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Non-existence of warped product slant lightlike submanifolds

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Abstract

We study hemi-slant lightlike submanifolds of indefinite Cosymplectic manifolds. We establish some necessary and sufficient conditions for such submanifolds to be minimal. We also obtain some characterization theorems for the non-existence of warped product slant lightlike submanifolds of indefinite Cosymplectic manifolds.

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1. Introduction

The theory of submanifolds as a field of differential geometry is as old as differential geometry itself. The geometry of submanifolds not only plays an important role in many diverse area of differential geometry and but also in physics. The geometry of four dimensional submanifold (model for our universe) embedded in ($4 + d$)-dimensional spacetime manifold attracted the attentions of many physicists. Lie theory approach of submanifolds is also applicable in solution of some partial differential equations (cf. [14]). During the last two decades, geometry of lightlike (degenerate) submanifolds of semi-Riemannian manifolds has become one of interesting topics in geometry of submanifolds as it produces models of different types of horizons like black hole horizons, conformal Killing horizons, dynamical horizons, event horizons, Cauchy's horizons etc., for more details, see [7].

Chen [5], [6] introduced the notion of slant submanifolds as a generalization of holomorphic and totally real submanifolds for complex geometry. Later, slant lightlike submanifolds of indefinite Hermitian manifolds were introduced by Sahin [22]. Haider et al. [11] introduced hemi-slant lightlike submanifolds of indefinite Kaehler manifolds. It is well known that the contact geometry has vital role in the theory of differential equations, optics, and phase spaces of a dynamical system (for detail see [1], [13], [14]), therefore contact geometry with definite and indefinite metric becomes the topic of main discussion. Hence, the notion of slant lightlike submanifolds of indefinite Sasakian manifolds was introduced by Sahin and Yildirim in [24] and further explored by Rashmi et al. in [18], [20]. Later, Rashmi et al. also studied hemi-slant lightlike submanifolds of indefinite Kaehler manifolds, Sasakian manifolds and Cosymplectic manifolds in [19], [21] and [17], respectively.

In present paper, we obtain some necessary and sufficient conditions for hemi-slant lightlike submanifold of indefinit Cosymplectic manifold to be a minimal hemi-slant lightlike submanifolds. It is known that warped product manifolds are generalization of Riemannian product manifolds and appears in differential geometric studies in a natural way. For instance, a surface of revolution is a warped product manifold and the space around a black hole or a massive star can be modeled on a warped product manifold. In view of its physical applications, many research articles have recently appeared exploring existence (or non-existence) of warped product submanifolds in known spaces. Hence, we obtain some characterization theorems for the non-existence of warped product slant lightlike submanifolds of indefinite Cosymplectic manifolds.

2. Lightlike Submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1, 1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$, see [7]. For a degenerate metric g on $M, TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{RadT}_xM = T_xM \cap T_xM^{\perp}$ which is known as the radical (null) subspace. If the mapping RadTM: $x \in M \longrightarrow \mathrm{RadT}T_xM$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and RadTM is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is, $TM = \mathrm{Rad}TM \perp S(TM)$, and $S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}TM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of RadTM, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $(S(TM))^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $\mathrm{ltr}(TM)$ locally spanned by $\{N_i\}$. Let $\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp})$, then $\mathrm{tr}(TM)$ is a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$ and we have $T\bar{M}|_M = TM \oplus \mathrm{tr}(TM) = (\mathrm{Rad}TM \oplus \mathrm{ltr}(TM)) \perp S(TM) \perp S(TM^{\perp})$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then for $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{tr}(TM))$, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \bar{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \#1$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla_X^{\perp} U\}$ belongs to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. Here, ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, called the second fundamental form, A_U is linear a operator on M, known as the shape operator.

Let $\mathcal{L}$ and $\mathcal{S}$ be the projection morphisms of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively then (1) becomes

$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \bar{\nabla}_X U = -A_U X + D_X^l U + D_X^s U, \#2$

where $h^l(X,Y) = \mathcal{L}(h(X,Y)), h^s(X,Y) = \mathcal{S}(h(X,Y)), D_X^l U = \mathcal{L}\left(\nabla\frac{1}{X}U\right), D_X^s U = \mathcal{S}\left(\nabla\frac{1}{X}U\right)$.

As h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma\big(S(TM^{\perp})\big)$-valued respectively, therefore, they are called as the lightlike second fundamental form and the screen second fundamental form on M, respectively. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X,N) \#(3)$$
$$\bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X,W) \#(4)$$

where $X \in \Gamma(TM), N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma\left(S(TM^{\perp})\right)$. Further, from (2), (3) and (4), we obtain

$$\bar{g}(h^{s}(X,Y),W) + \bar{g}(Y,D^{l}(X,W)) = g(A_{W}X,Y)\#(5)$$

Let $\bar{P}$ be a projection morphism of TM on $S(TM)$, we can write

$$\nabla_{X}\bar{P}Y = \nabla_{X}^{*}\bar{P}Y + h^{*}(X,\bar{P}Y), \nabla_{X}\xi = -A_{\xi}^{*}X + \nabla_{X}^{*t}\xi\#(6)$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{RadTM})$, where $\{\nabla_{X}^{*}\bar{P}Y, A_{\xi}^{*}X\}$ and $\{h^{*}(X,\bar{P}Y), \nabla_{X}^{*t}\xi\}$ belongs to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively.

Here, ∇^{*} and ∇_{X}^{*t} are linear connections on $S(TM)$ and $RadTM$, respectively. Using (3)-(4) and (6), we obtain

$$\bar{g}(h^{l}(X,\bar{P}Y),\xi) = g\left(A_{\xi}^{*}X,\bar{P}Y\right), \bar{g}(h^{*}(X,\bar{P}Y),N) = \bar{g}(A_{N}X,\bar{P}Y)\#(7)$$

An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors ($\phi, V, \eta, \bar{g}$), where ϕ is a (1,1) tensor field, V a vector field, called characteristic vector field, η a 1 -form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying (see [4], [12])

$$\phi^{2}X = -X + \eta(X)V, \eta \circ \phi = 0, \phi V = 0, \eta(V) = 1\#(8)$$
$$\bar{g}(\phi X,\phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \bar{g}(X,V) = \eta(X)\#(9)$$

for $X, Y \in \Gamma(T\bar{M})$.

