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On Clairaut semi-invariant Riemannian maps to Sasakian manifolds

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Abstract

In this paper, we define Clairaut semi-invariant Riemannian maps to Sasakian manifolds. We obtain the necessary and sufficient conditions for a curve on a base manifold to be geodesic. We also find the conditions for semi-invariant Riemannian maps to be Clairaut semi-invariant Riemannian map. Further, we calculate the necessary and sufficient conditions of these maps to be totally geodesic. Also, we discuss the biharmonicity condition of this map. Later, we obtain the inequality results for these maps. Finally, we give non-trivial examples to show the existence of these maps.

Mathematics Subject Classification: 53B20, 53C22, 53C25, 53D10.

Keywords: Riemannian maps, Clairaut Riemannian maps, semi-invariant Riemannian maps, Sasakian manifolds.

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1. Introduction

The concept of Riemannian submersion between Riemannian manifolds was introduced by O'Neill 14 and Gray 8 . For further study of Riemannian submersions, we refer 66. Later, Fishcher [7] introduced the concept of Riemannian map between Riemannian manifolds as the generalization of Riemannian submersion and isometric immersion. The geometry of different kind of Riemaanian maps such as invariant Riemannian maps, anti-invariant Riemannian maps, semi-invariant Riemannian maps etc. has been studied by many authors [17, 19, 21, 15, 18, 27, 28, 1, 11, 22, 26, 20, 10, 23.

In classical differential geometry, there is a famous theorem, known as Clairaut's theorem which helps us to find the geodesics on a surface of revolution. This theorem states that for any geodesic γ on a surface S, a function $r\sin\theta$ is constant along γ, where θ is the angle between γ and the meridian through γ and r is the distance from a point on the surface of rotation axis. Bishop [3] extended this idea to Riemannian manifolds and defined Clairaut submersion. He obtained conditions under which a curve becomes geodesic and also derived necessary and sufficient condition for a Riemannian submersion to be a Clairaut Riemannian submersion.

Şahin [24] proposed the concept of a Clairaut Riemannian map as a generlization of Clairaut submersion where he obtained the condition for a curve to be geodesic on the total manifold. Then, he derived the necessary and sufficient condition for a Riemannian map to be a Clairaut Riemannian map. Yadav et al. 12, 30, 29, introduced

Clairaut Riemannian map for base manifold, Clairaut invariant and anti-invariant Riemannian maps from and to Kähler manifolds. Later, Polat and Meena 16 introduced Clairaut semi-invariant Riemannian maps to Kähler manifolds. Clairaut anti-invariant Riemannian map to Sasakian manifold was introduced by Zafar et al. 32.

This paper is organised as follows. Section 2 contains some important definitions and results which are needed for this paper and section 3 deals with Clairaut semiinvariant Riemannian maps to Sasakian manifolds. In section 4, we obtain some basic inequalities related to these maps. Finally, we provide an illustrative example.

2. Preliminaries

In this section, we recall some the definitions and results which are needed for this paper.

Definition 2.1. [4] An odd dimensional smooth manifold N is said to have an almost-contact structure (ϕ,ξ,η) if there exists a tensor field ϕ of type (1,1), a vector field ξ (known as the Reeb vector field), and a 1-form η, which satisfy the following conditions:

$$\phi^2=-I+\eta\otimes\xi,\phi\xi=0,\eta\circ\phi=0,\eta(\xi)=1\#(2.1)$$

Furthermore, an almost-contact manifold N with a Riemannian metric g_N is said to be compatible with the almost-contact structure (ϕ,ξ,η) if it satisfies the following conditions for any vector fields $E,G\in\Gamma(TN)$:

$$g_N(\phi E,\phi G)=g_N(E,G)-\eta(E)\eta(G),\#(2.2)$$
$$g_N(\phi E,G)=-g_N(E,\phi G),\eta(E)=g_N(E,\xi),\#(2.3)$$

and the structure (ϕ,ξ,η,g_N) is referred as an almost contact metric structure. The almost-contact structure (ϕ,ξ,η) is called normal if $\mathcal{N}+d\eta\otimes\xi=0$, where $\mathcal{N}$ is the Nijenhuis tensor of ϕ.

Additionally, if $d\eta=\Phi$, where $\Phi(E,G)=g_N(\phi E,G)$ is a tensor field of type (0,2), then an almost contact metric structure is said to be a normal contact metric structure.

A normal contact metric manifold N is called a Sasakian manifold if it satisfies the following conditions:

$$(\nabla_E^N\phi)G=g_N(E,G)\xi-\eta(G)E\#(2.4)$$
$$\nabla_E^N\xi=-\phi X\#(2.5)$$

where ∇^N is the Levi-Civita connection on N.

A Sasakian manifold with constant sectional curvature c is called a Sasakian space form. The curvature tensor of a Sasakian space form $N(c)$ is given by [5]

$$R(X,Y)Z=\frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\}+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X$$
$$+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y$$

Definition 2.2. [7] A smooth map $F:(M^m,g_M)\to(N^n,g_N)$ between two Riemannian manifolds is said to be a Riemannian map at $p\in M$ if the horizontal restriction $F_p^h:\left(\ker F_{*p}\right)^\perp\to(\text{range } F_{*p})$ is a linear isometry between the inner product spaces $\left(\left(\ker F_{*p}\right)^\perp, g_M(p)|_{(kerF_{*p})^\perp}\right)$ and ($\left(\text{range}F_{*p}, g_N(q)\mid(\text{range } _{*p}\right)$), where $q=F(p)$.

Further, consider a smooth map $F:(M^m,g_M)\to(N^n,g_N)$ between Riemannian manifolds M and N. The differential map F_* can be interpreted as a section of the bundle $\mathrm{Hom}(TM,F^{-1}TN)\to M$, where $F^{-1}TN$ is the pullback bundle, with fibers at $y_1\in M$ given by $(F^{-1}TN)_{y_1}=T_{F(y_1)}N$. The bundle Hom $(TM,F^{-1}TN)$ has a connection induced from the Levi-Civita connection ∇^M and the pullback connection ∇^{N_F}. The second fundamental form of F is symmetric and can be expressed as [13]

$$(\nabla F_*)(X_1, Y_1) = \nabla^N_{F_*X_1} F_* Y_1 - F_*(\nabla^M_{X_1} Y_1), \#(2.7)$$

for all $X_1, Y_1 \in \Gamma(TM)$. It is proved in [18], if $X_1, Y_1 \in \Gamma(kerF_*)^\perp$ then $(\nabla F_*)(X, Y) \in \Gamma(\text{ range } F_*)^\perp$.

Definition 2.3. [2] Let $F: (M^m, g_M) \to (N^n, g_N)$ be a smooth map between two Riemannian manifolds. Then the tension field is the trace of the second fundamental form. i.e.,

$$\tau(F) = \sum_{i=1}^{n} (\nabla F_*)(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of TM.

Definition 2.4. [25] A map F between two Reimannian manifolds is called a harmonic map, if it has a vanishing tenson field, i.e., $\tau(F) = 0$.

