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Decomposition Theorems and Pluriharmonicity of Conformal Quasi Bi-Slant Submersions from Cosymplectic Manifolds

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Abstract

In this paper, we study conformal quasi bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds as a generalization of bi slant submersions and hemi-slant submersions. We discuss integrability conditions for distributions with the study of geometry of leaves of the distributions. Also, we explore some decomposition theorems and pluriharmonicity for conformal quasi bi-slant submersion and provide non-trivial examples to support the study.

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1. Introduction

Both mathematics and physics employ immersions and submersions exten sively. Yang-Mills theory ([?], [?]), Kaluza-Klein theory ([?], [?]) are the significant application of submersion. The characteristics of slant submersions have become a fascinating topic in differential geometry, as well as in complex and contact geometry. Riemannian submersion between Riemannian manifold was studied by B.O' Neill [?] and A. Gray [?], independently. In 1976, B. Watson [?], considered the submersion between almost Hermitian manifolds with name as almost Hermitian submersions. Since then, they have been widely used in differential geometry to study Riemannian manifolds having differentiable structures [?]. A step forward, R Prasad et. al. studied quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds [?], [?].

The notion of almost contact Riemannian submersions between almost contact metric manifold was introduced by D. Chinea in [?]. He studied the fibre space, base space and total space with differential geometric point of view. As a generalization of Riemannian submersions, Fuglede [?] and Ishihara [?], separately studied horizontally conformal submersions. Later on, many authors investigated different kinds of conformal Riemannian submersions like conformal anti-invariant submersions ([?], conformal slant submersions [?], conformal semi-slant submersions ([?], [?], [?]) and conformal hemi-slant submersions

([?], [?]) etc. from almost Hermitian manifolds onto a Riemannian manifold. Most of these Riemannian submersions and conformal submersions are also studied from almost contact metric manifolds onto a Riemannian manifold.

In this paper, we study conformal quasi bi-slant submersions from cosymplectic manifold onto a Riemannian manifold. This paper is divided into seven sections. Section 2 contains brief history of Riemannian and conformal submersions. Also, we recall almost contact metric manifolds and, in particular, cosymplectic manifolds. In section 3, we investigate some fundamental results for conformal quasi bi-slant submersions from cosymplectic manifolds onto a Riemannian manifold those are required for our main sections. The results of integrability and totally geodesicness of distributions are presented in Section 4. In section 5, we obtain some conditions under which a Riemannian submersion become totally geodesic and we work out on decomposition theorems for the fibres and the total space of such submersions. Section 6 is devoted to the study of pluriharmonicity of conformal quasi bi-slant submersion while last section contains some non-trivial example of conformal quasi bi-slant submersions from cosymplectic manifold.

Note: We will use some abbreviations throughout the paper as follows: Riemannian submersion- RS, Riemannian Manifold- RM, Almost contact metric manifold-ACM manifold, Quasi bi-slant conformal submersion- CQBS submersion, gradient- G.

2. Preliminaries

Let M be a ($2n+1$)-dimensional almost contact manifold with almost contact structures (ϕ, ξ, η), where a (1,1) tensor field ϕ, a vector field ξ and a 1 -form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta \circ \phi = 0, \eta(\xi) = 1, \#(1)$$

where I is the identity tensor. An almost contact structure on M is said to be normal if the induced almost complex structure J on the product manifold $M \times R$, defined by

$$J\left(\widehat{U}, \lambda \frac{d}{dt}\right) = \left(\phi\widehat{U} - \lambda\xi, \eta(\widehat{U}) \frac{d}{dt}\right), \#(2)$$

is integrable, where $\widehat{U}$ is a vector field tangent to M, t is the co-ordinate function on R and λ is a smooth function on $M \times R$. There exists a Riemannian metric g on almost contact manifold which is compatible with the almost contact structure (ϕ, ξ, η) in such a way that

$$g(\phi\widehat{U}_1, \phi\widehat{V}_1) = g(\widehat{U}_1, \widehat{V}_1) - \eta(\widehat{U}_1)\eta(\widehat{V}_1), \#(3)$$

from which it can be observed that

$$\eta(\widehat{U}_1) = g(\widehat{U}_1, \xi), \#(4)$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(TM)$. Then (ϕ, ξ, η, g)-structure is called an almost contact metric structure. An almost contact metric manifold (ACM manifold) with almost contact structure (ϕ, ξ, η, g) is called a cosymplectic manifold if

$$(\nabla_{\widehat{U}_1}\phi)\widehat{V}_1 = 0, \#(5)$$

where ∇ is the Levi-Civita connection of g. From above formula, for cosymplectic manifold, we have

$$\nabla_{\widehat{U}_1}\xi = 0, \#(6)$$

for any vector fields U_1, V_1 on M_1.

We recall the following example from [?]:

Example 1 [?] R^{2n+1} with Cartesian coordinates $(x_i, y_i, z)(i = 1,2, \dots, n)$ and its usual contact form $\eta = dz$. The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric g and tensor field- ϕ are given by

$$g=\sum_{i=1}^{n}((dx_i)^2+(dy_i)^2)+(dz)^2,\phi=\begin{pmatrix}0 & \delta_{ij} & 0\\ -\delta_{ij} & 0 & \\ 0 & 0 & 0\end{pmatrix} i=1,2,\dots,n.$$

This gives a cosymplectic structure on R^{2n+1}. The vector fields $e_i=\frac{\partial}{\partial y_i},e_{n+i}=\frac{\partial}{\partial x_i},\xi$ form a ϕ-basis for the cosymplectic structure.

The covariant derivative of ϕ is defined by

$$(\nabla_{\hat{U}_1}\phi)\hat{V}_1=\nabla_{\hat{U}_1}\phi\hat{V}_1-\phi\nabla_{\hat{U}_1}\hat{V}_1,\#(7)$$

for any vector fields $\hat{U}_1,\hat{V}_1\in\Gamma(TM)$. Now, we provide a definition for conformal submersion and discuss some useful results that help us to achieve our main results.

Definition 1 Let $\bar{f}$ be a RS from an ACM manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then, $\bar{f}$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$g_1(\hat{U}_1,\hat{V}_1)=\frac{1}{\lambda^2}g_2(\bar{f}_*\hat{U}_1,\bar{f}_*\hat{V}_1),\#(8)$$

for any $\hat{U}_1,\hat{V}_1\in\Gamma(\ker\bar{f}_*)^{\perp}$. It is obvious that every RS is a particularly horizontally conformal submersion with $\lambda=1$.

Let $\bar{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a RS. A vector field $\bar{X}$ on M_1 is called a basic vector field if $\bar{X}\in\Gamma(\ker\bar{f}_*)^{\perp}$ and $\bar{f}$-related with a vector field $\bar{X}$ on M_2 i.e., $\bar{f}_*(\bar{X}(q))=\bar{X}\bar{f}(q)$ for $q\in M_1$.

The formulae below provide two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$, plays a crucial role in the theory of submersion

$$\mathcal{A}_{E_1}F_1=\mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}F_1+\mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}F_1,\#(9)$$

$$\mathcal{T}_{E_1}F_1=\mathcal{H}\nabla\mathcal{V}_{E_1}\mathcal{V}F_1+\mathcal{V}\nabla\mathcal{V}_{E_1}\mathcal{H}F_1,\#(10)$$

for any $E_1,F_1\in\Gamma(TM_1)$ and ∇ is Levi-Civita connection of g_1[?]. Note that a RS $\bar{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. From (??) and (??), we can deduce

$$\nabla_{\hat{U}_1}\hat{V}_1=\mathcal{T}_{\hat{U}_1}\hat{V}_1+\mathcal{V}\nabla_{\hat{U}_1}\hat{V}_1\#(11)$$

$$\nabla_{\hat{U}_1}\hat{X}_1=\mathcal{T}_{\hat{U}_1}\hat{X}_1+\mathcal{H}\nabla_{\hat{U}_1}\hat{X}_1\#(12)$$

$$\nabla_{\hat{X}_1}\hat{U}_1=\mathcal{A}_{\hat{X}_1}\hat{U}_1+\mathcal{V}_1\nabla_{\hat{X}_1}\hat{U}_1\#(13)$$

$$\nabla_{\hat{X}_1}\hat{Y}_1=\mathcal{H}\nabla_{\hat{X}_1}\hat{Y}_1+\mathcal{A}_{\hat{X}_1}\hat{Y}_1\#(14)$$

for any vector fields $\hat{U}_1,\hat{V}_1\in\Gamma(\ker\bar{f}_*)$ and $\hat{X}_1,\hat{Y}_1\in\Gamma(\ker\bar{f}_*)^{\perp}$.

It is observe that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, that is,

$$g(\mathcal{A}_{\hat{X}}E_1,F_1)=-g(E_1,\mathcal{A}_{\hat{X}}F_1),g(\mathcal{T}_{\hat{V}}E_1,F_1)=-g(E_1,\mathcal{T}_{\hat{V}}F_1),\#(15)$$

for any vector fields $E_1,F_1\in\Gamma(T_pM_1)$.

Definition 2 A horizontally conformally submersion $\bar{f}:M_1\to M_2$ is called horizontally homothetic if the gradient (G) of its dilation λ is vertical, i.e.,

$$\mathcal{H}(G\lambda)=0,\#(16)$$

where $\mathcal{H}$ is a projection map to horizontal distribution in $\Gamma(T_pM)$.

The second fundamental form of smooth map $\bar{f}$ is given by

$$(\nabla\bar{f}_*)(\hat{U}_1,\hat{V}_1)=\nabla^{\bar{f}}_{\hat{U}_1}\bar{f}_*\hat{V}_1-\bar{f}_*\nabla_{\hat{U}_1}\hat{V}_1\#(17)$$

and the map be totally geodesic if $(\nabla\bar{f}_*)(\hat{U}_1,\hat{V}_1)=0$ for all $\hat{U}_1,\hat{V}_1\in\Gamma(T_pM)$, where ∇ and $\nabla\bar{f}$ are Levi-Civita and pullback connections.

Lemma 1 Let $\bar{f}: M_1\to M_2$ be a horizontal conformal submersion. Then, we have

(i) $(\nabla\bar{f}_*)(\hat{X}_1,\hat{Y}_1)=\hat{X}_1(\ln\lambda)\bar{f}_*(\hat{Y}_1)+\hat{Y}_1(\ln\lambda)\bar{f}_*(\hat{X}_1)-g_1(\hat{X}_1,\hat{Y}_1)\bar{f}_*(\mathrm{grad}\ln\lambda)$,

(ii) $(\nabla\bar{f}_*)(\hat{U}_1,\hat{V}_1)=-\bar{f}_*(\mathcal{T}_{\hat{U}_1}\hat{V}_1)$,

(iii) $(\nabla\bar{f}_*)(\hat{X}_1,\hat{U}_1)=-\bar{f}_*(\nabla_{\hat{X}_1}\hat{U}_1)=-\bar{f}_*(\mathcal{A}_{\hat{X}_1}\hat{U}_1)$,

for any horizontal vector fields $\hat{X}_1,\hat{Y}_1$ and vertical vector fields $\hat{U}_1,\hat{V}_1$ [?].

3. Conformal quasi bi-slant submersions

Definition 3 [?] A RS $\bar{f}$ from an ACM manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) is called conformal quasi bi-slant (CQBS) submersion if there exists four mutually orthogonal distributions $D^T, D^{\theta_1}, D^{\theta_2}$ and $<\xi>$ such that

(i) $\ker\bar{f}_* = D^T\oplus D^{\theta_1}\oplus D^{\theta_2}\oplus<\xi>$,

(ii) D^T is p invariant. i.e., $\phi D^T=D^T$,

(iii) $\phi D^{\theta_1}\perp D^{\theta_2}$ and $\phi D^{\theta_2}\perp D^{\theta_1}$,

(iv) for any non-zero vector field $\hat{V}_1\in(D^{\theta_1})_p, p\in M_1$ the angle θ_1 between $(D^{\theta_1})_p$ and $\phi\hat{V}_1$ is constant and independent of the choice of the point p and $\hat{V}_1\in(D^{\theta_1})_p$,

(v) for any non-zero vector field $\hat{V}_1\in(D^{\theta_2})_q, q\in M_1$ the angle θ_2 between $(D^{\theta_2})_q$ and $\phi\hat{V}_1$ is constant and independent of the choice of the point q and $\hat{V}_1\in(D^{\theta_2})_q$,

where θ_1 and θ_2 are called the slant angles of submersion and $<\xi>$ is the one-dimensional distribution spanned by ξ.