An almost contact structure ($\phi, V, \eta, \bar{g}$) is said to be normal if the almost complex structure J on the product manifold $\widetilde{M} \times \mathrm{R}$ is given by $J\left(U, f\frac{d}{dt}\right) = \left(\phi U - fV, \eta(U)\frac{d}{dt}\right)$, where $J^{2} = -1$ and f is a differentiable function on $\widetilde{M} \times \mathbf{R}$ has no torsion and in terms of ϕ, the structure ($\phi, V, \eta, \bar{g}$) is normal if $[\phi,\phi] + 2d\eta \otimes V = 0$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ.

An odd-dimensional counterpart of a Kaehler manifold is given by a Cosymplectic manifold, which is locally a product of a Kaehler manifold with a circle or a line [3]. Indeed, a Cosymplectic structure on an odd dimensional manifold $\bar{M}$ is a normal almost contact metric structure such that the 1 -form η and the fundamental 2 -form Φ, given by $\Phi(X,Y) = \bar{g}(X,\phi Y)$, are closed. A trivial example of a Cosymplectic manifold is the product of a $2n$-dimensional Kaehler manifold with a 1 -dimensional manifold. In other words, an indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Cosymplectic manifold if (see [4]),

$$(\bar{\nabla}_{X}\phi)Y = 0, \bar{\nabla}_{X}V = 0\#(10)$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denote the Levi-Civita connection on $\bar{M}$.

3. Hemi-Slant Lightlike Submanifolds

Let M be an r-lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ of index q such that the characteristic vector field V is tangent to M. Then, V does not belong to $\mathrm{Rad}(TM)$, if we assume $V \in \Gamma(\mathrm{Rad}(TM))$ then there exists a vector field $N \in \Gamma(\mathrm{ltr}(TM))$ such that $g(N,V) = 1$ but using (8) and (9), $\bar{g}(N,V) = \bar{g}(\phi N,\phi V) = 0$, leads to a contradiction. Let the radical distribution be such that $\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM)$, then we have local quasi orthonormal field of frames on $\bar{M}$ along M as $\{X_{a}, V, \xi_{i}, N_{i}, W_{\alpha}\}$,

where $\{\xi_{i}\}_{i=1}^{r}$ and $\{N_{i}\}_{i=1}^{r}$ are lightlike basis of $\mathrm{Rad}(TM)$ and $\mathrm{ltr}(TM)$, respectively and $\{X_{a}\}_{a=1}^{k}$ and $\{W_{\alpha}\}_{\alpha=1}^{l}$ are orthonormal basis of $S(TM)$ (except $\{V\}$) and $S(TM^{\perp})$, respectively. Now, for lightlike basis $\{\xi_{1}, \ldots, \xi_{r}, N_{1}, \ldots, N_{r}\}$ of $\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)$, we can construct an orthonormal basis $\{U_{1}, \ldots, U_{2r}\}$ as

$$U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1), U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1)$$
$$U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2), U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2)$$
$$\dots \dots$$
$$\dots \dots$$
$$U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r), U_{2r} = \frac{1}{\sqrt{2}}(\xi_r - N_r)$$

Hence, $\mathrm{span}\{\xi_i, N_i\}$ is a non-degenerate space of constant index r implies $\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)$ is non-degenerate and of constant index r on $\bar{M}$. Therefore,

index $(T\bar{M}) = \mathrm{index}(\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)) + \mathrm{index}\big(S(TM) \perp S(TM^{\perp})\big)$,

implies $q = r +$ index $\big(S(TM) \perp S(TM^{\perp})\big)$. If $r = q$, then $S(TM) \perp S(TM^{\perp})$ is Riemannian and hence $S(TM)$ is Riemannian. Thus, we have the following important lemma.

Lemma 3.1. Let M be an r-lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ of index q such that the characteristic vector field V is tangent to M. Assume that the radical distribution Rad(TM) is such that $\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM)$. If $r = q$ then the screen distribution $S(TM)$ is Riemannian.

To define slant submanifolds, we need angle between two vector fields of the submanifold. The radical distribution is totally lightlike so it is not possible to define angle between two of its vector fields. From Lemma 3.1, the screen distribution is Riemannian and possible to define angle between two of its vector fields. Thus, we define hemi-slant lightlike submanifolds of indefinite Cosymplectic manifold as below:

Definition 3.2. Let M be a q-lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ of index q with characteristic vector field V tangent to M. Then the submanifold M is said to be hemi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\mathrm{Rad}(TM)$ is a distribution on M such that $\phi(\mathrm{Rad}(TM)) = \mathrm{ltr}(TM)$.

(ii) For all $x \in U \subset M$ and for each non-zero vector field X tangent to $S(TM) = D^{\theta} \perp V$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $S(TM)$ is constant, where D^{θ} is complementary distribution to V in screen distribution $S(TM)$.

A hemi-slant lightlike submanifold is said to be proper if $D^{\theta} \neq 0$ and $\theta \neq 0, \pi/2$. Hence, by using the definition of hemi-slant lightlike submanifold, the tangent bundle TM of M is decomposed as

$$TM = S(TM) \perp \mathrm{Rad}(TM) = D^{\theta} \perp \{V\} \perp \mathrm{Rad}(TM)$$

The following construction helps in understanding the example of hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold.

Let $\big(\mathbf{R}_q^{2m+1}, \phi, V, \eta, \bar{g}\big)$ be with its usual Cosymplectic structure and given by

$$\eta = dz, V = \partial z$$
$$\bar{g} = \eta \otimes \eta - \sum_{i=1}^{q/2} (dx^i \otimes dx^i + dy^i \otimes dy^i) + \sum_{i=q+1}^{m} (dx^i \otimes dx^i + dy^i \otimes dy^i)$$
$$\phi\left(\sum_{i=1}^{m} (X_i \partial x^i + Y_i \partial y^i) + z\partial z\right) = \sum_{i=1}^{m} (Y_i \partial x^i - X_i \partial y^i)$$

where (x^i, y^i, z) are cartesian coordinates.