Geodesics, constant maps, and identity maps are examples of harmonic maps.

Definition 2.5. [9] The bitension field $\tau_2(F)$ of F is defined as

$$\tau_2(F) = -\Delta^F \tau(F) - \text{trace}^N(F_*, \tau(F))F_*.$$

Additionally, the map F is called biharmonic map if and only bitension field vanishes for every point.

Lemma 2.1. [19] Let $F: (M^m, g_M) \to (N^n, g_N)$ be a Riemannian map between Riemannian manifolds. Then, F is an umbilical Riemannian map if and only if

$$(\nabla F_*)(X, Y) = g_M(X, Y)H_N, \#(2.8)$$

for $X, Y \in \Gamma(\ker F_*)^\perp$ and H_N is a nowhere zero, mean curvature vector field on $\Gamma(\text{ range } F_*)^\perp$. Let X be a vector field on M, and let Z be any section of $\Gamma(\text{ range } F_*)^\perp$. Then, the orthogonal projection of $\nabla^N_X Z$ onto $\Gamma(\text{ range } F_*)^\perp$, is given by $\nabla^{F\perp}_X Z$, where $\nabla^{F\perp}$ is a linear connection on $\Gamma(\text{ range } F_*)^\perp$ that satisfies $\nabla^{F\perp} g_N = 0$.

Now, for a Riemannian map, we have the following relation [25] (p. 188)

$$\nabla^N_{F_*X} Z = -S_Z F_* X + \nabla^{F\perp}_X Z, \#(2.9)$$

where $S_Z F_* X$ is the tangential component of $\nabla^N_{F_*X} Z$. Thus at $p \in M$, we have $\nabla^N_{F_*X} Z(p) \in T_{F(p)}P, S_Z F_* X(x) \in F_{*p}(T_p M)$ and $\nabla^{F\perp}_X Z(p) \in \left(F_{*p}(T_p N)\right)^\perp$.

Additionally, $S_Z F_* X$ is bilinear in Z and $F_* X$, and $S_Z F_* X$ at p depends only on Z_p and $F_{*p} X(p)$.

Definition 2.6. [12] Let $F: (M^m, g_M) \to (N^n, g_N)$ be a Riemannian map between two Riemannian manifolds and $s: N \to \mathbb{R}^+$be a positive function on N. Then F is called Clairaut Riemannian map if for every geodesic the function $s \circ \sigma \sin \omega(t)$ is constant, where $F_* X, Z$ are the components of $\dot{\sigma}$ and $\omega(t)$ is the angle between horizontal space and $\dot{\sigma}$.

If the distribition $(\text{ range } F_*)^\perp$ is totally geodesic (for more details see [31]), then we have the following:

Theorem 2.1. [12] Let $F: (M^m, g_M) \to (N^n, g_N)$ be a Riemannian map between Riemannian manifolds and $\gamma, \sigma = F \circ \gamma$, are geodesic curves on M and N, respectively. Then F is Clairaut Riemannian map with $\tilde{s} = e^g$ if and only if one of the following conditions hold.

1. F is umbilical map, and has $H_N = -\nabla^N g$, where g is a smooth function on N and H_N is the mean curvature vector field of range F_*.
2. $S_V F_* X = -V(g) F_* X$, where $F_* X, V$ are the vertical and horizontal components of $\dot{\sigma}$.

Lemma 2.2. Let $F:(M^m,g_M)\to(N^n,g_N)$ be a Clairaut Riemannian map between Riemannian manifolds. Then the tension field $\tau(F)$ of F is

$$\tau(F)=-rF_*(\mu^{kerF_*})-(m-r)\nabla^N g$$

Proof. Using $H_N=-\nabla^N g$ in Lemma 49 of [25], we get tension field for Clairaut Riemannian map.

3. Clairaut semi-invariant Riemannian map to Sasakian manifolds

In this section, first we define semi-invariant Riemannian map from a Riemannian manifold to an almost contact manifold and then, Clairaut semi-invariant Riemannian map from a Riemannian manifold to a Sasakian manifold admitting horizontal Reeb vector field. Since the idea of Clairaut Riemannian maps is based on geodesic therefore we give a neccessary and sufficient condition for a curve on the base manifold to be geodesic. Then, we discuss the case when a semi-invariant Riemannian map becomes Clairaut semi-invariant Riemannian map. Later, we obtain some results on these maps like totally geodesic and geometric inequalities. At last, we construct an example to show the existence of such maps. Throughout this section, it is assumed that the distribution $(\text{ range }_*)^\perp$ is totally geodesic.

Definition 3.1. Let F be a Riemannian map from a Riemannaian manifold (M^m,g_M) to an almost contact manifold $(N^n,g_N,\phi,\xi,\eta,)$. Then, we say that F is a semiinvariant Riemannian map at $p\in M$ if the following conditions are satisfied:

(i) There exists a subbundle of range F_* such that $\phi(D_1)=D_1$.

(ii) There exists a complementary subbundle D_2 to D_1 in range F_* such that $\phi(D_2)\subseteq(\text{ range } F_*)^\perp$.

If F is semi-invariant Riemannian map at every $p\in M$, then we say that F is a semi-invariant Riemannian map.

Then, for $F_*X\in\Gamma$ (range F_*), we can write

$$\phi F_*X=\omega_1F_*X+\omega_2F_*X\#(3.1)$$

where $\omega_1F_*X\in\Gamma(D_1)$ and $\omega_2F_*X\in\Gamma(\phi D_2)$. Also, for $Z\in\Gamma(\text{ range } F_*)^\perp$, we have

$$\phi Z=BZ+CZ,\#(3.2)$$

where $BZ\in\Gamma(D_1)$ and $CZ\in\Gamma(\mu)$.

Definition 3.2. A semi-invariant Riemannian map $F:(M^m,g_M)\to(N^n,\phi,\xi,\eta,g_N)$ from a Riemannian manifold to a Sasakian manifold is called a Clairaut semiinvariant Riemannian map if there is a function $s:N\to\mathbb{R}^+$such that for every geodesic σ on N, the function ($s\circ\sigma$) $\sin\theta(t)$ is constant, where $F_*X\in\Gamma$ (range F_*) and $Z\in\Gamma(\text{ range } F_*)^\perp$ are components of $\dot\sigma(t)$, and $\theta(t)$ is the angle between $\dot\sigma(t)$ and Z for all t.