If we denote the dimensions of D^T, D^{θ_1} and D^{θ_2} by m_1, m_2 and m_3 respectively, then we have the following observations:

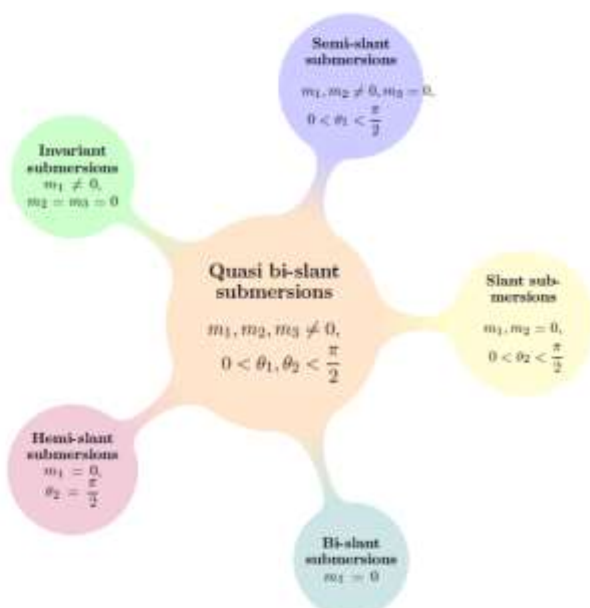

Fig. 1: Classification of Quasi Bi-Slant Submersion

Hence, it is clear that CQBS submersions are generalized version of conformal quasi hemi-slant submersions.

Let $\bar{f}$ be a CQBS submersion from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2). Then, for any $U \in (\ker \bar{f}_*)$, we have

$$\hat{U} = \mathfrak{A}\hat{U} + \mathfrak{B}\hat{U} + \mathfrak{C}\hat{U} + \eta(\hat{U})\xi, \#(18)$$

where $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are the projections morphism onto D^T, D^{θ_1}, and D^{θ_2}, respectively. Now, for any $\hat{U} \in (\ker \bar{f}_*)$, we have

$$\phi\hat{U} = \delta\hat{U} + \zeta, \hat{U}\#(19)$$

where $\delta\hat{U} \in \Gamma(\ker \bar{f}_*)$ and $\zeta\hat{U} \in \Gamma(\ker \bar{f}_*)^{\perp}$. From equations (??) and (??), we have

$$\begin{aligned}\phi\hat{U} &= \phi(\mathfrak{A}\hat{U}) + \phi(\mathfrak{B}\hat{U}) + \phi(\mathfrak{C}\hat{U}) \\ &= \delta(\mathfrak{A}\hat{U}) + \zeta(\mathfrak{A}\hat{U}) + \delta(\mathfrak{B}\hat{U}) + \zeta(\mathfrak{B}\hat{U}) + \delta(\mathfrak{C}\hat{U}) + \zeta(\mathfrak{C}\hat{U})\end{aligned}$$

Since $\phi D^T = D^T$ and $\zeta(\mathfrak{A}\hat{U}) = 0$, we have

$$\phi\hat{U} = \delta(\mathfrak{A}\hat{U}) + \delta(\mathfrak{B}\hat{U}) + \zeta(\mathfrak{B}\hat{U}) + \delta(\mathfrak{C}\hat{U}) + \zeta(\mathfrak{C}\hat{U})$$

Hence, we have the decomposition as

$$\phi(\ker \bar{f}_*) = \delta D^T \oplus \delta D^{\theta_1} \oplus \delta D^{\theta_2} \oplus \zeta D^{\theta_1} \oplus \zeta D^{\theta_2}. \#(20)$$

From equations (??), we get

$$(\ker \bar{f}_*)^{\perp} = \zeta D^{\theta_1} \oplus \zeta D^{\theta_2} \oplus \mu, \#(21)$$

where μ is the orthogonal complement to $\zeta D^{\theta_1} \oplus \zeta D^{\theta_2}$ in $(ker \bar{f}_*)^{\perp}$ and μ is invariant with respect to ϕ. Now, for any $\hat{X} \in \Gamma(\ker \bar{f}_*)^{\perp}$, we have

$$\phi\hat{X} = P\hat{X} + Q\hat{X}\#(22)$$

where $P\hat{X} \in \Gamma(\ker \bar{f}_*)$ and $Q\hat{X} \in \Gamma(\mu)$.

Lemma 2 Let $(M_1, \phi, \xi, \eta, g_1)$ be an ACM manifold and (M_2, g_2) be a RM. If $\bar{f}: M_1 \to M_2$ is a CQBS submersion, then we have

$$\begin{gathered}-\hat{U} + \eta(\hat{U})\xi = \delta^2\hat{U} + P\delta\hat{U}, \zeta\hat{U} + Q\zeta\hat{U} = 0 \\ -\hat{X} = \zeta P\hat{X} + Q^2\hat{X}, \delta P\hat{X} + PQ\hat{X} = 0\end{gathered}$$

for $\hat{U} \in \Gamma(\ker \bar{f}_*)$ and $\hat{X} \in \Gamma(\ker \bar{f}_*)^{\perp}$.

Proof On using equations (??), (??) and (??), we get the desired results.

Since $\bar{f}: M_1 \to M_2$ is a CQBS submersion, Here we give some useful results that will be used throughout the paper.

Lemma 3 [?] Let $\bar{f}$ be a CQBS submersion from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a $RM(M_2, g_2)$, then we have

(i) $\delta^2\hat{U} = -\cos^2\theta_1$,

(ii) $g_1(\delta\hat{U}, \delta\hat{V}) = \cos^2\theta_1 g_1(\hat{U}, \hat{V})$,

(iii) $g(\zeta\hat{U}, \zeta\hat{V}) = \sin^2\theta_1 g_1(\hat{U}, \hat{V})$,

for any vector fields $\hat{U}, \hat{V} \in \Gamma(D^{\theta_1})$.

Lemma 4 [?] Let $\bar{f}$ be a CQBS submersion from an ACM manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a $RM(M_2, g_2)$, then we have

(i) $\delta^2\hat{Z} = -\cos^2\theta_2\hat{Z}$,

(ii) $g_1(\delta\hat{Z}, \delta\hat{W}) = \cos^2\theta_2 g_1(\hat{Z}, \hat{W})$,

(iii) $g_1(\zeta\hat{Z}, \zeta\hat{W}) = \sin^2\theta_2 g_1(\hat{Z}, \hat{W})$, for any vector fields $\hat{Z},\widehat{W} \in \Gamma(D^{\theta_2})$.

The proof of above Lemmas is similar to the proof of the Theorem 3.5 of [?]. Thus, we omit the proofs.

Let (M_2, g_2) be a Riemannian manifold and that (M_1,ϕ,ξ,η,g_1) is a cosymplectic manifold. We now observe how the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a CQBS submersion $\bar{f}:(M_1,\phi,\xi,\eta,g_1) \to (M_2,g_2)$ are affected by the cosymplectic structure on M_1.

Lemma 5 Let $\bar{f}$ be a CQBS submersion from an ACM manifold (M_1,ϕ,ξ,η,g_1) onto a $RM(M_2,g_2)$, then we have

$$\mathcal{A}_{\hat{X}}P\hat{Y} + \mathcal{H}\nabla_{\hat{X}}Q\hat{Y} \ = \delta\mathcal{H}\nabla_{\hat{X}}\hat{Y} + P\mathcal{A}_{\hat{X}}\hat{Y} \#(23)$$
$$\mathcal{V}\nabla_{\hat{X}}P\hat{Y} + \mathcal{A}_{\hat{X}}Q\hat{Y} \ = \zeta\mathcal{H}\nabla_{\hat{X}}\hat{Y} + Q\mathcal{A}_{\hat{X}}\hat{Y} \#(24)$$
$$\mathcal{V}\nabla_{\hat{X}}\delta\hat{V} + \mathcal{A}_{\hat{X}}\zeta\hat{V} \ = P\mathcal{A}_{\hat{X}}\hat{V} + \delta\mathcal{V}\nabla_{\hat{X}}\hat{V} \#(25)$$
$$\mathcal{A}_{\hat{X}}\delta\hat{V} + \mathcal{H}\nabla_{\hat{X}}\zeta\hat{V} \ = \mathfrak{C}\mathcal{A}_{\hat{X}}\hat{V} + \zeta\mathcal{V}\nabla_{\hat{X}}\hat{V} \#(26)$$
$$\mathcal{V}\nabla_{\hat{V}}P\hat{X} + \mathcal{T}_{\hat{V}}Q\hat{X} \ = \delta\mathcal{T}_{\hat{V}}Q\hat{X} + P\mathcal{H}\nabla_{\hat{V}}\hat{X} \#(27)$$
$$\mathcal{T}_{\hat{V}}P\hat{X} + \mathcal{H}\nabla_{\hat{V}}Q\hat{X} \ = \zeta\mathcal{T}_{\hat{V}}\hat{X} + Q\mathcal{H}\nabla_{\hat{V}}\hat{X} \#(28)$$
$$\mathcal{V}\nabla_{\hat{U}}\delta\hat{V} + \mathcal{T}_{\hat{U}}\zeta\hat{V} \ = \mathfrak{B}\mathcal{T}_{\hat{U}}\hat{V} + \delta\mathcal{V}\nabla_{\hat{U}}\hat{V} \#(29)$$
$$\mathcal{T}_{\hat{U}}\delta\hat{V} + \mathcal{H}\nabla_{\hat{U}}\zeta\hat{V} \ = Q\mathcal{T}_{\hat{U}}\hat{V} + \zeta\mathcal{V}\nabla_{\hat{U}}\hat{V} \#(30)$$

for any vector fields $\widehat{U},\hat{V} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X},\hat{Y} \in \Gamma(\ker\bar{f}_*)^\perp$.

Proof By the direct calculation, using (??), (??) and (??), we can easily obtain relations given by (??) and (??). Remaining relations can be obtained similarly by using (??), (??), (??)-(??) and (??).

Now, we discuss some basic results which are useful to explore the geometry of CQBS submersion $\bar{f}: M_1 \to M_2$. For this, define the following :

$$(\nabla_{\hat{U}}\delta)\hat{V} \ = \mathcal{V}\nabla_{\hat{U}}\delta\hat{V} - \delta\mathcal{V}\nabla_{\hat{U}}\hat{V} \#(31)$$
$$(\nabla_{\hat{U}}\zeta)\hat{V} \ = \mathcal{H}\nabla_{\hat{U}}\zeta\hat{V} - \zeta\mathcal{V}\nabla_{\hat{U}}\hat{V} \#(32)$$
$$(\nabla_{\hat{X}}P)\hat{Y} \ = \mathcal{V}\nabla_{\hat{X}}P\hat{Y} - P\mathcal{H}\nabla_{\hat{X}}\hat{Y} \#(33)$$
$$(\nabla_{\hat{X}}Q)\hat{Y} \ = \mathcal{H}\nabla_{\hat{X}}Q\hat{Y} - Q\mathcal{H}\nabla_{\hat{X}}\hat{Y} \#(34)$$

for any vector fields $\widehat{U},\hat{V} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X},\hat{Y} \in \Gamma(\ker\bar{f}_*)^\perp$.

Lemma 6 Let (M_1,ϕ,ξ,η,g_1) be cosymplectic manifold and (M_2,g_2) be a RM. If $\bar{f}: M_1 \to M_2$ is a CQBS submersion, then we have

$$(\nabla_{\hat{U}}\delta)\hat{V} = P\mathcal{T}_{\hat{U}}\hat{V} - \mathcal{T}_{\hat{U}}\zeta\hat{V}$$
$$(\nabla_{\hat{U}}\zeta)\hat{V} = Q\mathcal{T}_{\hat{U}}\hat{V} - \mathcal{T}_{\hat{U}}\delta\hat{V}$$
$$(\nabla_{\hat{X}}P)\hat{Y} = \delta\mathcal{A}_{\hat{X}}\hat{Y} - \mathcal{A}_{\hat{X}}Q\hat{Y}$$
$$(\nabla_{\hat{X}}Q)\hat{Y} = \zeta\mathcal{A}_{\hat{X}}\hat{Y} - \mathcal{A}_{\hat{X}}P\hat{Y}$$

for all vector fields $\widehat{U},\hat{V} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X},\hat{Y} \in \Gamma(\ker\bar{f}_*{}^\perp)$.