Example 1. Let M be a submanifold of $\bar{M} = (\mathbf{R}_2^9, \bar{g})$ defined by

$$x_1 = s, x_2 = t, x_3 = u\sin v, x_4 = \sin u$$
$$y_1 = t, y_2 = s, y_3 = u\cos v, y_4 = \cos u$$

where $u, v \in (0, \pi/2)$ and $\mathbf{R}_2^9$ is a semi-Euclidean space of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Then, the local frame $\{\xi_1, \xi_2, Z_1, Z_2, V\}$ of TM is given by

$$\xi_1 = \partial x_1 + \partial y_2, \xi_2 = \partial x_2 + \partial y_1$$
$$Z_1 = \sin v \partial x_3 + \cos u \partial x_4 + \cos v \partial y_3 - \sin u \partial y_4$$
$$Z_2 = u\cos v \partial x_3 - u\sin v \partial y_3$$
$$V = \partial z$$

Hence, clearly M is a 2-lightlike submanifold with $\mathrm{Rad}(TM) = \mathrm{span}\{\xi_1, \xi_2\}$ and $S(TM) = \mathrm{span}\{Z_1, Z_2\} \perp V$, which is Riemannian. It can be easily see

that $S(TM)$ is a slant distribution with slant angle $\theta = \pi/4$. Further, the screen transversal bundle $S(TM^\perp)$ is spanned by

$$W_1 = \sin u \partial x_4 + \cos u \partial y_4$$
$$W_2 = \sin v \partial x_3 - \cos u \partial x_4 + \cos v \partial y_3 + \sin u \partial y_4$$

The transversal lightlike bundle ltr(TM) is spanned by

$$N_1 = -\frac{1}{2}(-\partial x_1 - \partial y_2), N_2 = \frac{1}{2}(\partial x_2 - \partial y_1).$$

Clearly $\phi\xi_1 = 2N_2, \phi\xi_2 = -2N_1$. Hence, M is a hemi-slant lightlike submanifold of $\mathbf{R}_2^9$.

If the projections on the distributions D^θ and $\mathrm{Rad}(TM)$ are denoted by P and Q respectively, then for any X tangent to M can be written as $X = PX + \eta(X)V + QX$. Applying ϕ and using $\phi V = 0$, we obtain

$$\phi X = \mathcal{T}PX + \mathcal{F}PX + \mathcal{F}QX \#(11)$$

Then, using the definition of hemi-slant lightlike submanifolds, we get $\mathcal{T}PX \in \Gamma(D^\theta), \mathcal{F}PX \in \Gamma(\mathrm{tr}(TM))$ and $\mathcal{F}QX \in \Gamma(\mathrm{ltr}(TM))$. Similarly, for any $U \in \Gamma(\mathrm{tr}(TM))$, we have

$$\phi U = \mathcal{B}U + \mathcal{C}U \#(12)$$

where $\mathcal{B}U$ and $\mathcal{C}U$ are the tangential and the transversal components of ϕU, respectively.

Lemma 3.3. Let M be a hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ then $\mathcal{F}PX \in \Gamma(S(TM^\perp))$, for any $X \in \Gamma(TM)$.

Proof. Clearly $\mathcal{F}PX \in \Gamma(S(TM^\perp))$, if and only if, $\bar{g}(\mathcal{F}PX, \xi) = 0$, for any $\xi \in \Gamma(\mathrm{Rad}(TM))$. Using (8), (9) and (11), we have

$$\bar{g}(\mathcal{F}PX, \xi) = \bar{g}(\phi X - \mathcal{T}PX, \xi) = \bar{g}(\phi PX, \xi) = -\bar{g}(PX, \phi\xi) = 0$$

hence, the result follows.

Thus, using the last Lemma 3.3 and (10), it follows that $\mathcal{F}(S(TM))$ is a subspace of $S(TM^\perp)$. Therefore, there exists an invariant subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^\perp) = \mathcal{F}\left(S(T_pM)\right) \perp \mu_p \#(13)$$

thus $T_p\bar{M} = S(T_pM) \perp \{\mathrm{Rad}(T_pM) \oplus \mathrm{ltr}(T_pM)\} \perp \left\{\mathcal{F}\left(S(T_pM)\right) \perp \mu_p\right\}$. Now, differentiating (11) and using (2)-(4) and (12), we have

$$(\nabla_X\mathcal{T})PY = A_{\mathcal{F}PY}X + A_{\mathcal{F}QY}X + Bh^s(X,Y) + \phi h^l(X,Y),$$
$$(\nabla_X\mathcal{F})PY = \mathcal{C}h^s(X,Y) - h^s(X,\mathcal{T}PY) - D^s(X,\mathcal{F}QY), \#(14)$$

$$(\nabla_X \mathcal{F})QY = -h^l(X, \mathcal{T}PY) - D^l(X, \mathcal{F}PY), \#(15)$$

for any $X, Y \in \Gamma(TM)$, where $(\nabla_X \mathcal{T})PY = \nabla_X \mathcal{T}PY - \mathcal{T}P\nabla_X Y, (\nabla_X \mathcal{F})PY = \nabla_X^s \mathcal{F}PY - \mathcal{F}P\nabla_X Y$ and $(\nabla_X \mathcal{F})QY = \nabla_X^l \mathcal{F}QY - \mathcal{F}Q\nabla_X Y$.

In [16], we derived the following theorem for the existence of hemi-slant lightlike submanifold of indefinite Cosymplectic manifolds.

Theorem 3.4 ([16]) The necessary and sufficient conditions for a q-lightlike submanifold of an indefinite Cosymplectic manifold of index q to be a hemislant lightlike submanifold are

(a) $\phi(\text{ltr}(TM))$ is a distribution on M.

(b) For any vector field X tangent to M, there exists a constant $\lambda \in [-1,0]$ such that $(\mathcal{T}P)^2 PX = -\lambda PX$, where $\lambda = -\cos^2 \theta$.

Corollary 3.5. Let M be a hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ of index q. Then

$$g(\mathcal{T}PX, \mathcal{T}PX) = \cos^2 \theta g(PX, PX), \bar{g}(\mathcal{F}PX, \mathcal{F}PX) = \sin^2 \theta g(PX, PX) \#(16)$$

for any $X, Y \in \Gamma(TM)$.

Theorem 3.6. Let M be a q-dimensional lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ of index q. Then, any Coisotropic semitransversal lightlike submanifold is a hemi-slant lightlike submanifold with $\theta = 0$. Moreover, any semi-transversal lightlike submanifold ([23]) of $\bar{M}$ with $D = \{0\}$ is a hemi-slant submanifold with $\theta = \pi/2$.