Lemma 3.1 Let $F:(M^m,g_M)\to(N^n,\phi,\xi,\eta,g_N)$ be a semi-invariant Riemannian map from a Riemannian manifold to a Sasakian manifold and let $\gamma:I\to M$ be a geodesic on M. Then, the curve $\sigma=F\circ\gamma$ is geodesic on N if and only if

$$F_*\left(\nabla_X^{M*}F_*(\omega_1F_*X)\right)-S_{\omega_2F_*X}F_*X+\nabla_Z^N\omega_1F_*X+F_*(\nabla_X^{M*}F_*BZ)-S_{CZ}F_*X$$

and

$$(\nabla F_*)\left(X,{}^*F_*(\omega_1F_*X)\right)+\nabla_X^{F\perp}\omega_2F_*X+\nabla_Z^{F\perp}\omega_2F_*X+(\nabla F_*)(X,{}^*F_*BZ)$$

where $F_*X\in\Gamma\left(range\ F_*\right),Z\in\Gamma(\text{ range } F_*)^\perp$ are components of $\dot\sigma,\nabla^N$ is the Levi-Civita connection on N, and $\nabla^\perp$ is a linear connection on (range $F_*)^\perp$.

Proof. Let $\gamma:I\to M$ be a geodesic on M and $\sigma=F\circ\gamma$ be a geodesic with speed $\sqrt{c}$ on N, where $F_*Y\in\Gamma(\text{range}F_*)$ and $Z\in\Gamma(\text{range}F_*)^\perp$ are components of $\dot\sigma(t)$. Since N is a Sasakian manifold, we have

$$\phi\nabla^N_{\dot\sigma}\dot\sigma = \nabla^N_{\dot\sigma}\phi\dot\sigma - g_N(\dot\sigma,\dot\sigma)\xi + \eta(\dot\sigma)\dot\sigma.$$

Since $\dot\sigma = F_*X + Z$ and $g_N(\dot\sigma,\dot\sigma) = c$, we get

$$\phi\nabla^N_{\dot\sigma}\dot\sigma = \nabla^N_{F_*X+Z}\phi(F_*X + Z) - c\xi - \eta(F_*X + Z)(F_*X + Z).$$

Since $\eta(F_*X) = g_N(F_*X,\xi) = 0$, above equation takes the form

$$\phi\nabla^N_{\dot\sigma}\dot\sigma = \nabla^N_{F_*X}\phi F_*X + \nabla^N_Z\phi F_*X + \nabla^N_{F_*X}\phi Z + \nabla^N_Z\phi Z - c\xi + \eta(Z)Z + \eta(Z)F_*X. \#(3.5)$$

Using (3.1) and (3.2) in above equation, we get

$$\phi\nabla^N_{\dot\sigma}\dot\sigma = \nabla^N_{F_*X}\omega_1F_*X + \nabla^N_{F_*X}\omega_2F_*X + \nabla^N_Z\omega_1F_*X + \nabla^N_Z\omega_2F_*X + \nabla^N_{F_*X}BZ$$

Since $(\text{range }_*)^\perp$ is totally geodesic, $\nabla^N_Z CZ = \nabla^{F\perp}_Z CZ, \nabla^N_Z\omega_2F_*X = \nabla^{F\perp}_Z\omega_2F_*X$. Since ∇^N is Levi-Civita connection on $N, g_N(\nabla^N_Z BZ, U) = 0$, where $Z, U \in \Gamma(\text{range } F_*)^\perp$ which implies that $\nabla^N_Z BZ \in \Gamma$ (range F_*). Now from (2.7), we obtain

$$\nabla^N_{F_*X}BZ = (\nabla F_*)(X, {}^*F_*BZ) + F_*(\nabla^{M*}_X {}^*F_*BZ)\#(3.7)$$

and

$$\nabla^N_{F_*X}\omega_1F_*X = (\nabla F_*)\big(X, {}^*F_*(\omega_1F_*X)\big) + F_*\big(\nabla^{M*}_X {}^*F_*(\omega_1F_*X)\big)\#(3.8)$$

Also, from 2.9), we have

$$\nabla^N_{F_*X}Z = -S_ZF_*X + \nabla^{F\perp}_X Z\#(3.9)$$
$$\nabla^N_{F_*X}\omega_2F_*X = -S_{\omega_2F_*X}F_*X + \nabla^{F\perp}_X\omega_2F_*X\#(3.10)$$

and

$$\nabla^N_{F_*X}CZ = -S_{CZ}F_*X + \nabla^{F\perp}_X CZ\#(3.11)$$

Using (3.7), (3.8), (3.9), (3.10), (3.11) in (3.6), we get

$$\begin{aligned}\phi\nabla^N_{\dot\sigma}\dot\sigma =& (\nabla F_*)\big(X, {}^*F_*(\omega_1F_*X)\big) + F_*\big(\nabla^{M*}_X {}^*F_*(\omega_1F_*X)\big) - S_{\omega_2F_*X}F_*X\\ &+F_*(\nabla^{M*}_X {}^*F_*BZ) - S_{CZ}F_*X + \nabla^{F\perp}_X CZ + \nabla^N_Z BZ + \nabla^{F\perp}_Z CZ\\ &-c\xi + \eta(Z)F_*X + \eta(Z)Z\end{aligned}$$

Comparing horizontal and vertical components of 3.12, we get

$$\mathcal{V}\nabla^N_{\dot\sigma}\dot\sigma = F_*\big(\nabla^{M*}_X {}^*F_*(\omega_1F_*X) - S_{\omega_2F_*X}F_*X + \nabla^N_Z\omega_1F_*X + F_*(\nabla^{M*}_X {}^*F_*BZ)$$

and

$$\mathcal{H}\nabla^N_{\dot\sigma}\dot\sigma = (\nabla F_*)\big(X, {}^*F_*(\omega_1F_*X)\big) + \nabla^{F\perp}_X\omega_2F_*X + \nabla^{F\perp}_Z\omega_2F_*X + \nabla^{F\perp}_X CZ$$

One knows that σ is geodesic on N if and only if $\nabla^N_{\dot\sigma}\dot\sigma = 0$, therefore, we must have $\mathcal{V}\nabla^N_{\dot\sigma}\dot\sigma = 0 = \mathcal{H}\nabla^N_{\dot\sigma}\dot\sigma$. From (3.13) and (3.14), we obtain the required result.

Theorem 3.1. Let $F: (M^m, g_M) \to (N^n, \phi, \xi, \eta, g_N)$ be a semi-invariant Riemannian map from a Riemannian manifold to a Sasakian manifold and $\gamma, \sigma = F\circ\gamma$ are geodesics on M and N, respectively. Then, F is a Clairaut semi-invariant Riemannian map with $s = e^g$ if and only if

$$\begin{aligned}&g_N\big((\nabla F_*)(X, {}^*F_*BZ + \nabla^{F\perp}_X CZ + \nabla^{F\perp}_Z CZ - c\xi + \eta(Z)Z, \omega_2F_*X)\\ &+\eta(Z)F_*Y, \omega_1F_*X) = 0\end{aligned}$$

where g is a smooth function on N and $F_*X \in \Gamma$ (range F_*), $Z \in \Gamma(\text{range } F_*)^\perp$ are components of $\dot\sigma$.

Proof. Let $\gamma: I \to M$ be a geodesic curve on M and $\sigma = F\circ\gamma$ be geodesic with speed $\sqrt{c}$ on N with F_*X in Γ (range F_*) and $Z \in \Gamma(\text{range } F_*)^\perp$ components of $\dot\sigma(t)$ and $\theta(t)$ denote the angle in $\left[0, \frac{\pi}{2}\right]$ between $\dot\sigma$ and Z.