Proof On using equations (??), (??)- (??) and equations (??)-(??), we get the proof of the lemma.

If the tenors δ and ζ are parallel with respect to the connection ∇ of M_1 then, we have

$$P\mathcal{T}_{\hat{U}}\hat{V} = \mathcal{T}_{\hat{U}}\zeta\hat{V}, Q\mathcal{T}_{\hat{U}}\hat{V} = \mathcal{T}_{\hat{U}}\delta\hat{V}$$

for any vector fields $\widehat{U},\hat{V} \in \Gamma(TM_1)$.

4. Integrability and totally geodesicness of distributions

Since, $\bar{f}: M_1 \to M_2$ is a CQBS submersion which ensure the existence of three orthogonal complementary distributions. We investigate the integrability conditions for these distributions. Now, we initiate with invariant distribution.

Theorem 1 Let $\bar{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a CQBS submersion, where (M_1,ϕ,ξ,η,g_1) a cosymplectic manifold and (M_2,g_2) a RM. Then the invariant distribution D^T is integrable if and only if

$$g_1\big(\mathcal{T}_{\hat{U}}\delta\mathfrak{A}\hat{V}-\mathcal{T}_{\hat{V}}\delta\mathfrak{A}\hat{U},\zeta\mathfrak{B}\hat{Z}+\zeta\mathfrak{C}\hat{W}\big)$$

for any $\hat{U},\hat{V}\in\Gamma(D^T)$ and $\hat{Z}\in\Gamma\big(D^{\theta_1}\oplus D^{\theta_2}\oplus<\xi>\big)$.

Proof For all $\hat{U},\hat{V}\in\Gamma(D^T)$ and $\hat{Z}\in\Gamma\big(D^{\theta_1}\oplus D^{\theta_2}\oplus<\xi>\big)$ with using equations (??), (??), (??) and decomposition (??), we have

$$g_1([\hat{U},\hat{V}],\hat{Z})=g_1\big(\nabla_{\hat{U}}\delta\mathfrak{A}\hat{V},\phi\mathfrak{B}\hat{Z}+\phi\mathfrak{C}\hat{Z}\big)-g_1\big(\nabla_{\hat{V}}\delta\mathfrak{A}\hat{U},\phi\mathfrak{B}\hat{Z}+\phi\mathfrak{C}\hat{Z}\big)$$

On using equation (??), we finally have

$$\begin{aligned}g_1([\hat{U},\hat{V}],\hat{Z})=&g_1\big(\mathcal{T}_{\hat{U}}\delta\mathfrak{A}\hat{V}-\mathcal{T}_{\hat{V}}\delta\mathfrak{A}\hat{U},\zeta\mathfrak{B}\hat{Z}+\zeta\mathfrak{C}\hat{Z}\big)\\&+g_1\big(\mathcal{V}\nabla_{\hat{U}}\delta\mathfrak{A}\hat{V}-\mathcal{V}\mathcal{A}_{\hat{V}}\delta\mathfrak{A}\hat{U},\delta\mathfrak{B}\hat{Z}+\delta\mathfrak{C}\hat{Z}\big)\end{aligned}$$

This completes the proof of theorem.

In a similar way, we can examine the condition of integrability for slant distribution as follows:

Theorem 2 Let $\bar{f}$ be a CQBS submersion from cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a $RM(M_2,g_2)$. Then slant distribution D^{θ_1} is integrable if and only if

$$\begin{aligned}&\frac{1}{\lambda^2}\Big\{g_2\Big(\nabla^{\bar{f}}_{\hat{U}_1}\bar{f}_*\zeta\hat{V}_1+\nabla^{\bar{f}}_{\hat{V}_1}\bar{f}_*\zeta\hat{U}_1,\bar{f}_*\zeta\mathfrak{C}\hat{Z}\Big)\Big\}\\&-g_1\big(\nabla_{\hat{V}_1}\zeta\delta\hat{U}_1-\nabla_{\hat{U}_1}\zeta\delta\hat{V}_1,\hat{Z}\big)-g_1\big(\mathcal{T}_{\hat{U}_1}\zeta\hat{V}_1-\mathcal{T}_{\hat{V}_1}\zeta\hat{U}_1,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{C}\hat{Z}\big)\end{aligned}$$

for any $\hat{U}_1,\hat{V}_1\in\Gamma\big(D^{\theta_1}\big)$ and $\hat{Z}\in\Gamma\big(D^T\oplus D^{\theta_2}\oplus<\xi>\big)$.

Proof For $\hat{U}_1,\hat{V}_1\in\Gamma\big(D^{\theta_1}\big)$ and $\hat{Z}\in\Gamma\big(D^T\oplus D^{\theta_2}\oplus<\xi>\big)$ with using equations (??), (??), (??) and (??), we get

$$g_1\big([\hat{U}_1,\hat{V}_1],\hat{Z}\big)=g_1\big(\nabla_{\hat{U}_1}\delta\hat{V}_1,\phi\hat{Z}\big)+g_1\big(\nabla_{\hat{U}_1}\zeta\hat{V}_1,\phi\hat{Z}\big)-g_1\big(\nabla_{\hat{V}_1}\delta\hat{U}_1,\phi\hat{Z}\big)-g_1\big(\nabla_{\hat{V}_1}\zeta\hat{U}_1,\phi\hat{Z}\big).$$

By using equations (??), (??) and (??), we have

$$\begin{aligned}g_1\big([\hat{U}_1,\hat{V}_1],\hat{Z}\big)=&-g_1\big(\nabla_{\hat{U}_1}\delta^2\hat{V}_1,\hat{Z}\big)-g_1\big(\nabla_{\hat{U}_1}\zeta\delta\hat{V}_1,\hat{Z}\big)+g_1\big(\nabla_{\hat{V}_1}\delta^2\hat{U}_1,\hat{Z}\big)\\&+g_1\big(\nabla_{\hat{V}_1}\zeta\delta\hat{U}_1,\hat{Z}\big)+g_1\big(\nabla_{\hat{U}_1}\zeta\hat{V}_1,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{C}\hat{Z}+\zeta\mathfrak{C}\hat{Z}\big)\\&-g_1\big(\nabla_{\hat{V}_1}\zeta\hat{U}_1,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{C}\hat{Z}+\zeta\mathfrak{C}\hat{Z}\big)\end{aligned}$$

Taking account the fact of Lemma ?? with using equation (??), we get

$$\begin{aligned}g_1\big([\hat{U}_1,\hat{V}_1],\hat{Z}\big)=&\cos^2\theta_1 g_1\big([\hat{U}_1,\hat{V}_1],\hat{Z}\big)+g_1\big(\nabla_{\hat{V}_1}\zeta\delta\hat{U}_1-\nabla_{\hat{U}_1}\zeta\delta\hat{V}_1,\hat{Z}\big)\\&+g_1\big(\mathcal{T}_{\hat{U}_1}\zeta\hat{V}_1-\mathcal{T}_{\hat{V}_1}\zeta\hat{U}_1,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{C}\hat{Z}\big)\\&+g_1\big(\mathcal{H}\nabla_{\hat{U}_1}\zeta\hat{V}_1-\mathcal{H}\nabla_{\hat{V}_1}\zeta\hat{U}_1,\zeta\mathfrak{C}\hat{Z}\big)\end{aligned}$$

On using equation (??), formula (??) with Lemma ??, we finally get

$$\begin{aligned}\sin^2\theta_1 g_1\big([\hat{U}_1,\hat{V}_1],\hat{Z}\big)=&\frac{1}{\lambda^2}\Big\{g_2\Big(\nabla^{\bar{f}}_{\hat{U}_1}\bar{f}_*\zeta\hat{V}_1-\nabla^{\bar{f}}_{\hat{V}_1}\bar{f}_*\zeta\hat{U}_1,\bar{f}_*\zeta\mathfrak{C}\hat{Z}\Big)\Big\}\\&+\frac{1}{\lambda^2}\Big\{g_2\Big(\big(\nabla\bar{f}_*\big)\big(\hat{U}_1,\zeta\hat{V}_1\big),\bar{f}_*\zeta\mathfrak{C}\hat{Z}\Big)+g_2\Big(\big(\nabla\bar{f}_*\big)\big(\hat{V}_1,\zeta\hat{U}_1\big),\bar{f}_*\zeta\mathfrak{C}\hat{Z}\Big)\Big\}\\&+g_1\big(\mathcal{T}_{\hat{U}_1}\zeta\hat{V}_1-\mathcal{T}_{\hat{V}_1}\zeta\hat{U}_1,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{C}\hat{Z}\big)+g_1\big(\nabla_{\hat{V}_1}\zeta\delta\hat{U}_1-\nabla_{\hat{U}_1}\zeta\delta\hat{V}_1,\hat{Z}\big)\end{aligned}$$

Theorem 3 Let $\bar{f}:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a CQBS submersion, where (M_1,ϕ,ξ,η,g_1) a cosymplectic manifold and (M_2,g_2) a RM. Then slant distribution D^{θ_2} is integrable if and only if

$$-\frac{1}{\lambda^2}\left\{g_2\left((\nabla\bar{f}_*)(\hat{U}_2,\zeta\hat{V}_2)-(\nabla\bar{f}_*)(\hat{V}_2,\zeta\hat{U}_2),\bar{f}_*\zeta\mathfrak{B}\hat{Z}\right)\right\}$$
$$=g_1\big(\mathcal{T}_{\hat{V}_2}\zeta\delta\hat{U}_2-\mathcal{T}_{\hat{U}_2}\zeta\delta\hat{V}_2,\hat{Z}\big)+g_1\big(\mathcal{T}_{\hat{U}_2}\zeta\hat{V}_2-\mathcal{T}_{\hat{V}_2}\zeta\hat{U}_2,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{B}\hat{Z}\big)$$
$$+\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar{f}}_{\hat{U}_2}\bar{f}_*\zeta\hat{V}_2-\nabla^{\bar{f}}_{\hat{V}_2}\bar{f}_*\zeta\hat{U}_2,\bar{f}_*\zeta\mathfrak{B}\hat{Z}\right)\right\}$$

for any $\hat{U}_2,\hat{V}_2\in\Gamma(D^{\theta_2})$ and $\hat{Z}\in\Gamma(D^T\oplus D^{\theta_1}\oplus<\xi>)$.