Proof. Let M be a q-dimensional semi-transversal lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then, by the definition of semitransversal lightlike submanifold, $\text{Rad}(TM)$ is a distribution on M such that $\phi(\text{Rad}(TM)) = \text{ltr}(TM)$. If M is a Coisotropic semi-transversal lightlike submanifold then $S(TM^\perp) = \{0\}$ thus $D^\perp = \{0\}$. Using the Lemma 3.1, the screen distribution $S(TM)$ is Riemannian and moreover D is invariant with respect to ϕ therefore it follows that $\theta = 0$. Second assertion is clear.

Theorem 3.7. There do not exist proper hemi-slant totally lightlike or isotropic submanifolds in indefinite Cosymplectic manifolds.

Proof. Let M be a totally lightlike submanifold of an indefinite Cosymplectic manifold then $TM = \text{Rad}(TM)$ and hence $S(TM) = \{0\}$. The other assertion follows similarly.

4. Minimal Hemi-Slant Lightlike Submanifolds

Definition 4.1. [25]. If the second fundamental form h of a submanifold tangent to characteristic vector field V, of an indefinite Sasakian manifold $\bar{M}$ is of the form

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\mathcal{H} + \eta(X)h(Y, V) + \eta(Y)h(X, V), \#(17)$$

for any $X, Y \in \Gamma(TM)$, where $\mathcal{H}$ is a vector field transversal to M, then M is called a totally contact umbilical and totally contact geodesic if $\mathcal{H} = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\mathcal{H}^l + \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V), \#(18)$$
$$h^s(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}\mathcal{H}^s + \eta(X)h^s(Y, V) + \eta(Y)h^s(X, V), \#(19)$$

where $\mathcal{H}^l \in \Gamma(\text{ltr}(TM))$ and $\mathcal{H}^s \in \Gamma(S(TM^\perp))$.

Now, we recall the following result from [16].

Theorem 4.2. ([16]) Every totally contact umbilical proper hemi-slant lightlike submanifold M of an indefinite Cosymplectic manifold $\bar{M}$ is totally contact geodesic.

Theorem 4.3. Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then the induced connection ∇ is a metric connection on M.

Proof. Since M is a totally contact umbilical proper hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ therefore using (18) and the fact that every totally contact umbilical proper hemi-slant lightlike submanifold is totally contact geodesic, we have $h^l(X,Y) = \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V)$, for any $X,Y \in \Gamma(TM)$. Then, in particular, for any $X,Y \in \Gamma(D^\theta), h^l(X,Y) = 0$ and moreover $h^l(V,V) = 0, h^l(Z,\xi) = 0, h^l(Z,V) = 0$, for any $Z \in \Gamma(TM)$. Hence, h^l vanishes identically on M. Thus, using the Theorem 2.2 in [7], at page 159, the induced connection ∇ becomes a metric connection on M.

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then by straightforward calculations (see [7]), we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y$$
$$-A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z)$$
$$+D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)) + (\nabla_X h^s)(Y,Z)$$
$$-(\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z))$$
$$-D^s\left(Y,h^l(X,Z)\right)\#20$$

where

$$(\nabla_X h^l)(Y,Z) = \nabla_X^l h^l(Y,Z) - h^l(\nabla_X Y,Z) - h^l(Y,\nabla_X Z)(\nabla_X h^s)(Y,Z) = \nabla_X^s h^s(Y,Z) - h^s(\nabla_X Y,Z) - h^s(Y,\nabla_X Z)\#21$$

An indefinite Cosymplectic space form is a connected indefinite Cosymplectic manifold of constant holomorphic sectional curvature c and denoted by $\bar{M}(c)$. Then, the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(X,Y)Z = \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \eta(X)\eta(Z)Y$$
$$+\bar{g}(\phi Y,Z)\phi X + \bar{g}(\phi Z,X)\phi Y - 2\bar{g}(\phi X,Y)\phi Z\} - \eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V\#22$$

for any vector fields X, Y, and $Z \in \bar{M}$, [4].

Theorem 4.4. There do not exist totally contact umbilical proper hemislant lightlike submanifolds of an indefinite Cosymplectic space form $\bar{M}(c)$ such that $c \neq 0$.

Proof. Suppose M be a totally contact umbilical proper hemi-slant lightlike submanifold of $\bar{M}(c)$ such that $c \neq 0$. Then using (22), for any $X \in \Gamma(D^\theta)$ and $\xi,\xi' \in \Gamma(\mathrm{Rad}(TM))$, we obtain

$$\bar{g}(\bar{R}(X,\phi X)\xi',\xi) = -\frac{c}{2}g(X,X)g(\phi\xi',\xi)\#(23)$$

On the other hand using (20), we get

$$\bar{g}(\bar{R}(X,\phi X)\xi',\xi) = \bar{g}\left((\nabla_X h^l)(\phi X,\xi'),\xi\right) - \bar{g}\left(\left(\nabla_{\phi X} h^l\right)(X,\xi'),\xi\right)\#(24)$$

Now, using (18) and (21), we have

$$(\nabla_X h^l)(\phi X,\xi') = \bar{g}(h^l(X,\mathcal{T}X),\xi')\mathcal{H}^l = g(X,\phi X)\bar{g}(\mathcal{H}^l,\xi') = 0$$
$$= -g(\nabla_X \phi X,\xi')\mathcal{H}^l - g(\phi X,\nabla_X \xi')\mathcal{H}^l = \bar{g}(\bar{\nabla}_X \mathcal{T}X,\xi')\mathcal{H}^l\#25$$

Similarly

$$\left(\nabla_{\phi X} h^l\right)(X,\xi') = \bar{g}(h^l(\phi X,X),\xi')\mathcal{H}^l = g(\phi X,X)\bar{g}(\mathcal{H}^l,\xi') = 0$$

$$= -g(\nabla_{\phi X}X, \xi')\mathcal{H}^l - g(X, \nabla_{\phi X}\xi')\mathcal{H}^l = \bar{g}(\bar{\nabla}_{\phi X}X, \xi')\mathcal{H}^l \#26$$

Thus, from (23)-(26), we obtain $\frac{c}{2}g(X,X)g(\phi\xi',\xi) = 0$. Since g is a Riemannian metric on D^θ, implies that $g(\phi\xi',\xi) \neq 0$, therefore $c = 0$. This contradiction completes the proof.