Hence, we obtain

$$g_N(F_*X, F_*X) = c\sin^2\theta(t), \#(3.16)$$

and

$$g_N(Z, Z) = c\cos^2\theta(t)\#(3.17)$$

Differentiating 3.16), we get

$$\frac{d}{dt}g_N(F_*X, F_*X) = 2g_N(\nabla^N_{\dot\sigma}F_*X, F_*X) = 2c\sin\theta(t)\cos\theta(t)\frac{d\theta}{dt}.$$

Hence by using Sasakian structure in above equation, we get

$$g_N(\phi\nabla^N_{\dot\sigma}F_*X, \phi F_*X) = c\sin\theta(t)\cos\theta(t)\frac{d\theta}{dt}\#(3.18)$$

From (2.4), we obtain

$$\nabla^N_{\dot\sigma}\phi F_*X - \phi\nabla^N_{\dot\sigma}F_*X = g_N(\dot\sigma, F_*X)\xi,$$

therefore

$$\phi\nabla^N_{\dot\sigma}F_*X = \nabla^N_{\dot\sigma}\phi F_*X - g_N(\dot\sigma, F_*X)\xi.$$

Taking inner product of above equation with ϕF_*X, we get

$$g_N(\phi\nabla^N_{\dot\sigma}F_*X, \phi F_*X) = g_N(\nabla^N_{\dot\sigma}\phi F_*X, \phi F_*X) - g_N(\dot\sigma, F_*X)g_N(\xi, \phi F_*X).\#(3.19)$$

Using (3.1) and putting $\dot\sigma = F_*X + Z$ in (3.19), we get

$$g_N(\phi\nabla^N_{\dot\sigma}F_*X, \phi F_*X) = g_N\big(\nabla^N_{F_*X+Z}(\omega_1F_*X + \omega_2F_*X), \omega_1F_*X + \omega_2F_*X\big)\#(3.20)$$
$$-g_N(F_*X, F_*X)g_N(\xi, \omega_2F_*X)\#(3.20)$$

After simplification, we obtain

$$\begin{aligned} g_N(\phi\nabla^N_{\dot\sigma}F_*X, \phi F_*X) =& g_N\big(\nabla^N_{F_*X}\omega_1F_*X + \nabla^N_Z\omega_1F_*X + \nabla^N_{F_*X}\omega_2F_*X \\ &+\nabla^N_Z\omega_2F_*X, \omega_1F_*X) + g_N\big(\nabla^N_{F_*X}\omega_1F_*X + \nabla^N_Z\omega_1F_*X \\ &+\nabla^N_Z\omega_2F_*X, \omega_2F_*X) - g_N(F_*X, F_*X)g_N(\xi, \omega_2F_*X) \end{aligned}$$

Using 2.7 and 2.9 in above equation, we get

$$\begin{aligned} g_N(\phi\nabla^N_{\dot\sigma}F_*X, \phi F_*X) =& g_N\big(F_*(\nabla^{M*}_X F_*\omega_1F_*X) + \nabla^N_Z\omega_1F_*X - S_{\omega_2F_*X}F_*X \\ &, \omega_1F_*X) + g_N\big((\nabla F_*)(X, {}^*F_*\omega_1F_*X) + \nabla^{F\perp}_X\omega_2F_*X \end{aligned}$$

With the help of (3.3) and (3.4) in above equation, we obtain

$$\begin{aligned} g_N(\phi\nabla^N_{\dot\sigma}F_*X, \phi F_*X) =& g_N(-F_*(\nabla^{M*}_X F_*BZ) + S_{CZ}F_*X - \nabla^N_Z BZ \\ &-\eta(Z)F_*X, \omega_1F_*X) + g_N(-(\nabla F_*)(X, {}^*F_*BZ) \\ &-g_N(F_*X, F_*X)g_N(\xi, \omega_2F_*X). \end{aligned}$$

Moreover, F is Clairaut semi-invariant map with $s = e^g$ if and only if

$$\frac{d}{dt}(e^{g\circ\sigma}\sin\theta(t)) = 0$$

which gives

$$e^{g\circ\sigma}\frac{d}{dt}(g\circ\sigma)\sin\theta(t) + e^{g\circ\sigma}\cos\theta(t)\frac{d\theta}{dt} = 0$$

Since $e^{g\circ\sigma}$ is positive function,

$$\frac{d}{dt}(g\circ\sigma)\sin\theta(t) + \cos\theta(t)\frac{d\theta}{dt} = 0\#(3.24)$$

Multiplying (3.24) with non-zero $k\sin\theta(t)$, we get

$$\frac{d}{dt}(g\circ\sigma)k\sin^2\theta(t) = -c\sin\theta(t)\cos\theta(t)\frac{d\theta}{dt}\#(3.25)$$

From (3.23) and (3.25), we get

$$g_N(\nabla^N g, Z) g_N(F_* X, F_* X) = g_N(F_*(\nabla_X^{M*} F_* BZ) - S_{CZ} F_* X + \nabla_Z^N BZ$$
$$+\eta(Z) F_* X, \omega_1 F_* X) + g_N\big((\nabla F_*)(X, {}^*F_* BZ)$$
$$+\nabla_X^{F\perp} CZ + \nabla_Z^{F\perp} CZ - c\xi + \eta(Z) Z, \omega_2 F_* X)$$

This completes the proof.

Theorem 3.2. Let $F: (M^m, g_M) \to (N^n, \phi, \xi, \eta, g_N)$ be a Clairaut semi-invaraint Riemannain map with $s = e^g$ from a Riemannain manifold to a Sasakian manifold. Then atleast one of the following statement is true:

1. g is constant on $\phi(D_2)$.
2. $\dim(\ker F_*)^\perp = 1$.

Proof. Since F is Clairaut Riemannian map with $s = e^g$, we have

$$\nabla_X^{N_F} F_* X - F_*(\nabla_X^M Y) = -g_M(X, Y) \nabla^N g \# (3.27)$$

For $X, Y \in \Gamma(ker F_*)^\perp$.

Taking inner product with $\phi F_* Z, F_* Z \in \Gamma(D_2)$, we get

$$g_N\big(\nabla_X^{N_F} F_* Y - F_*(\nabla_X^M Y), \phi F_* Z\big) = -g_M(X, Y) g_N(\nabla^N g, \phi F_* Z) \# (3.28)$$

Here, ∇^{N_F} is Pullback connection of the Levi-Civita connection ∇^N, therefore, ∇^{N_F} is also Levi-Civita connection.

Since $F_*(\nabla_X^M Y) \in \Gamma$ (range F_*) and $\nabla_X^{N_F} F_* Y - F_*(\nabla_X^M Y) \in \Gamma\big(\text{ range } F_*\big)^\perp$,

$$g_N\big(\nabla_{F_* X}^N F_* Y, \phi F_* Z\big) = -g_M(X, Y) g_N(\nabla^N g, \phi F_* Z), \# (3.29)$$

where $\phi F_* Z \in \Gamma\big(\text{ range } F_*\big)^\perp$.