Proof On using equations (??), (??), (??) and (??), we have

$$g_1([\hat{U}_2,\hat{V}_2],\hat{Z})=g_1(\nabla_{\hat{V}_2}\delta^2\hat{U}_2,\hat{Z})+g_1(\nabla_{\hat{V}_2}\zeta\delta\hat{U}_2,\hat{Z})-g_1(\nabla_{\hat{U}_2}\delta^2\hat{V}_2,\hat{Z})$$
$$-g_1(\nabla_{\hat{U}_2}\zeta\delta\hat{V}_2,\hat{Z})+g_1(\nabla_{\hat{U}_2}\zeta\hat{V}_2-\nabla_{\hat{V}_2}\zeta\hat{U}_2,\phi\hat{Z})$$

for any $\hat{U}_2,\hat{V}_2\in\Gamma(D^{\theta_2})$ and $\hat{Z}\in\Gamma(D^T\oplus D^{\theta_1}\oplus<\xi>)$. From equations (??) and Lemma ??, we get

$$\sin^2\theta_2 g_1([\hat{U}_2,\hat{V}_2],\hat{Z})=g_1\big(\mathcal{T}_{\hat{V}_2}\zeta\delta\hat{U}_2-\mathcal{T}_{\hat{U}_2}\zeta\delta\hat{V}_2,\hat{Z}\big)+g_1\big(\mathcal{T}_{\hat{U}_2}\zeta\hat{V}_2-\mathcal{T}_{\hat{V}_2}\zeta\hat{U}_2,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{B}\hat{Z}\big)$$
$$+g_1\big(\mathcal{H}\nabla_{\hat{U}_2}\zeta\hat{V}_2-\mathcal{H}\nabla_{\hat{V}_2}\zeta\hat{U}_2,\zeta\mathfrak{B}\hat{Z}\big)$$

Since $\bar{f}$ is CQBS submersion, using conformality condition with equations (??) and (??), we finally get

$$\sin^2\theta_2 g_1([\hat{U}_2,\hat{V}_2],\hat{Z})=g_1\big(\mathcal{T}_{\hat{V}_2}\zeta\delta\hat{U}_2-\mathcal{T}_{\hat{U}_2}\zeta\delta\hat{V}_2,\hat{Z}\big)+g_1\big(\mathcal{T}_{\hat{U}_2}\zeta\hat{V}_2-\mathcal{T}_{\hat{V}_2}\zeta\hat{U}_2,\phi\mathfrak{A}\hat{Z}+\delta\mathfrak{B}\hat{Z}\big)$$
$$+\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar{f}}_{\hat{U}_2}\bar{f}_*\zeta\hat{V}_2-\nabla^{\bar{f}}_{\hat{V}_2}\bar{f}_*\zeta\hat{U}_2,\bar{f}_*\zeta\mathfrak{B}\hat{Z}\right)\right\}$$
$$+\frac{1}{\lambda^2}\left\{g_2\left((\nabla\bar{f}_*)(\hat{U}_2,\zeta\hat{V}_2)-(\nabla\bar{f}_*)(\hat{V}_2,\zeta\hat{U}_2),\bar{f}_*\zeta\mathfrak{B}\hat{Z}\right)\right\}$$

This completes the proof of the theorem.

Now, we discuss the geometry of leaves of the distributions and initiate with horizontal distribution $(\ker\bar{f}_*)^\perp$. There is necessary and sufficient conditions under which horizontal distribution is defines totally geodesic foliation on M_1.

Theorem 4 Let $\bar{f}$ be a CQBS submersion from cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then ($\ker\bar{f}_*)^\perp$ defines totally geodesic foliation on M_1 if and only if

$$\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{B}\hat{Y}+\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{C}\hat{Y},\bar{f}_*\zeta\hat{Z}\right)\right\}$$
$$=g_1\big(\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{B}\hat{Y}+\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{C}\hat{Y}+\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{A}\hat{Y},\hat{Z}\big)$$
$$+g_1(\zeta\mathfrak{B}\hat{Y},\zeta\hat{Z})g_1(\hat{X},G\ln\lambda)+g_1(\hat{X},\zeta\hat{Z})g_1(\zeta\mathfrak{B}\hat{Y},G\ln\lambda)$$
$$-g_1(\hat{X},\zeta\mathfrak{B}\hat{Y})g_1(\zeta\hat{Z},G\ln\lambda)+g_1(\zeta\mathfrak{C}\hat{Y},\zeta\hat{Z})g_1(\hat{X},G\ln\lambda)$$
$$+g_1(\hat{X},\zeta\hat{Z})g_1(\zeta\mathfrak{C}\hat{Y},G\ln\lambda)-g_1(\hat{X},\zeta\mathfrak{C}\hat{Y})g_1(\zeta\hat{Z},G\ln\lambda)$$

for any $\hat{X},\hat{Y}\in\Gamma(\ker\bar{f}_*)^\perp$ and $\hat{Z}\in\Gamma(\ker\bar{f}_*)$.

Proof For $\hat{X},\hat{Y}\in\Gamma(\ker\bar{f}_*)^\perp$ and $\hat{Z}\in\Gamma(\ker\bar{f}_*)$ with using equations (??), (??), (??) with decomposition (??), we get

$$g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})=g_1(\nabla_{\hat{X}}\phi(\mathfrak{A}\hat{Y}),\phi\hat{Z})+g_1(\nabla_{\hat{X}}\phi(\mathfrak{B}\hat{Y}),\phi\hat{Z})+g_1(\nabla_{\hat{X}}\phi(\mathfrak{C}\hat{Y}),\phi\hat{Z}).$$

From equations (??) and (??) with Lemma ??, we have

$$g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})=g_1(\mathcal{V}\nabla_{\hat{X}}\mathfrak{A}\hat{Y},\hat{Z})+\cos^2\theta_1 g_1(\nabla_{\hat{X}}\mathfrak{B}\hat{Y},\hat{Z})+\cos^2\theta_2 g_1(\nabla_{\hat{X}}\mathfrak{C}\hat{Y},\hat{Z})$$
$$-g_1(\nabla_{\hat{X}}\zeta\delta\mathfrak{B}\hat{Y},\hat{Z})+g_1(\nabla_{\hat{X}}\zeta\mathfrak{B}\hat{Y},\phi\hat{Z})-g_1(\nabla_{\hat{X}}\zeta\delta\mathfrak{C}\hat{Y},\hat{Z})$$
$$+g_1(\nabla_{\hat{X}}\zeta\mathfrak{C}\hat{Y},\phi\hat{Z})-g_1(\nabla_{\hat{X}}\zeta\delta\mathfrak{A}\hat{Y},\hat{Z})$$

On using the equations (??) and (??), we get

$$g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})=g_1(\mathcal{V}\nabla_{\hat{X}}\mathfrak{A}\hat{Y}+\cos^2\theta_1\mathcal{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y}+\cos^2\theta_2\mathcal{V}\nabla_{\hat{X}}\mathfrak{C}\hat{Y},\hat{Z})$$
$$-g_1(\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{A}\hat{Y}+\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{B}\hat{Y}+\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{C}\hat{Y},\hat{Z})$$
$$+g_1(\mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{B}\hat{Y}+\mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{C}\hat{Y},\zeta\hat{Z})$$
$$+g_1(\mathcal{A}_{\hat{X}}\zeta\mathfrak{B}\hat{Y}+\mathcal{A}_{\hat{X}}\zeta\mathfrak{C}\hat{Y},\delta\hat{Z})$$

From formula (??) and (??), we yields that

$$\begin{aligned} g_1\big(\nabla_{\hat{X}}\hat{Y},\hat{Z}\big) =& g_1\big(\mathcal{V}\nabla_{\hat{X}}\mathfrak{A}\hat{Y} + \cos^2\theta_1\mathcal{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y} + \cos^2\theta_2\mathcal{V}\nabla_{\hat{X}}\mathfrak{C}\hat{Y},\hat{Z}\big)\\ &-g_1\big(\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{A}\hat{Y} + \mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{B}\hat{Y} + \mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{C}\hat{Y},\hat{Z}\big)\\ &+\frac{1}{\lambda^2}\Big\{g_2\Big(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{B}\hat{Y} + \nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{C}\hat{Y},\bar{f}_*\zeta\hat{Z}\Big)\Big\}\\ &-\frac{1}{\lambda^2}\Big\{g_2\Big(\big(\nabla\bar{f}_*\big)(\hat{X},\zeta\mathfrak{B}\hat{Y}) + \big(\nabla\bar{f}_*\big)(\hat{X},\zeta\mathfrak{C}\hat{Y}),\bar{f}_*\zeta\hat{Z}\Big)\Big\}. \end{aligned}$$

Since $\bar{f}$ is conformal submersion, then we finally get

$$\begin{aligned} g_1\big(\nabla_{\hat{X}}\hat{Y},\hat{Z}\big) =& g_1\big(\mathcal{V}\nabla_{\hat{X}}\mathfrak{A}\hat{Y} + \cos^2\theta_1\mathcal{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y} + \cos^2\theta_2\mathcal{V}\nabla_{\hat{X}}\mathfrak{C}\hat{Y},\hat{Z}\big)\\ &-g_1\big(\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{A}\hat{Y} + \mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{B}\hat{Y} + \mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{C}\hat{Y},\hat{Z}\big)\\ &+\frac{1}{\lambda^2}\Big\{g_2\Big(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{B}\hat{Y} + \nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{C}\hat{Y},\bar{f}_*\zeta\hat{Z}\Big)\Big\}\\ &-g_1(\zeta\mathfrak{B}\hat{Y},\zeta\hat{Z})g_1(\hat{X},G\ln\lambda) - g_1(\hat{X},\zeta\hat{Z})g_1(\zeta\mathfrak{B}\hat{Y},G\ln\lambda)\\ &+g_1(\hat{X},\zeta\mathfrak{B}\hat{Y})g_1(\zeta\hat{Z},G\ln\lambda) - g_1(\zeta\mathfrak{C}\hat{Y},\zeta\hat{Z})g_1(\hat{X},G\ln\lambda)\\ &-g_1(\hat{X},\zeta\hat{Z})g_1(\zeta\mathfrak{C}\hat{Y},G\ln\lambda) + g_1(\hat{X},\zeta\mathfrak{C}\hat{Y})g_1(\zeta\hat{Z},G\ln\lambda) \end{aligned}$$

This completes the proof of theorem.

In a similar way, we can study the necessary and sufficient condition under which vertical distribution defines totally geodesic foliation.

Theorem 5 Let $\bar{f}\colon(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a CQBS submersion, where (M_1,ϕ,ξ,η,g_1) a cosymplectic manifold and (M_2,g_2) a RM. Then ($\ker\bar{f}_*$) defines totally geodesic foliation on M_1 if and only if

$$\begin{aligned} &\frac{1}{\lambda^2}\Big\{g_2\Big(\nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\delta\mathfrak{B}\hat{V} + \nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\delta\mathfrak{C}\hat{V},\bar{f}_*\hat{X}\Big)\Big\}\\ &+\frac{1}{\lambda^2}\Big\{g_2\Big(\nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\hat{V} - \big(\nabla\bar{f}_*\big)(\hat{U},\zeta\hat{V}),\bar{f}_*Q\hat{X}\Big)\Big\} \end{aligned}$$

for any $\hat{U},\hat{V}\in\Gamma\big(\ker\bar{f}_*\big)$ and $\hat{X}\in\Gamma\big(\ker\bar{f}_*\big)^\perp$.

Proof For any $\hat{U},\hat{V}\in\Gamma\big(\ker\bar{f}_*\big)$ and $\hat{X}\in\Gamma\big(\ker\bar{f}_*\big)^\perp$ with using equations (??), (??), (??) with decomposition (??), we get

$$g_1\big(\nabla_{\hat{U}}\hat{V},\hat{X}\big) = g_1\big(\nabla_{\hat{U}}\phi\mathfrak{A}\hat{V},\phi\hat{X}\big) + g_1\big(\nabla_{\hat{U}}\phi\mathfrak{B}\hat{V},\phi\hat{X}\big) + g_1\big(\nabla_{\hat{U}}\phi\mathfrak{C}\hat{V},\phi\hat{X}\big)$$

On using equations (??) with Lemma ?? and Lemma ??, we have

$$\begin{aligned} g_1\big(\nabla_{\hat{U}}\hat{V},\hat{X}\big) =& g_1\big(\nabla_{\hat{U}}\mathfrak{A}\hat{V},\hat{X}\big) + \cos^2\theta_1 g_1\big(\nabla_{\hat{U}}\mathfrak{B}\hat{V},\hat{X}\big) + \cos^2\theta_2 g_1\big(\nabla_{\hat{U}}\mathfrak{C}\hat{V},\hat{X}\big)\\ &+g_1\big(\nabla_{\hat{U}}\zeta\mathfrak{B}\hat{V},\phi\hat{X}\big) - g_1\big(\nabla_{\hat{U}}\zeta\delta\mathfrak{B}\hat{V},\hat{X}\big) - g_1\big(\nabla_{\hat{U}}\zeta\delta\mathfrak{C}\hat{V},\hat{X}\big)\\ &+g_1\big(\nabla_{\hat{U}}\zeta\mathfrak{C}\hat{V},\phi\hat{X}\big). \end{aligned}$$