In [7], a minimal lightlike submanifold M is defined, when M is a hypersurface of a 4 -dimensional Minkowski space. Then in [2], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ is introduced as follows:

Definition 4.5. A lightlike submanifold ($M, g, S(TM)$) isometrically immersed in a semi-Riemannian manifold ($\bar{M}, \bar{g}$) is minimal if $h^s = 0$ on $\mathrm{Rad}(TM)$ and trace $h = 0$, where trace is written with respect to g restricted to $S(TM)$.

We use the orthonormal basis of M given by $\{\xi_1, \dots, \xi_r, e_1, \dots, e_k\}$, such that $\{\xi_1, \dots, \xi_r\}$ and $\{e_1, \dots, e_k\}$ form a basis of $\mathrm{Rad}(TM)$ and $S(TM)$, respectively.

Theorem 4.6. Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then M is minimal.

Proof. The assertion follows directly using the Theorem 4.2.

Lemma 4.7. Let M be a proper hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ such that $\dim(S(TM)) = \dim(S(TM^\perp))$. If $\{e_1, \dots, e_k\}$ is a local orthonormal basis of $S(TM)$ then

$$\{\csc\theta\mathcal{F}e_1, \dots, \csc\theta\mathcal{F}e_k\}$$

is a orthonormal basis of $S(TM^\perp)$.

Proof. From the Lemma 3.3 and (16), the proof is complete.

Definition 4.8. [8]. A lightlike submanifold is called an irrotational submanifold, if and only if, $\bar{\nabla}_X\xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$.

This implies that, if M is an irrotational lightlike submanifold then $\bar{\nabla}_X\xi = \nabla_X\xi, h^l(X,\xi) = 0$ and $h^s(X,\xi) = 0$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$.

Theorem 4.9. Let M be an irrotational hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then M is minimal, if and only if,

$$\mathrm{trace}A_{W_q}\Big|_{S(TM)} = 0,\ \mathrm{trace}A^*_{\xi_j}\Big|_{S(TM)} = 0$$

where $\{W_q\}_{q=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_j\}_{j=1}^{r}$ is a basis of $\mathrm{Rad}(TM)$.

Proof. Since M is an irrotational lightlike submanifold then $h^s(X,\xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$. Thus h^s vanishes on $\mathrm{Rad}(TM)$. Hence M is minimal, if and only if, trace $h = 0$ on $S(TM)$, that is, M is minimal if and only $\sum_{i=1}^{k} h(e_i, e_i) = 0$. Using (5) and (7), we obtain

$$(27)\ \sum_{i=1}^{k} h(e_i, e_i) = \sum_{i=1}^{k}\left\{\frac{1}{r}\sum_{j=1}^{r} g\left(A^*_{\xi_j}e_i, e_i\right)N_j + \frac{1}{l}\sum_{q=1}^{l} g\left(A_{W_q}e_i, e_i\right)W_q\right\}.$$

Thus the assertion follows from (27).

Theorem 4.10. Let M be a proper hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then M is minimal, if and only if, $\mathrm{trace}A_{W_q}\Big|_{S(TM)} = 0$, $\mathrm{trace}A^*_{\xi_j}\Big|_{S(TM)} = 0$, and $\bar{g}(D^l(X,W),Y) = 0$, for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$, where $\{W_q\}_{q=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_j\}_{j=1}^{r}$ is a basis of $\mathrm{Rad}(TM)$.

Proof. Let $X, Y \in \Gamma(\text{Rad}(TM))$ then using (5), it is clear that $h^s = 0$ on $\text{Rad}(TM)$, if and only if, $\bar{g}(D^l(X,W),Y) = 0$. Moreover, using the Proposition 3.1 of [2], $h^l = 0$ on $\text{Rad}(TM)$. Therefore, M is minimal, if and only if, $\sum_{i=1}^{k} h(e_i,e_i) = 0$. Using (5) and (7), we obtain

$$\sum_{i=1}^{k} h(e_i,e_i) = \sum_{i=1}^{k} \left\{ \frac{1}{r} \sum_{j=1}^{r} g\left(A^*_{\xi_j} e_i, e_i\right) N_j + \frac{1}{l} \sum_{q=1}^{l} g\left(A_{W_q} e_i, e_i\right) W_q \right\}.$$

Thus, the proof is complete.

Theorem 4.11. Let M be a proper hemi-slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ such that $\dim(S(TM)) = \dim(S(TM^{\perp}))$. Then M is minimal, if and only if,

$$\text{trace} A_{\csc\theta \mathcal{F} e_i}\big|_{S(TM)} = 0,\ \text{trace} A^*_{\xi_j}\big| S(TM) = 0, \bar{g}(D^l(X,\mathcal{F}e_i),Y) = 0 \quad ,$$

for any $X, Y \in \Gamma(\text{Rad}(TM))$, where $\{e_i\}_{i=1}^{k}$ is a basis of $S(TM)$.

Proof. Let $\{e_i\}_{i=1}^{k}$ is a basis of $S(TM)$, then using the Lemma 4.7 we know $\{\csc\theta \mathcal{F} e_i\}_{i=1}^{k}$ is a basis of $S(TM^{\perp})$. Therefore, we can write

$$h^s(X,X) = \sum_{i=1}^{k} \lambda_i \csc\theta \mathcal{F} e_i \#(28)$$

for any $X \in \Gamma(TM)$ and for some functions $\lambda_i, i \in \{1, \ldots, k\}$. Using (5), we have $\bar{g}(h^s(X,X),W) = \bar{g}(A_W X, X)$, for any $X \in \Gamma(S(TM))$. Then, using (16) and (28), we obtain $\lambda_i = \bar{g}\left(A_{\csc\theta \mathcal{F} e_i} X, X\right)$ and hence we get

$$h^s(X,X) = \sum_{i=1}^{k} \csc\theta \mathcal{F} e_i \bar{g}\left(A_{\csc\theta \mathcal{F} e_i} X, X\right),$$

for any $X \in \Gamma(S(TM))$. Then the assertion comes from the Theorem 4.10.

