+s}((c))_{s+t}((d))_{r+t}}{((a'))_{r+s+t}((b'))_{r+s}((c'))_{s+t}((d'))_{r+t}}$$

$$\frac{((e))_r\,((f))_s\,((g))_t\,(z_1q)^r\,(z_2q)^s\,(z_3q)^t}{((e'))_r\,((f'))_s\,((g'))_t(q)_s\,(q)_t\,(q)_r} \times \sum_{i,j,k=0}^{\infty} q^{i(\eta_1+s_1+r)+j(\eta_2+s_2+s)+k(\eta_3+s_3+t)}$$

$$\times M_q\,[f(x,y,z)]$$

$$= q^{s_1+s_2+s_3} \sum_{r,s,t=0}^{\infty} \frac{((a))_{r+s+t}\,((b))_{r+s}\,((c))_{s+t}\,((d))_{r+t}}{((a'))_{r+s+t}\,((b'))_{r+s}\,((c'))_{s+t}\,((d'))_{r+t}}$$

$$\times \frac{((e))_r\,((f))_s\,((g))_t\,(z_1q)^r\,(z_2q)^s\,(z_3q)^t}{((e'))_r\,((f'))_s\,((g'))_t(q)_s\,(q)_t\,(q)_r}$$

$$\times (1-q^{\eta_1+s_1+r})^{-1}\,(1-q^{\eta_2+s_2+s})^{-1}\,(1-q^{\eta_3+s_3+t})^{-1} \times M_q\,[f(x,y,z)]$$

$$= q^{s_1+s_2+s_3}\;(1-q^{\eta_1+s_1})^{-1}\,(1-q^{\eta_2+s_2})^{-1}\,(1-q^{\eta_3+s_3})^{-1} \times$$

$$\sum_{r,s,t=0}^{\infty} \frac{((a))_{r+s+t}\,((b))_{r+s}\,((c))_{s+t}\,((d))_{r+t}\,((e))_r}{((a'))_{r+s+t}\,((b'))_{r+s}\,((c'))_{s+t}\,((d'))_{r+t}\,((e'))_r}$$

$$\times \frac{(\eta_1+s_1)_r\,((f))_s\,(\eta_2+s_2)_s\,((g))_t\,(\eta_3+s_3)_t\,(z_1q)^r\,(z_2q)^s\,(z_3q)^t}{(1+\eta_1+s_1)_r\,((f'))_s\,(1+\eta_2+s_2)_s\,((g'))_t\,(1+\eta_3+s_3)_t\,(q)_r\,(q)_s\,(q)_t}$$

$$\times M_q\,[f(x,y,z)]$$

which is the right hand side of result (15). The Proof of result (16) is similar to proof of result (15).

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