From equations (??), (??) and (??), we may yields

$$\begin{aligned} g_1\big(\nabla_{\hat{U}}\hat{V},\hat{X}\big) =& g_1\big(\mathcal{T}_{\hat{U}}\mathfrak{A}\hat{V} + \cos^2\theta_1\mathcal{T}_{\hat{U}}\mathfrak{B}\hat{V} + \cos^2\theta_2\mathcal{T}_{\hat{U}}\mathfrak{C}\hat{V},\hat{X}\big)\\ &-g_1\big(\mathcal{H}\nabla_{\hat{U}}\zeta\delta\mathfrak{B}\hat{V} + \mathcal{H}\nabla_{\hat{U}}\zeta\delta\mathfrak{C}\hat{V},\hat{X}\big) + g_1\big(\mathcal{T}_{\hat{U}}\zeta\mathfrak{B}\hat{V} + \mathcal{T}_{\hat{U}}\zeta\mathfrak{C}\hat{V},P\hat{X}\big)\\ &+g_1\big(\mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{B}\hat{V} + \mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{C}\hat{V},Q\hat{X}\big) \end{aligned}$$

From decomposition (??), the above equation takes the form

$$\begin{aligned} g_1\big(\nabla_{\hat{U}}\hat{V},\hat{X}\big) =& g_1\big(\mathcal{T}_{\hat{U}}\mathfrak{A}\hat{V} + \cos^2\theta_1\mathcal{T}_{\hat{U}}\mathfrak{B}\hat{V} + \cos^2\theta_2\mathcal{T}_{\hat{U}}\mathfrak{C}\hat{V},\hat{X}\big) + g_1\big(\mathcal{T}_{\hat{U}}\zeta\hat{V},P\hat{X}\big)\\ &-g_1\big(\mathcal{H}\nabla_{\hat{U}}\zeta\delta\mathfrak{B}\hat{V} + \mathcal{H}\nabla_{\hat{U}}\zeta\delta\mathfrak{C}\hat{V},\hat{X}\big) + g_1\big(\mathcal{H}\nabla_{\hat{U}}\zeta\hat{V},Q\hat{X}\big) \end{aligned}$$

Using the conformality of $\bar{f}$ with equations (??) and (??), we have

$$
\begin{aligned}
g_1(\nabla_{\hat U}\hat V,\hat X) =& g_1(\mathcal{T}_{\hat U}\mathfrak{A}\hat V + \cos^2\theta_1\mathcal{T}_{\hat U}\mathfrak{B}\hat V + \cos^2\theta_2\mathcal{T}_{\hat U}\mathfrak{C}\hat V,\hat X) + g_1(\mathcal{T}_{\hat U}\zeta\hat V, P\hat X)\\
&+\frac{1}{\lambda^2}\left\{g_2\left((\nabla\bar f_*)(\hat U,\zeta\delta\mathfrak{B}\hat V) - (\nabla\bar f_*)(\hat U,\zeta\delta\mathfrak{C}\hat V),\bar f_*\hat X\right)\right\}\\
&-\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar f}_{\hat U}\bar f_*\zeta\delta\mathfrak{B}\hat V + \nabla^{\bar f}_{\hat U}\bar f_*\zeta\delta\mathfrak{C}\hat V,\bar f_*\hat X\right)\right\}\\
&+\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar f}_{\hat U}\bar f_*\zeta\hat V - (\nabla\bar f_*)(\hat U,\zeta\hat V),\bar f_* Q\hat X\right)\right\}
\end{aligned}
$$

This completes the proof of the theorem.

Now, we discuss the geometry of leaves of the distributions and initiate with invariant distribution D^T. There is necessary and sufficient conditions under which slant distribution D^T is defines totally geodesic foliation on M.

Theorem 6 Let $\bar f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a CQBS submersion, where (M_1,ϕ,ξ,η,g_1) a cosymplectic manifold and (M_2,g_2) a RM. Then invariant distribution D^T defines totally geodesic foliation on M_1 if and only if

(i) $\lambda^{-2}g_2\left\{\left((\nabla\bar f_*)(\hat U,\phi\hat V),\bar f_*\zeta\hat Z\right)\right\} = g_1(\mathcal{V}\nabla_{\hat U}\phi\hat V,\delta\hat Z)$

(ii) $\lambda^{-2}\left\{g_2\left((\nabla\bar f_*)(\hat U,\phi\hat V),\bar f_* Q\hat X\right)\right\} = g_1(\mathcal{V}\nabla_{\hat U}\phi\hat V, P\hat X)$,

for any $\hat U,\hat V\in\Gamma(D^T)$ and $\hat Z\in\Gamma(D^{\theta_1}\oplus D^{\theta_2}\oplus<\xi>$.

Proof For any $\hat U,\hat V\in\Gamma(D^T)$ and $\hat Z\in\Gamma(D^{\theta_1}\oplus D^{\theta_2}\oplus<\xi>)$ with using equations (??), (??), (??) and (??), we may write

$$g_1(\nabla_{\hat U}\hat V,\hat Z) = g_1(\mathcal{V}\nabla_{\hat U}\phi\hat V,\delta\hat Z) + g_1(\mathcal{T}_{\hat U}\phi\hat V,\zeta\hat Z)$$

On using the conformality of $\bar f$ with equation (??) and (??), we get

$$g_1(\nabla_{\hat U}\hat V,\hat Z) = g_1(\mathcal{V}\nabla_{\hat U}\phi\hat V,\delta\hat Z) - \lambda^{-2}g_2\left((\nabla\bar f_*)(\hat U,\phi\hat V),\bar f_*\zeta\hat Z\right)$$

On the other hand, using equations (??), (??), (??) with conformality of $\bar f$, we finally have

$$g_1(\nabla_{\hat U}\hat V,\hat X) = g_1(\mathcal{V}\nabla_{\hat U}\phi\hat V,P\hat X) - \lambda^{-2}g_2\left((\nabla\bar f_*)(\hat U,\phi\hat V),\bar f_* Q\hat X\right)$$

This completes the proof of the theorem.

In similar way, we can discuss the geometry of leaf of slant distribution as follows:

Theorem 7 Let $\bar f$ be a CQBS submersion from cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a $RM(M_2,g_2)$. Then slant distribution D^{θ_1} defines totally geodesic foliation on M_1 if and only if

$$
\begin{aligned}
\lambda^{-2}g_2\left(\nabla^{\bar f}_{\hat Z}\bar f_*\zeta\mathfrak{B}\widehat W,\bar f_*\mathfrak{C}\hat U\right) =& g_1(\mathcal{T}_{\hat Z}\zeta\delta\mathfrak{B}\widehat W,\hat U) - g_1(\mathcal{T}_{\hat Z}\zeta\mathfrak{B}\widehat W,\phi\mathfrak{A}\hat U)\\
&-g_1(\mathcal{T}_{\hat Z}\zeta\mathfrak{B}\widehat W,\delta\mathfrak{C}\hat U) - \cos^2\theta_1 g_1(\mathcal{V}\nabla_{\hat Z}\mathfrak{B}\widehat W,\hat U(\mathfrak{W},9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda^{-2}\left\{g_2\left(\nabla^{\bar f}_{\hat Z}\bar f_*\zeta\delta\mathfrak{B}\widehat W,\bar f_*\hat X\right) - g_2\left(\nabla^{\bar f}_{\hat Z}\bar f_*\zeta\delta\mathfrak{B}\widehat W,\bar f_* Q\hat X\right)\right\}\\
&=\frac{1}{\lambda^2}g_2\left((\nabla\bar f_*)(\hat Z,\zeta\delta\mathfrak{B}\widehat W),\bar f_*\hat X\right)\Big)\\
&\quad+\cos^2\theta_1 g_1(\nabla_{\hat Z}\mathfrak{B}\widehat W,\hat X) + g_1(\mathcal{T}_{\hat Z}\zeta\delta\mathfrak{B}\widehat W,P\hat X),
\end{aligned}
$$

for any $\hat Z,\widehat W\in\Gamma(D^{\theta_1}),\hat U\in\Gamma(D^T\oplus D^{\theta_2}\oplus<\xi>)$ and $\hat X\in\Gamma(\ker\bar f_*)^{\perp}$.

Proof By using equations (??), (??), (??) and (??), we get

$$g_1(\nabla_{\hat Z}\widehat W,\hat U) = g_1(\nabla_{\hat Z}\zeta\mathfrak{B}\widehat W,\phi(\mathfrak{A}\hat U+\mathfrak{C}\hat U)) - g_1(\phi\nabla_{\hat Z}\delta\mathfrak{B}\widehat W,\hat U),$$

for $\hat{Z}, \widehat{W} \in \Gamma(D^{\theta_1})$ and $\widehat{U} \in \Gamma(D^T \oplus D^{\theta_2} \oplus < \xi >)$. Again using equations (??), (??), (??), (??), (??) with Lemma ??, we may write

$$g_1(\nabla_{\hat{Z}}\widehat{W}, \widehat{U}) = \cos^2\theta_1 g_1(\nabla_{\hat{Z}}\mathfrak{B}\mathfrak{W}, \widehat{U}) - g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, \widehat{U}) + g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, \phi\mathfrak{A}\widehat{U})$$
$$+ g_1(\mathcal{T}_{\hat{Z}}\zeta\mathfrak{B}\widehat{W}, \delta\mathfrak{C}\widehat{U}) + g_1(\mathcal{H}\nabla_{\hat{Z}}\zeta\mathfrak{B}\widehat{W}, \zeta\mathfrak{C}\widehat{U})$$

Since, $\bar{f}$ is conformal, using Lemma ?? with equations (??) and (??), we have

$$g_1(\nabla_{\hat{Z}}\widehat{W}, \widehat{U}) = \cos^2\theta_1 g_1(\nabla_{\hat{Z}}\mathfrak{B}\widehat{W}, \widehat{U}) - g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, \widehat{U}) + g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, \phi\mathfrak{A}\widehat{U})$$
$$+ g_1(\mathcal{T}_{\hat{Z}}\zeta\mathfrak{B}\widehat{W}, \delta\mathfrak{C}\widehat{U}) - \frac{1}{\lambda^2} g_2\left(\nabla_{\hat{Z}}^{\bar{f}}\bar{f}_*\zeta\mathfrak{B}\widehat{W}, \bar{f}_*\zeta\mathfrak{C}\widehat{W}\right)$$

On the other hand, for $\hat{Z}, \widehat{W} \in \Gamma(D^{\theta_1})$ and $\hat{X} \in \Gamma(\ker \bar{f}_*)^{\perp}$, with using equations (??), (??), (??) and (??), we get

$$g_1(\nabla_{\hat{Z}}\widehat{W}, \hat{X}) = g_1(\nabla_{\hat{Z}}\delta\mathfrak{B}\widehat{W}, \phi\hat{X}) + g_1(\nabla_{\hat{Z}}\zeta\mathfrak{B}\widehat{W}, \phi\hat{X})$$

From Lemma ?? with equations (??) and (??), the above equation takes the form

$$g_1(\nabla_{\hat{Z}}\widehat{W}, \hat{X}) = \cos^2\theta_1 g_1(\nabla_{\hat{Z}}\mathfrak{B}\widehat{W}, \hat{X}) - g_1(\mathcal{H}\nabla_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, \hat{X})$$
$$+ g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, P\hat{X}) + g_1(\mathcal{H}\nabla_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, Q\hat{X})$$

Since $\bar{f}$ is conformal and from equations (??) and (??), we have

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\widehat{W}, \hat{X}) = {} & \cos^2\theta_1 g_1(\nabla_{\hat{Z}}\widehat{W}, \hat{X}) + g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, P\hat{X}) \\ & + \frac{1}{\lambda^2} g_2\left((\nabla\bar{f}_*)(\zeta\delta\mathfrak{B}\widehat{W}, \hat{Z}), \bar{f}_*\hat{X}\right) \\ & - \frac{1}{\lambda^2} g_2\left(\nabla_{\hat{Z}}^{\bar{f}}\bar{f}_*\zeta\delta\mathfrak{B}\widehat{W}, \bar{f}_*\hat{X}\right) \\ & - \frac{1}{\lambda^2} g_2\left((\nabla\bar{f}_*)(\zeta\delta\mathfrak{B}\widehat{W}, \hat{Z}), \bar{f}_*Q\hat{X}\right) \\ & + \frac{1}{\lambda^2} g_2\left(\nabla_{\hat{Z}}^{\bar{f}}\bar{f}_*\zeta\delta\mathfrak{B}\widehat{W}, \bar{f}_*Q\hat{X}\right) \end{aligned}$$

This completes the proof of theorem.