5. Slant Lightlike Submanifolds

Definition 5.1. [10]. Let M be an r-lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ of index $2r$. Then M is called a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\text{Rad}(TM)$ is a distribution on M such that $\phi\text{Rad}(TM) \cap \text{Rad}(TM) = \{0\}$.

(ii) For each non-zero vector field X tangent to $\bar{D} = D \perp \{V\}$ at $x \in U \subset M$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $\bar{D}_x$ is constant, that is, it is independent of the choice of $x \in U \subset M$ and $X \in \bar{D}_x$, where $\bar{D}$ is complementary distribution to $\phi\text{ltr}(TM) \oplus \phi\text{Rad}(TM)$ in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called the slant angle of the distribution $\bar{D}$. A slant lightlike submanifold M is said to be proper if $\bar{D} \neq \{0\}$, and $\theta \neq 0, \frac{\pi}{2}$. Then the tangent bundle TM of M is decomposed as $TM = \text{Rad}(TM) \perp S(TM) = \text{Rad}(TM) \perp (\phi\text{Rad}(TM) \oplus \phi\text{ltr}(TM)) \perp \bar{D}$, where $\bar{D} = D \perp \{V\}$. Therefore for any $X \in \Gamma(TM)$, we write

$$\phi X = \mathcal{T}X + \mathcal{F}X, \#(29)$$

where $\mathcal{T}X$ is the tangential component of ϕX and $\mathcal{F}X$ is the transversal component of ϕX. Similarly, for any $U \in \Gamma(\text{tr}(TM))$, we write

$$\phi U = \mathcal{B}U + \mathcal{C}U, \#(30)$$

where $\mathcal{B}U$ (respectively, $\mathcal{C}U$) is the tangential (respectively, transversal) component of ϕU. Let P_1, P_2, Q_1, Q_2 and $\bar{Q}_2$ be the projections on $\mathrm{Rad}(TM)$, $\phi\mathrm{Rad}(TM), \phi\mathrm{ltr}(TM), D$ and $\bar{D} = D \perp V$, respectively. Then for any $X \in \Gamma(TM)$, we can write $X = P_1X + P_2X + Q_1X + \bar{Q}_2X$, where $\bar{Q}_2X = Q_2X + \eta(X)V$ and on applying ϕ, we obtain

$$\phi X = \phi P_1X + \phi P_2X + \mathcal{F}Q_1X + \mathcal{T}Q_2X + \mathcal{F}Q_2X.$$

Then using (29) and (30), we get

$$\phi P_1X = \mathcal{T}P_1X \in \Gamma(\phi\mathrm{Rad}(TM)), \phi P_2X = \mathcal{T}P_2X \in \Gamma(\mathrm{Rad}(TM)),$$
$$\mathcal{F}P_1X = \mathcal{F}P_2X = 0, \mathcal{T}Q_2X \in \Gamma(D), \mathcal{F}Q_1X \in \Gamma(\mathrm{ltr}(TM)).$$

Let M be a slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Similar to the proof of Lemma 3.3, we have $\mathcal{F}Q_2X \in \Gamma\big(S(TM^{\perp})\big)$, for any $X \in \Gamma(TM)$. Thus, it follows that $\mathcal{F}\big(D_p\big)$ is a subspace of $S(TM^{\perp})$. Therefore, there exists an invariant subspace μ_p of $T_p\bar{M}$ such that $S\big(T_pM^{\perp}\big) = \mathcal{F}\big(D_p\big) \perp \mu_p$.

Theorem 5.2. [10]. Let M be a q-lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then M is a slant lightlike submanifold, if and only if,

(i) $\phi(\mathrm{Rad}(TM))$ is a distribution on M such that $\phi\mathrm{Rad}(TM) \cap \mathrm{Rad}(TM) = \{0\}$.

(ii) For any X tangent to $\bar{D}$, there exists a constant $\lambda \in [-1,0]$ such that

$$\mathcal{T}^2X = \lambda(X - \eta(X)V)\#(31)$$

where $\bar{D}$ is a complementary distribution such to $\phi\mathrm{ltr}(TM) \oplus \phi\mathrm{Rad}(TM)$ in TM and $\lambda = -\cos^2\theta$.

Corollary 5.3. [10]. Let M be a slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then we have

$$g(\mathcal{T}\bar{Q}_2X, \mathcal{T}\bar{Q}_2Y) = \cos^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)\}, \#(32)$$

and

$$g(\mathcal{F}\bar{Q}_2X, \mathcal{F}\bar{Q}_2Y) = \sin^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)\}$$

for any $X, Y \in \Gamma(TM)$.

Theorem 5.4. [10]. Let M be a q-lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$. Then M is a slant lightlike submanifold, if and only if,

(i) $\phi(\mathrm{Rad}(TM))$ is a distribution on M such that $\phi\mathrm{Rad}(TM) \cap \mathrm{Rad}(TM) = \{0\}$.

(ii) For any X tangent to $\bar{D}$, there exists a constant $\mu \in [-1,0]$ such that

$$\mathcal{B}\mathcal{F}X = \mu(X - \eta(X)V), \#(33)$$

where $\bar{D}$ is a complementary distribution such to $\phi\mathrm{ltr}(TM) \oplus \phi\mathrm{Rad}(TM)$ in TM and $\mu = -\sin^2\theta$.

Definition 5.5. [9]. Let (M, g) be a lightlike submanifold, tangent to characteristic vector field V, of an indefinite Sasakian manifold ($\bar{M}, \bar{g}$). Then M is said to be a contact Screen Cauchy Riemann (SCR)-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) There exist real non-null distributions $D \subset S(TM)$ and $D^{\perp}$ such that

$$S(TM) = D \oplus D^{\perp} \perp \{V\}, \phi D^{\perp} \subset \big(S(TM^{\perp})\big), D \cap D^{\perp} = \{0\}$$

where $D^{\perp}$ is orthogonal complementary to $D \perp \{V\}$ in $S(TM)$.

(i) The distributions D and $\mathrm{Rad}(TM)$ are invariant with respect to ϕ.

Theorem 5.6. A contact SCR-lightlike submanifold M, of an indefinite Sasakian manifold $\bar{M}$, is a holomorphic or complex (resp. screen real) lightlike submanifold, if and only if, $D^{\perp} = \{0\}$ (resp. $D = \{0\}$).