Using metric compatibility, we get

$$-g_N\big(\nabla_{F_* X}^N \phi F_* Z, F_* Y\big) = -g_M(X, Y) g_N(\nabla^N g, \phi F_* Z).$$

Using Sasakian structure, we obtain

$$-g_N\big(\phi \nabla_{F_* X}^N F_* Z, F_* Y\big) = -g_M(X, Y) g_N(\nabla^N g, \phi F_* Z).$$

With the help of (2.3), we get

$$g_N\big(\nabla_{F_* X}^N F_* Z, \phi F_* Y\big) = -g_M(X, Y) g_N(\nabla^N g, \phi F_* Z)$$

From 3.29 and above equation, we get

$$g_M(X, Z) g_N(\nabla^N g, \phi F_* Y) = g_M(X, Y) g_N(\nabla^N g, \phi F_* Z) \# (3.30)$$

Taking $X = Y$ in above equation, we get

$$g_M(X, Z) g_N(\nabla^N g, \phi F_* X) = g_M(X, X) g_N(\nabla^N g, \phi F_* Z). \# (3.31)$$

After interchanging X and Z, we obtain

$$g_M(Z, X) g_N(\nabla^N g, \phi F_* Z) = g_M(Z, Z) g_N(\nabla^N g, \phi F_* X) \# (3.32)$$

From 3.31) and (3.32), we get

$$g_M(\nabla^N g, \phi F_* Z) = \frac{g_M(Z, Z) g_M(X, X)}{g_M(Z, X) g_M(X, Z)} g_N(\nabla^N g, \phi F_* Z) = 0$$

From above equation, we get required result.

Theorem 3.3. Let $F: (M^m, g_M) \to (N^n, g_N, \phi, \xi, \eta)$ be a Clairaut semi-invariant Riemannian map from a Riemannian manifold to a Sasakian manifold. Then F is a totally geodesic map if and only if following conditions are satisfied.

1. $\ker F_*$ is totally geodesic.

2. $(\ker F_*)^\perp$ is totally geodesic.
3. $g_N(X, {}^*F_*\omega_1 F_* Y)(B\nabla f + C\nabla f) + \omega_1\big(F_*(\nabla_X^{M*} F_*\omega_1 F_* Y)\big) - F_*(\nabla_X^M Y)$
$+\omega_1\big(S_{\omega_2 F_* Y} F_* X\big) + \omega_2\big(S_{\omega_2 F_* Y} F_* X\big) + B(\nabla_X^{F\perp}\omega_2 F_* Y) + C(\nabla_X^{F\perp}\omega_2 F_* Y)$
$-g_N(F_* X, F_* Y)\xi(f)\xi + \omega_2\big(F_*(\nabla_X^{M*} F_*\omega_1 F_* Y)\big) = 0$

Proof. Since F is a totally geodesic map, $(\nabla F_*)(X_1, Y_1) = 0\ \forall\ X_1, Y_1 \in \Gamma(TM)$. We discuss totally geodesicness under following cases:

1. If $X_1, Y_1 \in \Gamma(\ker F_*)$, i.e., $X_1 = U_1$ and $Y_1 = V_1$, then $(\nabla F_*)(U_1, V_1) = 0$, this implies that $\ker F_*$ is totally geodesic.
2. If $X_1 \in \Gamma(\ker F_*)$ and $Y_1 \in \Gamma(\ker F_*)^\perp$, i.e., $X_1 = U_1$ and $Y' = Y$, then $(\nabla F_*)(U_1, Y) = 0$, which implies that $(\ker F_*)^\perp$ is totally geodesic.
3. If $X_1, Y_1 \in \Gamma(ker F_*)^\perp$, i.e., $X_1 = X$ and $Y_1 = Y$, then $(\nabla F_*)(X, Y) = 0$, $\nabla_{F_* X}^N F_* Y - F_*(\nabla_X^M Y) = 0$

Using ϕ structure of Sasakian manifold, we get

$$\phi\nabla_{F_* X}^N \phi F_* Y - \eta\big(\nabla_{F_* X}^N F_* Y\big)\xi + F_*(\nabla_X^M Y) = 0.$$

With the help of (3.1), above equation gives

$$\phi\nabla_{F_* X}^N(\omega_1 F_* Y + \omega_2 F_* Y) + g_N\big(\nabla_{F_* X}^M F_* Y, \xi\big)\xi + F_*(\nabla_X^M Y) = 0.$$

Using 2.7 and 2.9 in above equation, we get

$$\begin{aligned}&\phi\big((\nabla F_*)(X, {}^*F_*\omega_1 F_* Y) - F_*(\nabla_X^{M*} F_*\omega_1 F_* Y)\big) + \phi\big(-S_{\omega_2 F_* Y} F_* X\\ &+\nabla_X^{F\perp}\omega_2 F_* Y) + g\big((\nabla F_*)(X, Y), \xi\big)\xi + F_*(\nabla_X^M Y) = 0\end{aligned}$$

Using Clairaut condition in above equation, we get

$$\begin{aligned}&g_N(X, {}^*F_*\omega_1 F_* Y)(B\nabla f + C\nabla f) + \omega_1\big(F_*(\nabla_X^{M*} F_*\omega_1 F_* Y)\big) - F_*(\nabla_X^M Y)\\ &+\omega_1\big(S_{\omega_2 F_* Y} F_* X\big) + \omega_2\big(S_{\omega_2 F_* Y} F_* X\big) - B(\nabla_X^{F\perp}\omega_2 F_* Y) - C(\nabla_X^{F\perp}\omega_2 F_* Y)\\ &-g_N(F_* X, F_* Y)\xi(f)\xi + \omega_2\big(F_*(\nabla_X^{M*} F_*\omega_1 F_* Y)\big) = 0\end{aligned}$$

This completes the proof.