Theorem 8 Let $\bar{f}: (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a CQBS submersion, where $(M_1, \phi, \xi, \eta, g_1)$ a cosymplectic manifold and (M_2, g_2) a RM. Then slant distribution D^{θ_2} defines totally geodesic foliation on M_1 if and only if

$$\lambda^{-2} g_2\left(\nabla_{\hat{Z}}^{\bar{f}}\bar{f}_*\zeta\mathfrak{B}\widehat{W}, \bar{f}_*\mathfrak{C}\widehat{V}\right) = g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, \widehat{V}) - g_1(\mathcal{T}_{\hat{Z}}\zeta\mathfrak{B}\widehat{W}, \phi\mathfrak{A}\widehat{V})$$
$$- g_1(\mathcal{T}_{\hat{Z}}\zeta\mathfrak{B}\widehat{W}, \delta\mathfrak{C}\widehat{V}) - \cos^2\theta_1 g_1(\mathcal{V}\nabla_{\hat{Z}}\mathfrak{B}\widehat{W}, \widehat{V}\chi_2)$$

and

$$\lambda^{-2}\left\{g_2\left(\nabla_{\hat{Z}}^{\bar{f}}\bar{f}_*\zeta\delta\mathfrak{B}\widehat{W}, \bar{f}_*\hat{Y}\right) - g_2\left(\nabla_{\hat{Z}}^{\bar{f}}\bar{f}_*\zeta\delta\mathfrak{B}\widehat{W}, \bar{f}_*Q\hat{Y}\right)\right\}$$
$$= \frac{1}{\lambda^2} g_2\left((\nabla\bar{f}_*)(\hat{Z}, \zeta\delta\mathfrak{B}\widehat{W}), \bar{f}_*\hat{Y}\right)\Big)$$
$$+ \cos^2\theta_2 g_1(\nabla_{\hat{Z}}\mathfrak{B}\widehat{W}, \hat{Y}) + g_1(\mathcal{T}_{\hat{Z}}\zeta\delta\mathfrak{B}\widehat{W}, P\hat{Y}),$$

for any $\hat{Z}, \widehat{W} \in \Gamma(D^{\theta_2}), \widehat{V} \in \Gamma(D^T \oplus D^{\theta_1} \oplus < \xi >)$ and $\hat{Y} \in \Gamma(\ker \bar{f}_*)^{\perp}$.

Proof The proof of above theorem is similar to the proof of Theorem ??.

We now have some necessary and sufficient conditions for a conformal CQBS submersion $\bar{f}: M_1 \to M_2$ to be totally geodesic map. In this regard, we are presenting the following theorem.

Theorem 9 Let $\bar{f}$ be a CQBS submersion from cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a $RM(M_2, g_2)$. Then $\bar{f}: (M_1, \phi, \xi, \eta, g_1) \to (M_1, g_2)$ is totally geodesic map if and only if

(i) $\bar{f}_*\{\cos^2\theta_1 \nabla_{\widehat{U}}\mathfrak{B}\widehat{V} + \cos^2\theta_2 \nabla_{\widehat{U}}\mathfrak{C}\widehat{V} + \nabla_{\widehat{U}}\zeta\delta\mathfrak{B}\widehat{V} + \nabla_{\widehat{U}}\zeta\delta\mathfrak{C}\widehat{V}\}$

$$= \bar{f}_*\{Q(\mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{B}\hat{V} + \mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{C}\hat{V} - \mathcal{T}_{\hat{U}}\delta\mathfrak{A}\hat{V})\} + \bar{f}_*\{\zeta(\mathcal{T}_{\hat{U}}\zeta\mathfrak{B}\hat{V} - \mathcal{T}_{\hat{U}}\zeta\mathfrak{C}\hat{V} - \mathcal{V}\nabla_{\hat{U}}\delta\mathfrak{A}\hat{V})\},$$

(ii) $\bar{f}_*\{\cos^2\theta_1\nabla_{\hat{X}}\mathfrak{B}\hat{U} + \cos^2\theta_2\nabla_{\hat{X}}\mathfrak{C}\hat{U} - \nabla_{\hat{X}}\zeta\delta\mathfrak{B}\hat{U} - \nabla_{\hat{X}}\zeta\delta\mathfrak{C}\hat{U}\} = \bar{f}_*\{Q(\mathcal{A}_{\hat{X}}\delta\mathfrak{A}\hat{U} + \mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{B}\hat{U} + \mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{C}\hat{U})\}$
$+\bar{f}_*\{\zeta(\mathcal{V}\nabla_{\hat{X}}\delta\mathfrak{A}\hat{U} + \mathcal{A}_{\hat{X}}\zeta\mathfrak{B}\hat{U} + \mathcal{A}_{\hat{X}}\zeta\mathfrak{C}\hat{U})\}$,

for any $\hat{U},\hat{V} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X},\hat{Y} \in \Gamma(\ker\bar{f}_*)^{\perp}$.

Proof Now, using equations (??), (??), (??) and (??).

$$(\nabla\bar{f}_*)(\hat{U},\hat{V}) = -\bar{f}_*\{\eta(\nabla_{\hat{U}}\hat{V})\xi - \phi\nabla_{\hat{U}}\phi\hat{V}\},$$

for any $\hat{U},\hat{V} \in \Gamma(\ker\bar{f}_*)$. From decomposition (??) and equation (??), we may write

$$(\nabla\bar{f}_*)(\hat{U},\hat{V}) = \bar{f}_*\{\phi\nabla_{\hat{U}}\delta\mathfrak{A}\hat{V} - \phi\nabla_{\hat{U}}\delta\mathfrak{B}\hat{V} - \phi\nabla_{\hat{U}}\zeta\mathfrak{B}\hat{V} \\ -\phi\nabla_{\hat{U}}\delta\mathfrak{C}\hat{V} - \phi\nabla_{\hat{U}}\zeta\mathfrak{C}\hat{V}\}$$

By using equations (??) and (??), the above equation takes the form

$$(\nabla\bar{f}_*)(\hat{U},\hat{V}) = \bar{f}_*\{\phi\mathcal{T}_{\hat{U}}\delta\mathfrak{A}\hat{V} - \phi\mathcal{V}\nabla_{\hat{U}}\delta\mathfrak{A}\hat{V}\} - \bar{f}_*(\nabla_{\hat{U}}\phi\delta\mathfrak{B}\hat{V}) \\ -\bar{f}_*\{\phi\mathcal{T}_{\hat{U}}\zeta\mathfrak{B}\hat{V} - \phi\mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{B}\hat{V}\} - \bar{f}_*(\nabla_{\hat{U}}\phi\delta\mathfrak{B}\hat{V}) \\ -\bar{f}_*\{\phi\mathcal{T}_{\hat{U}}\zeta\mathfrak{C}\hat{V} - \phi\mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{C}\hat{V}\}.$$

Since $\bar{f}$ is conformal submersion, by using Lemma ?? and Lemma ?? with equation (??), we finally get

$$(\nabla\bar{f}_*)(\hat{U},\hat{V}) = \bar{f}*\{Q(\mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{B}\hat{V} + \mathcal{H}\nabla_{\hat{U}}\zeta\mathfrak{C}\hat{V} - \mathcal{T}_{\hat{U}}\delta\mathfrak{A}\hat{V}) \\ +\zeta(\mathcal{V}\nabla_{\hat{U}}\delta\mathfrak{A}\hat{V} - \mathcal{T}_{\hat{U}}\delta\mathfrak{B}\hat{V} - \mathcal{T}_{\hat{U}}\delta\mathfrak{C}\hat{V})\} \\ -\bar{f}_*\{\cos^2\theta_1\nabla_{\hat{U}}\mathfrak{B}\hat{V} + \cos^2\theta_2\nabla_{\hat{U}}\mathfrak{C}\hat{V} + \nabla_{\hat{U}}\zeta\delta\mathfrak{B}\hat{V} + \nabla_{\hat{U}}\zeta\delta\mathfrak{C}\hat{V}\}.$$

From this, the (i) part of theorem proved. On the other hand, for $\hat{U} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X} \in \Gamma(\ker\bar{f}_*)^{\perp}$ with using equations (??), (??), (??) and (??), we can write

$$(\nabla\bar{f}_*)(\hat{X},\hat{U}) = \bar{f}_*(\phi\nabla_{\hat{X}}\phi\hat{U})$$

On using decomposition (??) with equation (??), we have

$(\nabla\bar{f}_*)(\hat{X},\hat{U}) = \bar{f}_*\{\phi(\nabla_{\hat{X}}\delta\mathfrak{A}\hat{U} + \nabla_{\hat{X}}\delta\mathfrak{B}\hat{U} + \nabla_{\hat{X}}\zeta\mathfrak{B}\hat{U} + \nabla_{\hat{X}}\delta\mathfrak{C}\hat{U} + \nabla_{\hat{X}}\zeta\mathfrak{C}\hat{U})\}$.

By taking account the fact from equations (??) and (??), we get

$$(\nabla\bar{f}_*)(\hat{X},\hat{U}) = \bar{f}_*\{\phi(\mathcal{A}_{\hat{X}}\delta\mathfrak{A}\hat{U} + \mathcal{V}\nabla_{\hat{X}}\delta\mathfrak{A}\hat{U} + \nabla_{\hat{X}}\phi\delta\mathfrak{B}\hat{U} \\ +\phi(\mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{B}\hat{U} + \mathcal{A}_{\hat{X}}\zeta\mathfrak{B}\hat{U}) + \nabla_{\hat{X}}\phi\delta\mathfrak{C}\hat{U} \\ +\phi(\mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{C}\hat{U} + \mathcal{A}_{\hat{X}}\zeta\mathfrak{C}\hat{U})\}.$$

Finally, from conformality of RS $\bar{f}$ and Lemma ??, Lemma ??, we can write

$$(\nabla\bar{f}_*)(\hat{X},\hat{U}) = \bar{f}_*\{Q(\mathcal{A}_{\hat{X}}\delta\mathfrak{A}\hat{U}\mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{B}\hat{U} + \mathcal{H}\nabla_{\hat{X}}\zeta\mathfrak{C}\hat{U})\} \\ +\bar{f}_*\{\zeta(\mathcal{V}\nabla_{\hat{X}}\delta\mathfrak{A}\hat{U} + \mathcal{A}_{\hat{X}}\zeta\mathfrak{B}\hat{U} + \mathfrak{A}_{\hat{X}}\zeta\mathfrak{C}\hat{U})\} \\ -\bar{f}_*(\cos^2\theta_1\nabla_{\hat{X}}\mathfrak{B}\hat{U} + \cos^2\theta_2\nabla_{\hat{X}}\mathfrak{C}\hat{U} - \nabla_{\hat{X}}\zeta\delta\mathfrak{B}\hat{U} - \nabla_{\hat{X}}\zeta\delta\mathfrak{C}\hat{U}).$$

From which we obtain (ii) part of theorem. This completes the proof of theorem.

5. Decomposition Theorems

In this section, we recall the following result from [?] and discuss some decomposition theorems. Let g be a Riemannian metric tensor on the manifold $M = M_1 \times M_2$, then

(i) $M = M_1 \times {}_\lambda M_2$ is a locally product if and only if M_1 and M_2 are totally geodesic foliations,

(ii) $M = M_1 \times {}_\lambda M_2$ is a twisted product if and only if M_1 is a totally geodesic foliation and M_2 is a totally umbilic foliation.

The presence of three orthogonal complementary distributions D^T, D^{θ_1}, and D^{θ_2}, which are integrable and totally geodesic under the conditions that we have stated previously, is ensured by the fact that $\bar{f}: (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is CQBS submersion. It makes sense to now look for the conditions in which the total space M_1 converts into locally product manifolds or locally twisted product manifolds. In order to explore the geometry of conformal bislant submersion $\bar{f}$, we are providing here a few decomposition theorems that state that M_1 converts into locally product manifolds in a variety of situations.