Definition 5.7. [23]. Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is said to be a transversal lightlike submanifold if the following conditions are satisfied:

(i) $\text{Rad}(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}(\text{Rad}(TM)) = \text{ltr}(TM)$.

(ii) $S(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}(S(TM)) \subseteq S(TM^{\perp})$.

6. Non-existence of Warped Product Slant Lightlike Submanifolds

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_{M_1} and g_{M_2} respectively and $f > 0$ a differentiable function on M_1. Assume the product manifold $M_1 \times M_2$ with its projection $\pi: M_1 \times M_2 \to M_1$ and $\psi: M_1 \times M_2 \to M_2$. The warped product $M = M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ equipped with the Riemannian metric g where $g = g_{M_1} + f^2 g_{M_2}$. If X is tangent to $M = M_1 \times_f M_2$ at (p, q) then we have $\|X\|^2 = \|\pi_* X\|^2 + f^2(\pi(X))\|\psi_* X\|^2$. The function f is called the warping function of the warped product. For differentiable function f on M , the gradient ∇f is defined by $g(\nabla f, X) = Xf$, for all $X \in T(M)$.

Lemma 6.1. [15]. Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in T(M_1)$ and $U, Z \in T(M_2)$ then

$$\nabla_X U = \nabla_U X = \frac{Xf}{f} U = X(\ln f)U \#(34)$$

Next, we derive characterization theorems for the non-existence of warped product slant lightlike submanifolds of indefinite Cosymplectic manifolds.

Theorem 6.2. Let $\bar{M}$ be an indefinite Cosymplectic manifold. Then there does not exist warped product submanifold $M = M_\theta \times_f M_T$ of $\bar{M}$ such that M_θ is a proper slant lightlike submanifold of $\bar{M}$ and M_T is a holomorphic Screen Cauchy-Riemann (SCR)-lightlike submanifold of $\bar{M}$.

Proof. Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then using (34), we get $g(\nabla_{\phi X} Z, X) = Z(\ln f) g(\phi X, X) = 0$. Therefore, using (2), (8) to (10) and (29), we get

$$\begin{aligned} 0 = \bar{g}(\bar{\nabla}_{\phi X} Z, X) &= -\bar{g}(\phi Z, \bar{\nabla}_{\phi X} \phi X) = \bar{g}(\bar{\nabla}_{\phi X} \mathcal{T}Z, \phi X) - \bar{g}(\mathcal{F}Z, \bar{\nabla}_{\phi X} \phi X) \\ &= \bar{g}(\nabla_{\phi X} \mathcal{T}Z, \phi X) - \bar{g}(\mathcal{F}Z, h^s(\phi X, \phi X)) \end{aligned}$$

Further by virtue of (34), we obtain $\mathcal{T}Z(\ln f) g(X, X) = \bar{g}(h^s(\phi X, \phi X), \mathcal{F}Z)$. Thus, using polarization identity, we get

$$\mathcal{T}Z(\ln f) g(X, Y) = \bar{g}(h^s(\phi X, \phi Y), \mathcal{F}Z), \#(35)$$

for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. On the other hand, using (4) and (34), we have

$$\begin{aligned} g(A_{\mathcal{F}Z} \phi X, \phi Y) = -g(\bar{\nabla}_{\phi X} \mathcal{F}Z, \phi Y) &= \bar{g}(Z, \bar{\nabla}_{\phi X} Y) - \bar{g}(\mathcal{T}Z, \bar{\nabla}_{\phi X} \phi Y) \\ &= -\bar{g}(\bar{\nabla}_{\phi X} Z, Y) + \bar{g}(\bar{\nabla}_{\phi X} \mathcal{T}Z, \phi Y) \\ &= -Z(\ln f) g(\phi X, Y) + \mathcal{T}Z(\ln f) g(X, Y) \end{aligned}$$

Now using (5), we have $\bar{g}(h^s(\phi X, \phi Y), \mathcal{F}Z) = g(A_{\mathcal{F}Z} \phi X, \phi Y)$, therefore we obtain

$$\bar{g}(h^s(\phi X, \phi Y), \mathcal{F}Z) = -Z(\ln f) g(\phi X, Y) + \mathcal{T}Z(\ln f) g(X, Y) \#(36)$$

Thus, (35) and (36) imply Z $(\ln f)$ $g(\phi X, Y) = 0$, for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Since $M_T \neq \{0\}$ is a Riemannian and invariant then $Z \ln f = 0$, this shows that f is constant. Hence, the proof is complete.

Theorem 6.3. Let $\bar{M}$ be an indefinite Cosymplectic manifold. Then there does not exist warped product submanifold $M = M_T \times_f M_\theta$ in $\bar{M}$ such that M_T is a holomorphic SCR-lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.

Proof. Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then using (34) we have

$$g(\nabla_{\mathcal{T}Z} X, Z) = X(\ln f) g(\mathcal{T}Z, Z) = 0$$

This further using with (4), (5) and (32) implies that

$$\begin{aligned} 0 = \bar{g}(\bar{\nabla}_{\mathcal{T}Z} X, Z) &= -\bar{g}(\phi X, \bar{\nabla}_{\mathcal{T}Z} \mathcal{T}Z) - \bar{g}(\phi X, \bar{\nabla}_{\mathcal{T}Z} \mathcal{F}Z) \\ &= \phi X(\ln f) g(\mathcal{T}Z, \mathcal{T}Z) + \bar{g}(h^s(\phi X, \mathcal{T}Z), \mathcal{F}Z) \\ &= \phi X(\ln f) \cdot \cos^2\theta g(Z, Z) + \bar{g}(h^s(\phi X, \mathcal{T}Z), \mathcal{F}Z) \end{aligned}$$

Replace X by ϕX, we get

$$X(\ln f) \cdot \cos^2\theta g(Z, Z) + \bar{g}(h^s(X, \mathcal{T}Z), \mathcal{F}Z) = 0 \# (37)$$

After replacing Z by $\mathcal{T}Z$ and using (31), (32), we obtain

$$\bar{g}(h^s(X, Z), \mathcal{F}\mathcal{T}Z) = X(\ln f) \cdot \cos^2\theta g(Z, Z) \# (38)$$