Theorem 3.4. Let $F: (M^m, g_M) \to (N^n, g_N, \phi, \xi, \eta)$ be a Clairaut semi-invariant Riemannian map from a Riemannian manifold to a Sasakian space form $N(c)$. Then F is biharmonic if and only if

$$\begin{aligned}&-rg_N\big(B(\nabla^N g), \omega_1 F_*(\mu^{kerF_*})\big)\nabla^N g + rg_N\left(F_* e_i, \omega_1 F_*\left(\nabla_{e_i}^{M*} F_*\omega_1 F_*(\mu^{kerF_*})\right)\right)\nabla^N g\\ &+rC(\nabla^N g)\left(g_N\left(F_* e_i, F_*\left(\nabla_{e_i}^{M*} F_*\omega_1 F_*(\mu^{kerF_*})\right)\right) + g_N\left(\nabla_{F_* e_i}^N F_* e_i, \omega_1 F_*(\mu^{kerF_*})\right)\right)\\ &-r\omega_2 F_*\left(\nabla_{e_i}^{M*} F_*\omega_1 F_*(\mu^{kerF_*})\right)(g)F_*(e_i) - r\nabla_{e_i}^{F\perp}\omega_2 F_*\left(\nabla_{e_i}^{M*} F_*\omega_1 F_*(\mu^{kerF_*})\right)\\ &-r\omega_2 F_*(\mu^{kerF_*})(g)\big(-g_N(F_* e_i, \omega_1 F_* e_i)\nabla^N g + \nabla_{e_i}^{F\perp}\omega_2 F_* e_i\big) - (m-r)^2\|\nabla^N g\|^2\nabla^N g\\ &-rg_N\left(F_* e_i, B\left(\nabla_{e_i}^{F\perp}\omega_2 F_*(\mu^{kerF_*})\right)\right)\nabla^N g - r\nabla_{e_i}^{F\perp} C\left(\nabla_{e_i}^{F\perp}\omega_2 F_*(\mu^{kerF_*})\right)\\ &-rg_N\big(F_* e_i, F_*(\mu^{kerF_*})\big)\xi(g)\nabla_{e_i}^{F\perp}\xi + (m-r)\nabla_{e_i}^{F\perp}\nabla_{e_i}^{F\perp}\nabla^N g + \frac{c+3}{4}((m-r)^2\nabla^N g)\\ &-r\xi g_N\big(F_* e_i, F_*(\mu^{kerF_*})\big)\left(g_N\big(\nabla_{e_i}^{F\perp}\nabla^N g, \xi\big) + g_N\big(\nabla^N g, \nabla_{e_i}^{F\perp}\xi\big)\right)\\ &-r\xi.\xi(f)\left(g_N\left(F_*\big(\nabla_{e_i}^M e_i\big), F_*(\mu^{kerF_*})\right) + g_N\left(F_* e_i, F_*\big(\nabla_{e_i}^M \mu^{kerF_*}\big)\right)\right)\\ &+\frac{(c-1)}{4}\Big(3rg_N\left(\omega_1 F_* e_j, F_*(\mu^{kerF_*})\right)\omega_2 F_* e_j + rg_N\big(\omega_1 F_* e_j), F_* e_j\big)\omega_2 F_*(\mu^{kerF_*})\\ &-(m-r)^2\eta(\nabla^N g)\xi + 3(m-r)g_N\big(\omega_2 F_* e_j, \nabla^N g\big)\omega_2 F_* e_j\Big)\end{aligned}$$

and

$$rB(\nabla^N g)\left(g_N\left(F_*e_i, F_*\left(\nabla_{e_i}^{M*}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right) + g_N\left(\nabla_{F_*e_i}^N F_*e_i, \omega_1 F_*(\mu^{kerF_*})\right)\right)$$
$$+F_*\left(\nabla_{e_i}^{M*}F_*B(\nabla^N g)\right)\right) - rF_*\left(\nabla_{e_i}^{M*}F_*\omega_1 F_*\left(\nabla_{e_i}^{M*}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right)$$
$$-r\omega_2 F_*\left(\nabla_{e_i}^{M*}F_*\omega_1 F_*(\mu^{kerF_*})\right)(g)F_*(e_i) - rg_N\left(F_*e_i, F_*(\mu^{kerF_*})\right)\xi(g)\xi(g)F_*e_i$$
$$-r\omega_2 F_*(\mu^{kerF_*})(g)\left(F_*\left(\nabla_{e_i}^{M*}F_*\omega_1 F_*e_i\right) + \omega_2 F_*e_i(g)F_*e_i\right)$$
$$-rF_*\left(\nabla_{e_i}^{M*}F_*B\left(\nabla_{e_i}^{F\perp}\omega_2 F_*(\mu^{kerF_*})\right)\right) - rC\left(\nabla_{e_i}^{F\perp}\omega_2 F_*(\mu^{kerF_*})\right)(g)F_*e_i$$
$$+3(m-r)F_*e_i g_N\left(\nabla_{e_i}^{F\perp}\nabla^N g, \nabla^N g\right) + \frac{(c+3)}{4}\{-rg_N\left(F_*(\mu^{kerF_*}), F_*e_j\right)F_*e_j$$
$$+r(m-r)F_*(\mu^{kerF_*})\} + \frac{c-1}{4}\left\{3rg_N\left(\omega_1 F_*e_j, F_*(\mu^{kerF_*})\right)\left(\omega_1 F_*e_j\right)\right.$$
$$+rg_N\left(\omega_1 F_*e_j, F_*e_j\right)\left(\omega_1 F_*(\mu^{kerF_*})\right) + 3(m-r)g_N\left(\omega_2 F_*e_j, \nabla^N g\right)\left(\omega_1 F_*e_j\right)$$

Proof. Let $\{e_1, \dots, e_r\}$ be a orthonormal basis of $\ker F_*$ and $\{e_{r+1}, \dots, e_m\}$ be an orthonormal basis of $(kerF_*)^\perp$ at $p \in M$. Here, we calculate Laplacian of tension field as follows.

$$\Delta\tau(F) = -\sum_{i=r+1}^{m} \nabla_{e_i}^F \nabla_{e_i}^F \tau(F).$$

Using the Lemma 2.2 and the condition of Clairaut Riemannian map in above equation, we get

$$\Delta\tau(F) = -\sum_{i=r+1}^{m} \nabla_{e_i}^F\left\{\nabla_{e_i}^F\left(-rF_*(\mu^{kerF_*}) - (m-r)\nabla^N g\right)\right\}$$

With the help of Sasakian structure and (2.9), we get

$$\Delta\tau(F) = -\sum_{i=r+1}^{m} \nabla_{e_i}^F\left\{r\phi\nabla_{F_*e_i}^N \phi F_*(\mu^{kerF_*}) - rg_N\left(\nabla_{F_*e_i}^N F_*(\mu^{kerF_*}), \xi\right)\xi\right\}\#(3.35)$$
$$-(m-r)\sum_{i=r+1}^{m} \nabla_{F_*e_i}^N\left(S_{\nabla^N g}F_*e_i - \nabla_{e_i}^{F\perp}\nabla^N g\right)\#(3.35)$$

Using (3.1), in above equation, we get

$$\Delta\tau(F) = -\nabla_{e_i}^F\left\{r\phi\left(\nabla_{F_*e_i}^N \omega_1 F_*(\mu^{kerF_*}) + \nabla_{F_*e_i}^N \omega_2 F_*(\mu^{kerF_*})\right)\right\}$$