Theorem 10 Let $\bar{f}: (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a CQBS submersion, where $(M_1, \phi, \xi, \eta, g_1)$ a cosymplectic manifold and (M_2, g_2) a RM. Then M_1 is a locally product manifold if and only if

$$\begin{aligned}
&\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{B}\hat{Y} + \nabla^{\bar{f}}_{\hat{X}}\bar{f}_*\zeta\mathfrak{C}\hat{Y}, \bar{f}_*\zeta\hat{Z}\right)\right\}\\
&= g_1\left(\mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{B}\hat{Y} + \mathcal{A}_{\hat{X}}\zeta\delta\mathfrak{C}\hat{Y} + \mathcal{A}_{\hat{X}}\zeta\delta A\hat{Y}, \hat{Z}\right)\\
&\quad + g_1(\zeta\mathfrak{B}\hat{Y}, \zeta\hat{Z}) g_1(\hat{X}, G\ln\lambda) + g_1(\hat{X}, \zeta\hat{Z}) g_1(\zeta\mathfrak{B}\hat{Y}, G\ln\lambda)\\
&\quad - g_1(\hat{X}, \zeta\mathfrak{B}\hat{Y}) g_1(\zeta\hat{Z}, G\ln\lambda) + g_1(\zeta\mathfrak{C}\hat{Y}, \zeta\hat{Z}) g_1(\hat{X}, G\ln\lambda)\\
&\quad + g_1(\hat{X}, \zeta\hat{Z}) g_1(\zeta\mathfrak{C}\hat{Y}, G\ln\lambda) - g_1(\hat{X}, \zeta\mathfrak{C}\hat{Y}) g_1(\zeta\hat{Z}, G\ln\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\delta\mathfrak{B}\hat{V} + \nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\delta\mathfrak{C}\hat{V}, \bar{f}_*\hat{X}\right)\right\}\\
&+ \frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\hat{V} - (\nabla\bar{f}_*)(\hat{U}, \zeta\hat{V}), \bar{f}_*Q\hat{X}\right)\right\}
\end{aligned}$$

for any $\hat{U}, \hat{V} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X}, \hat{Y} \in \Gamma(\ker\bar{f}_*)^\perp$.

Since we discussed in the previous theorem, given certain necessary and sufficient conditions, the total space M_1 transforms into a locally product manifold. Now, it's intriguing to investigate if there are any circumstances under which the total space M_1 could turn into a locally twisted product manifold. The conditions that turn total space M_1 into a locally twisted product manifold are found in the following result.

Theorem 11 Let f be a CQBS submersion from cosymplectic manifold $(M, \phi, \xi, \eta, g_1)$ onto a $RM(M_2, g_2)$. Then M_1 is locally twisted product of the form $M_1(\ker\bar{f}_*) \times M_1(ker\bar{f}_*)^\perp$ if and only if

$$\frac{1}{\lambda^2} g_2\left((\nabla\bar{f}_*)(\hat{U}, \zeta\hat{V}), \bar{f}_*Q\hat{X}\right) = g_1\left(\nabla_{\hat{U}}\delta\hat{V}, \phi\hat{X}\right) + g_1\left(\mathcal{T}_{\hat{U}}\zeta\hat{V}, P\hat{X}\right)$$

and

$$g_1(\hat{X}, \hat{Y})H = -P\mathcal{A}_{\hat{X}}P\hat{Y} - \delta\nabla_{\hat{X}}P\hat{Y} - \delta\mathcal{A}_{\hat{X}}Q\hat{Y} - \phi\bar{f}_*\left(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*Q\hat{Y}\right) + \hat{X}(\ln\lambda)PQ\hat{Y}$$

where H is a mean curvature vector and for any $\hat{U}, \hat{V} \in \Gamma(\ker\bar{f}_*)$ and $\hat{X}_1, \hat{X}_2 \in \Gamma(\ker\bar{f}_*)^\perp$.

Proof For any $\hat{X} \in \Gamma(\ker\bar{f}_*)^\perp$ and $\hat{U}, \hat{V} \in \Gamma(\ker\bar{f}_*)$ and using equations (??), (??), (??), (??), (??) (??) and (??), we have

$$g_1(\nabla_{\hat{U}}\hat{V}, \hat{X}) = g_1\left(\nabla_{\hat{U}}\delta\hat{V}, \phi\hat{X}\right) + g_1\left(\mathcal{T}_{\hat{U}}\zeta\hat{V}, P\hat{X}\right) + g_1\left(\mathcal{H}\nabla_{\hat{U}}\zeta\hat{V}, Q\hat{X}\right).$$

From using formula (??) and definition of conformality, the above equation takes place as

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = g_1(\nabla_{\hat{U}}\delta\hat{V},\phi\hat{X}) + g_1(\mathcal{T}_{\hat{U}}\zeta\hat{V},P\hat{X}) - \frac{1}{\lambda^2}g_2\left((\nabla\bar{f}_*)(\hat{U},\zeta\hat{V}),\bar{f}_*Q\hat{X}\right) + \frac{1}{\lambda^2}g_2\left(\nabla^{\bar{f}}_{\hat{U}}\bar{f}_*\zeta\hat{V},\bar{f}_*Q\hat{X}\right)$$

It follows that the equation (??) satisfies if and only if $M_{1(ker\bar{f}_*)}$ is totally geodesic. On the other hand, for $\hat{U} \in \Gamma(\ker\bar{f}_*), \hat{X},\hat{Y} \in \Gamma(\ker\bar{f}_*)^\perp$ with using equations (??), (??), (??) (??), (??) and (??), we get

$$g_1(\nabla_{\hat{X}}\hat{Y},\hat{U}) = g_1(\nabla_{\hat{X}}P\hat{Y},\phi\hat{U}) + g_1(\mathcal{A}_{\hat{X}}Q\hat{Y},\delta\hat{U}) + g_1(\mathcal{H}\nabla_{\hat{X}}Q\hat{Y},\zeta\hat{U})$$

By using the equation (??) with definition of conformality of $\bar{f}$, we deduce that

$$g_1(\nabla_{\hat{X}}\hat{Y},\hat{U}) = -\frac{1}{\lambda^2}g_2\left((\nabla\bar{f}_*)(\hat{X},Q\hat{Y}),\bar{f}_*\zeta\hat{U}\right) + \frac{1}{\lambda^2}g_2\left(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*Q\hat{Y},\bar{f}_*\zeta\hat{U}\right) + g_1(\nabla_{\hat{X}}P\hat{Y},\phi\hat{U}) + g_1(\mathcal{A}_{\hat{X}}Q\hat{Y},\delta\hat{U})$$

Considering the (i) part of Lemma ??, above equation turns in to

$$\begin{aligned} g_1(\nabla_{\hat{X}}\hat{Y},\hat{U}) =& \frac{1}{\lambda^2}g_2\left(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*Q\hat{Y},\bar{f}_*\zeta\hat{U}\right) + g_1(\nabla_{\hat{X}}P\hat{Y},\phi\hat{U}) + g_1(\mathcal{A}_{\hat{X}}Q\hat{Y},\delta\hat{U}) \\ &- g_1(\mathrm{grad}\ln\lambda,\hat{X})g_1(Q\hat{Y},\zeta\hat{U}) - g_1(\mathrm{grad}\ln\lambda,Q\hat{Y})g_1(\hat{X},\zeta\hat{U}) \\ &+ g_1(\mathrm{grad}\ln\lambda,\zeta\hat{U})g_1(\hat{X},Q\hat{Y}) \end{aligned}$$

By direct calculation, finally we get

$$\begin{aligned} g_1(\hat{X},\hat{Y})H =& -P\mathcal{A}_{\hat{X}}P\hat{Y} - \delta\nabla_{\hat{X}}P\hat{Y} - \delta\mathcal{A}_{\hat{X}}Q\hat{Y} - \phi\bar{f}_*\left(\nabla^{\bar{f}}_{\hat{X}}\bar{f}_*Q\hat{Y}\right) + \hat{X}(\ln\lambda)PQ\hat{Y} \\ &+ Q\hat{Y}(\ln\lambda)P\hat{X} - P(\mathrm{grad}\ln\lambda)g_1(\hat{X},Q\hat{Y}) \end{aligned}$$

From the above equation we conclude that $M_{1(\ker\bar{f}_*)^\perp}$ is totally umbilical if and only if equation (??) satisfied

We can provide these decomposition theorems by taking into consideration the prior theorems.

Theorem 12 Let $\bar{f}:(M_1,\phi,\xi,\eta,g_1) \to (M_2,g_2)$ be a CQBS submersion with slant angles θ_1 and θ_2, where (M_1,ϕ,ξ,η,g_1) a cosymplectic manifold and (M_2,g_2) a RM. Then the total space $M_{1D^{\theta_1}} \times M_{1D^{\theta_2}} \times M_{1(ker\bar{f}_*)^\perp}$ is locally product if and only if equations (??), (??), (??), (??) and (??) are holds where $M_{1_{D^{\theta_1}}}, M_{1_{D^{\theta_2}}}$ and $M_{1(ker\bar{f}_*)^\perp}$ are integral manifolds of the distributions $D^{\theta_1}, D^{\theta_2}$ and $(\ker\bar{f}_*)^\perp$ respectively.

Theorem 13 Let $\bar{f}$ be a CQBS submersion from cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a $RM(M_2,g_2)$ with slant angles θ_1 and θ_2. Then the total space $M_{1(ker\bar{f}_*)} \times M_{1(ker\bar{f}_*)^\perp}$ is locally product if and only if equations (??) and (??) are holds where $M_{1(ker\bar{f}_*)}$ and $M_{1(ker\bar{f}_*)^\perp}$ are integral manifolds of the distributions $(\ker\bar{f}_*)$ and $(\ker\bar{f}_*)^\perp$ respectively.

Theorem 14 Let $\bar{f}:(M_1,\phi,\xi,\eta,g_1) \to (M_2,g_2)$ be a CQBS submersion with slant angles θ_1 and θ_2, where (M_1,ϕ,ξ,η,g_1) a cosymplectic manifold and (M_2,g_2) a RM. Then vertical distribution ($\ker\bar{f}_*$) is locally Riemannian product of the form $M_{1D^{\theta_1}} \times M_{1D^{\theta_2}}$ if and only if equations (??), (??), (??) and (??) holds where $M_{1_{D^{\theta_1}}}$ and $M_{1_{D^{\theta_2}}}$ are integral manifolds of distributions D^{θ_1} and D^{θ_2} respectively.

6. ϕ-Pluriharmonicity of Conformal Quasi Bi-slant Submersion

Let $\bar{f}$ be a CQBS submersion from cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles θ_1 and θ_2. Then CQBS submersion is $D^T-\phi$ pluriharmonic ($D^{\theta_1}-\phi$-pluriharmonic, $D^{\theta_2}-\phi$-pluriharmonic, $(D^T-D^{\theta_1})-\phi$

pluriharmonic, $(D^T - D^{\theta_2}) - \phi$ pluriharmonic, $\ker\bar{f}_* - \phi$-pluriharmonic, $(\ker\bar{f}_*)^\perp - \phi$-pluriharmonic and $\left((\ker\bar{f}_*)^\perp - \ker\bar{f}_*\right) - \phi$-pluriharmonic) if

$$(\nabla\bar{f}_*)(U,V) + (\nabla\bar{f}_*)(\phi U, \phi V) = 0\#(48)$$

for any $U,V \in \Gamma(D^T)$ (for any $U,V \in \Gamma(D^{\theta_1})$, for any $U,V \in \Gamma(D^{\theta_2})$, for any $U \in \Gamma(D^T), V \in \Gamma(D^{\theta_1})$, for any $U \in \Gamma(D^T), V \in \Gamma(D^{\theta_2})$, for any $U,V \in \Gamma(\ker\bar{f}_*)$, for any $U,V \in \Gamma(\ker\bar{f}_*)^\perp$ and for any $U \in \Gamma(\ker\bar{f}_*)^\perp, V \in \Gamma(\ker\bar{f}_*)$, respectively).