Next, on the other hand using (2), (29), (31), (32) and (34), for any X, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Y, Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ, we have

$$\begin{aligned} \bar{g}(h^s(\mathcal{T}Z, X), \mathcal{F}Y) &= -\bar{g}(\mathcal{T}Z, \bar{\nabla}_X \phi Y) + \bar{g}(\mathcal{T}Z, \bar{\nabla}_X \mathcal{T}Y) \\ &= -\cos^2\theta X(\ln f) g(Z, Y) + \bar{g}(\mathcal{F}\mathcal{T}Z, h^s(X, Y)) \\ &\quad + X(\ln f) g(\mathcal{T}Z, \mathcal{T}Y) \\ &= \bar{g}(\mathcal{F}\mathcal{T}Z, h^s(X, Y)) \end{aligned}$$

Put $Y = Z$

$$\bar{g}(h^s(\mathcal{T}Z, X), \mathcal{F}Z) = \bar{g}(\mathcal{F}\mathcal{T}Z, h^s(X, Z)) \# (39)$$

Thus, from (37)-(39), we have $X(\ln f)\cos^2\theta g(Z, Z) = 0$. Since D^θ is a proper slant and Z is non-null, we obtain $X(\ln f) = 0$. This proves our assertion.

Theorem 6.4. Let $\bar{M}$ be an indefinite Cosymplectic manifold. Then there does not exist warped product submanifold $M = M_\perp \times_f M_\theta$ of $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.

Proof. Let $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and X independent of V and tangent to $S(TM)$ of a transversal lightlike submanifold $M_\perp$, then using (4), (8) to (10), (29), (32), and (34), we have

$$\begin{aligned} g(A_{\phi X} \mathcal{T}Z, Z) &= g(\nabla_{\mathcal{T}Z} X, \mathcal{T}Z) + \bar{g}(h^s(\mathcal{T}Z, X), \mathcal{F}Z) \\ &= X(\ln f)\cos^2\theta g(Z, Z) + \bar{g}(h^s(\mathcal{T}Z, X), \mathcal{F}Z) \end{aligned}$$

Using (5), in the left-hand side of above equation, we obtain

$$\bar{g}(h^s(\mathcal{T}Z, Z), \phi X) = X(\ln f)\cos^2\theta g(Z, Z) + \bar{g}(h^s(\mathcal{T}Z, X), \mathcal{F}Z) \# (40)$$

Replace Z by $\mathcal{T}Z$ in (40) and using (31) and (32), we get

$$\bar{g}(h^s(Z, \mathcal{T}Z), \phi X) = -X(\ln f)\cos^2\theta g(Z, Z) + \bar{g}(h^s(Z, X), \mathcal{F}\mathcal{T}Z) \# (41)$$

On the other hand, using (4), (8) to (10), (29), (31) and (34), we have

$$\begin{aligned} g(A_{\mathcal{F}Z}X,\mathcal{T}Z) &= \bar{g}(\bar{\nabla}_X Z,\phi\mathcal{T}Z) + \bar{g}(\bar{\nabla}_X \mathcal{T}Z,\mathcal{T}Z) \\ &= -\cos^2\theta X(\ln f)g(Z,Z) + \bar{g}(h^s(X,Z),\mathcal{F}\mathcal{T}Z) \\ &\quad + X(\ln f)\cos^2\theta g(Z,Z) \\ &= \bar{g}(h^s(X,Z),\mathcal{F}\mathcal{T}Z) \end{aligned}$$

Hence using (5), we obtain

$$\bar{g}(h^s(\mathcal{T}Z,X),\mathcal{F}Z) = \bar{g}(h^s(X,Z),\mathcal{F}\mathcal{T}Z). \#(42)$$

Thus, using (40)-(42), we get $2X(\ln f)\cos^2\theta g(Z,Z) = 0$. Since M_θ is proper slant lightlike submanifold and D^θ is Riemannian therefore we obtain $X(\ln f) = 0$. Hence, f is constant, this proves our assertion.

Thus, using the Theorems 6.2, 6.3, 6.4, we call $M = M_\theta \times_f M_\perp$ as a warped product slant lightlike submanifold, where M_θ is a proper slant lightlike submanifold and $M_\perp$ is a transversal lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$.

Theorem 6.5. Let $M = M_\theta \times_f M_\perp$ be a warped product slant lightlike submanifold of an indefinite Cosymplectic manifold $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$. Then

$$g(h^s(X,Y),\phi Z) = -\mathcal{T}X(\ln f)g(Y,Z),$$

for any $X \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and Y, Z, independent of V and tangent to $S(TM)$ of transversal lightlike submanifold $M_\perp$.

Proof. For any $X \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and Y, Z independent of V and tangent to $S(TM)$ of transversal lightlike submanifold $M_\perp$, using (4), (8) to (10) and (29), we have $g(h^s(\mathcal{T}X,Y),\phi Z) = g(\bar{\nabla}_Y \mathcal{T}X,\phi Z) = g(\nabla_Y X,Z) + g(\bar{\nabla}_Y \phi\mathcal{F}X,Z)$. Since $F(D^\theta) \subset S(TM^\perp)$ and μ is invariant, therefore using (30), we have $\phi\mathcal{F}X = \mathcal{B}\mathcal{F}X$ and $\mathcal{C}\mathcal{F}X = 0$. Hence, using (33) and (34), we obtain $g(h^s(\mathcal{T}X,Y),\phi Z) = X(\ln f)g(Y,Z) - \sin^2\theta g(\bar{\nabla}_Y X,Z)$. Again using (34), we have

$$g(h^s(\mathcal{T}X,Y),\phi Z) = (1-\sin^2\theta)X(\ln f)g(Y,Z) = \cos^2\theta X(\ln f)g(Y,Z)$$

Replacing X by $\mathcal{T}X$ and then using (31), the assertion follows.

7. Conclusion

It is known that the Cosymplectic manifolds are odd-dimensional counterpart of Kaehler manifolds and there are many physical uses of Kaehler manifolds. Calabi-Yau manifolds are particular complex manifolds with Ricci flat Kaehler metric and having significant applications in super string theory which is based on a 10 -dimensional manifold $\bar{M} = M \times V_4$, where M is a Ricci flat 6-dimensional manifold and V_4 is an ordinary spacetime. Since as a generalization of Kaehler manifolds, Cosymplectic manifolds have significant applications in mathematical physics therefore low dimension hemislant lightlike submanifolds of indefinite Cosymplectic manifolds may have possible significant applications in mathematical physics.

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