Making use of 2.7, 2.9 and metric compatibility in above equation, we get

$$\Delta\tau(F) = -\nabla_{e_i}^F\left\{r\phi\left((\nabla F_*)\left(e_i, {}^*F_*\omega_1 F_*(\mu^{kerF_*})\right) + F_*\left(\nabla_{e_i}^{M*}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right)\right\}$$
$$-\nabla_{e_i}^F\left\{r\phi\left(-S_{\omega_2 F_*(\mu^{kerF_*}}F_*e_i + \nabla_{e_i}^{F\perp}\omega_2 F_*(\mu^{kerF_*})\right)\right\}$$
$$+rg_N\left(\nabla_{F_*e_i}^N F_*(\mu^{kerF_*}), \xi\right)\left(-S_\xi F_*e_i + \nabla_{e_i}^{F\perp}\xi\right)$$
$$-r\xi g_N\left(F_*e_i, F_*(\mu^{kerF_*})\right)\left(g_N\left(\nabla_{F_*e_i}^N \nabla^N g, \xi\right) + g_N\left(\nabla^N g, \nabla_{F_*e_i}^N \xi\right)\right)$$
$$-r\xi.\xi(f)\left(g_N\left(F_*\left(\nabla_{e_i}^M e_i\right), F_*(\mu^{kerF_*})\right) + g_N\left(F_*e_i, F_*\left(\nabla_{e_i}^M \mu^{kerF_*}\right)\right)\right)$$

Using (3.2), (3.1) and (2.9) in above equation, we get

$$\Delta\tau(F) = -\nabla^N_{F_*e_i}\{-rg_N(F_*e_i, \omega_1 F_*(\mu^{kerF_*}))(B(\nabla^N g) + C(\nabla^N g))\}$$
$$-\nabla^N_{F_*e_i}\left\{r\omega_1 F_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*(\mu^{kerF_*})\right) + r\omega_2 F_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right\}$$
$$-rg(\omega_2 F_*(\mu^{kerF_*}), \nabla^N g)\left(\nabla^N_{F_*e_i}(\omega_1 F_*e_i + \omega_2 F_*e_i)\right)$$
$$-r(\nabla F_*)\left(e_i,^* F_*B\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)\right) - rF_*\left(\nabla^{M*}_{e_i}F_*B\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)\right)$$
$$+rS_{C\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)}F_*e_i - r\nabla^{F\perp}_{e_i}C\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)$$
$$+rg_N(\nabla^N_{F_*e_i}F_*(\mu^{kerF_*}), \xi)(-S_\xi F_*e_i + \nabla^{F\perp}_{e_i}\xi)$$
$$-r\xi g_N(F_*e_i, F_*(\mu^{kerF_*}))\left(g(\nabla^N_{F_*e_i}\nabla^N g, \xi) + g(\nabla^N g, \nabla^N_{F_*e_i}\xi)\right)$$
$$-r\xi.\xi(f)\left(g_N\left(F_*(\nabla^M_{e_i}e_i), F_*(\mu^{kerF_*})\right) + g_N\left(F_*e_i, F_*(\nabla^M_{e_i}\mu^{kerF_*})\right)\right)$$

Making use of Clairaut condition in above equation, we get

$$\Delta\tau(F) = -rg_N(B(\nabla^N g), \omega_1 F_*(\mu^{kerF_*}))\nabla^N g + r(B(\nabla^N g) + C(\nabla^N g))$$
$$\left(g_N\left(F_*e_i, F_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right) + g_N\left(\nabla^N_{F_*e_i}F_*e_i, \omega_1 F_*(\mu^{kerF_*})\right)\right)$$
$$+rg_N(F_*e_i, \omega_1 F_*(\mu^{kerF_*}))\left(C(\nabla^N g)(g)F_*e_i + \nabla^{F\perp}_{e_i}C(\nabla^N g)\right)$$
$$+F_*\left(\nabla^{M*}_{e_i}F_*B(\nabla^N g)\right)\right) + rg_N\left(F_*e_i, \omega_1 F_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right)\nabla^N g$$
$$-rF_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*(\mu^{kerF_*})\right)\right) - r\omega_2 F_*(\nabla^{M*}_{e_i}F_*\omega_1$$
$$F_*(\mu^{kerF_*}))(g)F_*(e_i) - r\nabla^{F\perp}_{e_i}\omega_2 F_*\left(\nabla^{M*}_{e_i}F_*\omega_1 F_*(\mu^{kerF_*})\right)$$
$$-r\omega_2 F_*(\mu^{kerF_*})(g)(-g_N(F_*e_i, \omega_1 F_*e_i)\nabla^N g$$
$$+F_*(\nabla^{M*}_{e_i}F_*\omega_1 F_*e_i) + \omega_2 F_*e_i(g)F_*e_i + \nabla^{F\perp}_{e_i}\omega_2 F_*e_i)$$
$$-rg_N\left(F_*e_i, B\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)\right)\nabla^N g - rF_*(\nabla^{M*}_{e_i}F_*B$$
$$\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)\right) - rC\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right)(g)F_*e_i$$
$$-r\nabla^{F\perp}_{e_i}C\left(\nabla^{F\perp}_{e_i}\omega_2 F_*(\mu^{kerF_*})\right) - rg_N(F_*e_i, F_*(\mu^{kerF_*}))\xi(g)(\xi(f)F_*e_i$$
$$+\nabla^{F\perp}_{e_i}\xi) - r\xi g_N(F_*e_i, F_*(\mu^{kerF_*}))\left(g_N(\nabla^{F\perp}_{e_i}\nabla^N g, \xi) + g_N(\nabla^N g, \nabla^{F\perp}_{e_i}\xi)\right)$$
$$-r\xi.\xi(f)\left(g_N\left(F_*(\nabla^M_{e_i}e_i), F_*(\mu^{kerF_*})\right) + g_N\left(F_*e_i, F_*(\nabla^M_{e_i}\mu^{kerF_*})\right)\right)$$
$$-(m-r)^2\|\nabla^N g\|^2\nabla^N g + 3(m-r)F_*e_i g_N(\nabla^{F\perp}_{e_i}\nabla^N g, \nabla^N g)$$

From (2.6), we get

$$R^N(F_*e_j, \tau(F))F_*e_j = R^N\left(F_*e_j, -rF_*(\mu^{kerF_*})\right)F_*e_j$$

Using Lemma 2.2 in above equation, we obtain

$$R^N(F_*e_j, \tau(F))F_*e_j = \frac{(c+3)}{4}\{-rg_N(F_*(\mu^{kerF_*}), F_*e_j)F_*e_j + r(m-r)$$
$$F_*(\mu^{kerF_*}) + (m-r)^2\nabla^N g\} + \frac{c-1}{4}\{-rg_N(\phi F_*(\mu^{kerF_*})$$
$$, F_*e_j)\phi F_*e_j + rg_N(\phi F_*e_j, F_*e_j)\phi F_*(\mu^{kerF_*}) + 2rg_N(\phi F_*e_j$$
$$, F_*(\mu^{kerF_*}))\phi F_*e_j - (m-r)^2\eta(\nabla^N g)\xi - (m-r)g_N(\phi\nabla^N g,$$
$$F_*e_j)\phi F_*e_j + (m-r)g_N(\phi F_*e_j, F_*e_j)\nabla^N g + 2(m-r)$$

After simplification, we obtain

$$R^N\big(F_*e_j,\tau(F)\big)F_*e_j = \frac{(c+3)}{4}\{-rg_N\big(F_*(\mu^{kerF_*}),F_*e_j\big)F_*e_j + r(m-r)F_*(\mu^{kerF_*})$$
$$+(m-r)^2\nabla^N g\} + \frac{c-1}{4}\{3rg_N(\omega_1 F_*e_j, F_*(\mu^{kerF_*}))\big(\omega_1 F_*e_j$$
$$+\omega_2 F_*e_j\big) + rg_N\big(\omega_1 F_*e_j, F_*e_j\big)\big(\omega_1 F_*(\mu^{kerF_*}) + \omega_2 F_*(\mu^{kerF_*})\big)$$
$$-(m-r)^2\eta(\nabla^N g)\xi + 3(m-r)g_N\big(\omega_2 F_*e_j, \nabla^N g\big)\big(\omega_1 F_*e_j$$

From Definition 2.5, 3.39 and (3.42), and taking vertical and horizontal components, we obtain the required result.