Theorem 15 Let $\bar{f}_*$ be a CQBS submersion from cosymplectic manifold ($M_1, \phi, \xi, \eta, g_1$) onto a RM (M_2, g_2) with slant angles θ_1 and θ_2. Suppose that the map f is D^T - ϕ-pluriharmonic. Then map is ker $\bar{f}_*$-geodesic map if and only if $\mathcal{T} = \{0\}$, which gives that the fibres are totally geodesic manifold.

Proof Since map $\bar{f}$ is D^T - ϕ-pluriharmonic, then by using equation (??) and (??) for any $U,V \in \Gamma(\ker\bar{f}_*)$, we have

$$\begin{aligned} 0 &= (\nabla\bar{f}_*)(U,V) + (\nabla\bar{f}_*)(\phi U, \phi V) \\ &= -\bar{f}_*(\mathcal{T}_U V) + (\nabla\bar{f}_*)(\phi U, \phi V) \\ &= -\bar{f}_*(\mathcal{T}_U V) - \bar{f}_*(\nabla_{\delta U}\delta V) \end{aligned}$$

This completes the proof of the theorem.

Theorem 16 Let $\bar{f}_*$ be a CQBS submersion from cosymplectic manifold ($M_1, \phi, \xi, \eta, g_1$) onto a RM (M_2, g_2) with slant angles θ_1 and θ_2. Suppose that the map $\bar{f}$ is D^{θ_1} - ϕ-pluriharmonic. Then map is ωD^{θ_1}-geodesic map if and only if

$$\mathcal{T}_U V + \mathcal{T}_{\delta U}\delta V + \mathcal{A}_{\zeta U}\delta V + \mathcal{H}\nabla_{\delta U}\zeta V = 0$$

for any $U,V \in \Gamma(D^{\theta_1})$.

Proof Since map $\bar{f}$ is D^{θ_1} - ϕ-pluriharmonic, then by using equation (??) and (??), we have

$$\begin{aligned} 0 &= (\nabla\bar{f}_*)(U,V) + (\nabla\bar{f}_*)(\phi U, \phi V) \\ &= -\bar{f}_*(\mathcal{T}_U V) + (\nabla\bar{f}_*)(\omega U, \omega V) - \bar{f}_*(\mathcal{T}_{\delta U}\delta V + \mathcal{A}_{\zeta U}\delta V + \mathcal{H}\nabla_{\delta U}\zeta V) \end{aligned}$$

which gives the proof.

Theorem 17 Let f_* be a CQBS submersion from cosymplectic manifold ($M_1, \phi, \xi, \eta, g_1$) onto a RM (M_2, g_2) with slant angles θ_1 and θ_2. Suppose that the map $\bar{f}$ is ($D^T - D^{\theta_1}$)- ϕ-pluriharmonic. Then the following are equivalent.

(i) The invariant distribution D^T defines totally geodesic foliation on M_1.

(ii) $\bar{f}_*(\nabla_U W) = \nabla_{\phi U}\bar{f}_*(\omega W) - \bar{f}_*(Q\mathcal{T}_{\phi U}W + \zeta\mathcal{V}\nabla_{\phi U}W)$,
for any $U \in \Gamma(D^T)$ and $W \in \Gamma(D^{\theta_1})$.

Proof For any $U \in \Gamma(D^T)$ and $W \in \Gamma(D^{\theta_1})$ and since map is $(D^T - D^{\theta_1})$ - ϕ pluriharmonic, then by virtue of (??), (??) and (??), we have

$$\begin{aligned} 0 &= (\nabla\bar{f}_*)(U,W) + (\nabla\bar{f}_*)(\phi U, \phi W) \\ &= -\bar{f}_*(\nabla_U W) + \nabla_{\phi U}\bar{f}_*(\omega W) - \bar{f}_*(\nabla_{\phi U}\phi W) \\ &= -\bar{f}_*(\nabla_U W) + \nabla_{\phi U}\bar{f}_*(\omega W) - \bar{f}_*(\phi\nabla_{\phi U}W). \end{aligned}$$

By usig equations (??), (??) and (??), the above equations takes the form

$$\begin{aligned} 0 &= -\bar{f}_*(\nabla_U W) + \nabla_{\phi U}\bar{f}_*(\omega W) - \bar{f}_*\{\phi(\mathcal{T}_{\phi U}W + \mathcal{V}\nabla_{\phi U}W)\} \\ &= -\bar{f}_*(\nabla_U W) + \nabla_{\phi U}\bar{f}_*(\omega W) - \bar{f}_*(Q\mathcal{T}_{\phi U}W + \zeta\mathcal{V}\nabla_{\phi U}W) \\ \bar{f}_*(\nabla_U W) &= \nabla_{\phi U}\bar{f}_*(\omega W) - \bar{f}_*(Q\mathcal{T}_{\phi U}W + \zeta\mathcal{V}\nabla_{\phi U}W), \end{aligned}$$

from which we get desired result.

Theorem 18 Let $\bar{f}_*$ be a CQBS submersion from cosymplectic manifold ($M_1, \phi, \xi, \eta, g_1$) onto a $RM(M_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that the map $\bar{f}$ is $\left((\ker\bar{f}_*)^\perp - \ker\bar{f}_*\right) - \phi$-pluriharmonic. Then the following assertion are equivalent.

(i) The horizontal distribution $(\ker\bar{f}_*)^\perp$ defines totally geodesic foliation on M_1.

(ii) $(\nabla\bar{f}_*)(QX, \zeta Y) = \bar{f}_*\left(\mathcal{T}_{PX}\delta Y + \mathcal{H}\nabla_{PX}\zeta Y + \mathcal{A}_{QX}\delta Y\right) - \nabla_{PX}\bar{f}_*(\zeta Y)$,

for any $X \in \Gamma(\bar{f}_*)^\perp$ and $Y \in \Gamma(\ker\bar{f}_*)$.

Proof For any $X \in \Gamma(\bar{f}_*)^\perp$ and $Y \in \Gamma(\ker\bar{f}_*)$, since map $\bar{f}$ is $\left((\ker\bar{f}_*)^\perp - \ker\bar{f}_*\right) - \phi$-pluriharmonic, then by using (??), (??) and (??), we get

$$\begin{aligned}
0 =& (\nabla\bar{f}_*)(X, Y) + (\nabla\bar{f}_*)(\phi X, \phi Y)\\
=& -\bar{f}_*(\nabla_X Y) + (\nabla\bar{f}_*)(PX, \delta Y) + (\nabla\bar{f}_*)(PX, \zeta Y)\\
&+(\nabla\bar{f}_*)(QX, \delta Y) + (\nabla\bar{f}_*)(QX, \zeta Y)\\
=& -\bar{f}_*(\nabla_X Y) - \bar{f}_*(\nabla_{PX}\delta Y) + \nabla_{PX}\bar{f}_*(\zeta Y) - \bar{f}_*\nabla_{PX}\zeta Y\\
&-\bar{f}_*\left(\nabla_{QX}\delta Y\right) + (\nabla\bar{f}_*)(QX, \zeta Y)
\end{aligned}$$

$$(\ker\bar{f}_*)^\perp = \left\langle \frac{\partial}{\partial x_1}\cos\theta_1 - \frac{\partial}{\partial y_2}\sin\theta_1, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial x_3}\sin\theta_2 - \frac{\partial}{\partial y_4}\cos\theta_2, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6}, \frac{\partial}{\partial x_7} \right\rangle$$

Furthermore,

More, explicitly

$$\begin{aligned}
D^T &= \langle V_1, V_2, V_6, V_8 \rangle\\
D^{\theta_1} &= \langle V_3, V_4 \rangle\\
D^{\theta_2} &= \langle V_5, V_6 \rangle
\end{aligned}$$

such that

$$(\ker\bar{f}_*) = D^T \oplus D^{\theta_1} \oplus D^{\theta_2} \oplus \langle\xi\rangle$$

Taking account tha fact from (??) and (??), we have

$$\begin{aligned}
0 =& -\bar{f}_*(\nabla_X Y) - \bar{f}_*(\mathcal{T}_{PX}\delta Y) + \nabla_{PX}\bar{f}_*(\zeta Y)\\
&-\bar{f}_*(\mathcal{H}\nabla_{PX}\zeta Y) - \bar{f}_*\left(\mathcal{A}_{QX}\delta Y\right) + (\nabla\bar{f}_*)(QX, \zeta Y)\\
(\nabla\bar{f}_*)(QX, \zeta Y) =& \bar{f}_*\left(\mathcal{T}_{PX}\delta Y + \mathcal{H}\nabla_{PX}\zeta Y + \mathcal{A}_{QX}\delta Y\right) - \nabla_{PX}\bar{f}_*(\zeta Y)
\end{aligned}$$

7. Examples

Now, we provide some non-trivial examples. We will use same structure mentioned in Example ??.

Example 2 Consider a map

$$\begin{gathered}
\bar{f}: \mathbb{R}^{15} \to \mathbb{R}^6\\
(x_1, x_2, \dots, x_7, y_1, y_2, \dots, y_7, z) \longrightarrow e^9(x_1\cos\theta_1 - y_2\sin\theta_1, y_3, x_3\sin\theta_2 - y_4\cos\theta_2, x_6, x_7, y_6)
\end{gathered}$$

such that $\bar{f}$ is a conformal submersion with dilation $\lambda = e^9$.

By the direct calculation, we obtain

$$\begin{gathered}
V_1 = \frac{\partial}{\partial x_2}, V_2 = \frac{\partial}{\partial y_1}, V_3 = \frac{\partial}{\partial x_1}\sin\theta_1 + \frac{\partial}{\partial y_2}\cos\theta_1, V_4 = \frac{\partial}{\partial x_4}\\
V_5 = \frac{\partial}{\partial x_3}\cos\theta_2 + \frac{\partial}{\partial y_4}\sin\theta_2, V_6 = \frac{\partial}{\partial x_5}, V_7 = \frac{\partial}{\partial y_5}, V_8 = \frac{\partial}{\partial y_7}\\
V_9 = \xi = \frac{\partial}{\partial z}
\end{gathered}$$

More, explicitly

Thus, $\bar{f}$ defines a CQBS submersions from cosymplectic manifold ($\mathbb{R}^{15}, \varphi, \xi, \eta, g$) onto a RM$(\mathbb{R}^6, g_{\mathbb{R}^6})$ with invariant distribution D^T and slant distributions D^{θ_1} and D^{θ_2} having slant angles θ_1 and θ_2, respectively.

Example 3 Consider a map

$$\bar{f}: \mathbb{R}^{13} \to \mathbb{R}^6$$
$$(x_1, x_2, \dots, x_6, y_1, y_2, \dots, y_6, z) \longrightarrow \pi^5\left(\frac{\sqrt{3}x_1 - x_3}{2}, y_2, \frac{x_5 - x_6}{\sqrt{2}}, x_4, y_4, y_6\right)$$

such that $\bar{f}$ is conformal submersion.

By the direct computation, we get

$\ker\bar{f}_* = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1} \right\rangle \oplus \left\langle \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_5} \right\rangle \oplus \left\langle \frac{\partial}{\partial x_1} + \sqrt{3}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle \oplus \left\langle \frac{\partial}{\partial z} \right\rangle$,

with $\left(\ker\bar{f}_*\right)^\perp = \left\langle \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_1} - \frac{1}{2}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_2}, \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_6} \right\rangle$.

More, explicitly,

$$D^T = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1} \right\rangle$$
$$D^{\theta_1} = \left\langle \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_5} \right\rangle$$
$$D^{\theta_2} = \left\langle \frac{\partial}{\partial x_1} + \sqrt{3}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle$$
$$\langle \xi \rangle = \left\langle \frac{\partial}{\partial z} \right\rangle$$

such that

$$\left(\ker\bar{f}_*\right) = D^T \oplus D^{\theta_1} \oplus D^{\theta_2} \oplus \langle \xi \rangle$$

Thus, $\bar{f}$ defines a CQBS submersions from cosymplectic manifold ($\mathbb{R}^{13}, \varphi, \xi, \eta, g$) onto a RM$(\mathbb{R}^6, g_{\mathbb{R}^6})$ with slant distributions D^{θ_1} and D^{θ_2} having slant angels $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{\pi}{4}$, respectively and the dilation $\lambda = \pi^5$.

8. Conflict of Interest

The authors declare that there is no conflict of interest.

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