4. Some basic inequalities

In this section, we obtain some inequalities which involve Ricci and sectional curvature of horizontal and vertical distributions of the Clairaut semi-invariant Riemannian map and discuss the equality case.

Let (M^m, g_M) be a Riemannian manifold and ($N^n(c), g_N, \phi, \xi, \eta$) be a Sasakian space form. Consider $F:(M^m, g_M) \to (N^n, g_N, \phi, \xi, \eta)$ be a Clairaut semi-invariant Riemannian map. For a point $q \in N$, let $F_*X_1, \dots, F_*X_{m-r}, Z_1, \dots, Z_{n-m+r}$ be an orthonormal basis of $T_q(N(k))$ such that range $F_*p = \text{Span}\{F_*X_1, \dots, F_*X_{m-r}\}$ and $\big(\text{ range }_* p\big)^\perp = \text{Span}\{Z_1, \dots, Z_{n-m+r}\}$ for any tangent vector of N.

Riemannian curvature tensor(denoted by R) and sectional curvature(denoted by K) of M is given by

$$R^{(kerF_*)^\perp(X_1,X_2,X_3,X_4)=}\frac{c+3}{4}\{g_N(F_*X_2,F_*X_3)g_N(F_*X_1,F_*X_4)$$
$$-g_N(F_*X_1,F_*X_3)g_N(F_*X_2,F_*X_4)\}$$
$$-g_N(\omega_1 F_*X_1,F_*X_3)g_N(\omega_1 F_*X_2,F_*X_4)$$
$$-2g_N(\omega_1 F_*X_1,F_*X_2)g_N(\omega_1 F_*X_3,F_*X_4)\}$$
$$-g_N(F_*X_1,F_*X_3)g_N(F_*X_2,F_*X_4)|\nabla^N g|^2$$
$$+g_N(F_*X_2,F_*X_3)g_N(F_*X_1,F_*X_4)|\nabla^N g|^2)$$
$$K(X_1,X_2) = \frac{c+3}{4}\{g_N^2(F_*X_1,F_*X_2) - ||F_*X_1||^{\,2}\|F_*X_2\|^2\} - ||F_*X_1||^{\,2}\|F_*X_2\|^2$$
$$|\nabla^N g|^2 + \frac{c-1}{4}\{-g_N^2(F_*X_2,\phi F_*X_1) - g_N(\omega_1 F_*X_1,F_*X_1)$$
$$g_N(\omega_1 F_*X_2,F_*X_2) - 2g_N(\omega_1 F_*X_1,F_*X_2)g_N(\omega_2 F_*X_1,F_*X_2)\}$$

where $X_1, X_2, X_3, X_4 \in \Gamma(kerF_*)^\perp$.

Hence, we obtain the following result.

Theorem 4.1. Let $F:(M^m, g_M) \to (N^n(k), g_N, \phi, \xi, \eta)$ be a Clairaut semi-invariant Riemannian map from a Riemannian manifold to a Sasakian space form. Then, we have

$$\text{Ric}^{(kerF_*)^\perp}(X_1,X_2) \le \frac{(c-1)}{4}\{g_N(\phi F_*X_1,F_*X_2)g_N\big(\phi F_*X_j,F_*X_j\big)\#(4.3)$$
$$-3g_N\big(\phi F_*X_j,F_*X_2\big)g_N\big(\phi F_*X_1,F_*X_j\big)\}\#(4.3)$$

and

$$K(X_1,X_2) \le \frac{c+3}{4}\{g_N^2(F_*X_1,F_*X_2) - \|F_*X_1\|^2\|F_*X_2\|^2\}$$
$$g_N(\omega_1 F_*X_2,F_*X_2) - 2g_N(\omega_1 F_*X_1,F_*X_2)g_N(\omega_2 F_*X_1,F_*X_2)\}$$

For both Ricci curvature and sectional curvature, equality holds if F is a totally geodesic map.

Example 4.1. Let $M = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ be a Riemannian manifold with Riemannian metric

$$g_M = \begin{bmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $N = (y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$ be a Sasakian manifold with Sasakian structure (ϕ, ξ, η, g_N) where η is the 1 -form defined by $\eta(E) = g_N(E, \xi)$ for all $E \in \Gamma(TN)$ and ϕ is the $(1,1)$ tensor field defined by

$$\phi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\xi = \frac{\partial}{\partial y_2}$ and

$$g_N = \begin{bmatrix} \frac{7}{2} & 0 & 0 & 0 & 0 \\ 0 & 8 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $F: (M, g_M) \to (N, \phi, \xi, \eta, g_N)$ be a smooth map defined as

$$F(x_1, x_2, x_3, x_4, x_5) = \left(\frac{x_1 + x_2}{2}, 0, x_5, x_4, 0\right)$$

Clearly, one can check

$$\ker F_* = \text{span}\left\{\frac{e_1 - e_2}{2}, e_3\right\}$$
$$(\ker F_*)^\perp = \text{span}\left\{\frac{e_1 + e_2}{2}, e_4, e_5\right\}$$
$$\text{range} F_* = \text{span}\{e'_1, e'_3, e'_4\}$$

and

$$\left(\text{range}_*\right)^\perp = \text{span}\{e'_2, e'_5\}$$

We can show that

$$g_M(X, X) = g_N(F_*X, F_*X)\ \forall X \in (ker F_*)^\perp.$$

Consider $D_1 = \text{span}\{e'_1, e'_3\}$ and $D_2 = \text{span}\{e'_4\}$ Here, $\phi e'_1 = -e'_3 \in D_1, \phi e'_3 = e'_1 \in D_1$ and $\phi e'_4 = e'_5 \in \left(\text{range}_*\right)^\perp$. Therefore F is semi-invariant Riemannian map. In order to show that the defined map is a Clairaut Riemannian map, we find a smooth function g satisfying equation $(\nabla F_*)(X, X) = -g_N(X, X)\nabla^N g$. Here, it can be seen that $(\nabla F_*)(X, X) = 0$ hence taking g as a constant function, we can say that given map is a Clairaut semi-invariant Riemannian map.

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