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Chen Inequalities for Slant Submanifolds of Conformal Sasakian Space Forms

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Abstract

The present paper is devoted to obtain some basic inequalities for submanifolds of conformal Sasakian space from.

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1. Introduction

B. Y. Chen pioneered the notion of Chen invariants in a seminal study published in 1933 [5] and it is currently one of the most attractive research topics in differential geometry. The need to provide answers to Chern's open question about the existence of minimal immersions into an Euclidean space of arbitrary dimension, was the author's original motivation for introducing new types of Riemannian invariants, also known as δ-invariants or Chen invariants. Due to the lack of control of the extrinsic properties of the submanifolds by the known intrinsic invariants, no solutions to Chern's problem were known before the invention of Chen invariants. Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold M into Euclidean space and established a sharp inequality for a submanifold in a real space form using the scalar and sectional curvature and squared mean curvature. On the other hand, in [8] Chen found inequality between the K-Ricci curvature, the squared mean curvature, and the shape operator for submanifolds in real space forms with arbitrary codimensions. These inequalities are also sharp, and many attractive submanifold classes achieve equality in all of the above inequalities. Many works on Chen invariants and inequalities have appeared in the literature since then like Riemannian space forms [15], complex space forms[7], generalized complex space forms [2, 19], locally conformal Kaehler manifolds [13], locally conformal almost cosymplectic manifolds [3, 14, 22], Sasakian space forms [10, 11, 12, 18, 17, Kenmotsu space forms 4], generalized Sasakian space form [16], quaternionic space forms [20, 21].
The main purpose of this paper is to extend chen inequalities for submanifolds of conformal Sasakian space forms.

2. Preliminaries

The Riemannian invariants of a Riemannian manifold are the intrinsic characteristics of the Riemannian manifold (see [6]). Let M be a Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_x M, x \in M$. Let $\{E_1, \ldots, E_n\}$ be orthonormal basis of $T_x(M)$. The scalar curvature τ at x is defined as:

$$\tau(x) = \sum_{i<j} K\left(E_i \wedge E_j\right) \# (2.1)$$

One denotes by

$$(\inf K)(x)=\inf\left\{K(\pi):\text{ plane sections } \pi\subset T_xM\right\}$$

Where $K(\pi)$ denotes the sectional curvature of M associated with π and

$$\delta_M(x)=\tau(x)-\inf(K(x))$$

τ being the scalar curvature of M associated with π and δ_M is a well defined Riemannian invariant, which was introduced by Chen [5, 6]. Let L be a subspace of T_xM of dimension $r\geq 2$ and $\{E_1,E_2,\ldots,E_r\}$ an orthonormal basis of L. The scalar curvature of L is given by

$$\tau(L)=\sum_{1\leq\alpha\leq\beta\leq r}K\left(E_\alpha\wedge E_\beta\right).\#(2.2)$$

For an integer $k\geq 0$, we denote by $S(z,k)$ the finite set wich consists of k-tuples $(z_1,z_2,\ldots,z_k)$ of integers ≥ 2 satisfying $z_1<z$ and $z_1+\ \ldots+z_k\leq z$. Denote by $S(z)$ the set of k-tuples with $k\geq 0$ for a fixed z. For each k-tuples $(z_1,\ldots,z_k)\in S(z)$, one introduces a Riemannian invariant defined by:

$$\delta(z_1,\ldots,z_k)(x)=\tau(p)-S(z_1,\ldots,z_k)(x)\#(2.3)$$

where

$$S(z_1,\ldots,z_k)(x)=inf\left\{\tau(L_1)+\ldots+\tau(L_k)\right\}\#(2.4)$$

and $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j=z_j$, $j\in\{1,\ldots,k\}$. One puts

$$d(z_1,\ldots,z_k)=\frac{z^2\left(z+k-1-\sum_{j=1}^{k}z_j\right)}{2\left(z+k-\sum_{j=1}^{k}z_j\right)}$$

$$b(z_1,\ldots,z_k)=\frac{1}{2}\left[z(z-1)-\sum_{j=1}^{k}z_j\left(z_j-1\right)\right].$$

We recall the following lemma of Chen (see [5])

Lemma 2.1. Let $b_1,\ldots,b_n,c$ be $(n+1)(n\geq 2)$, real numbers such that

$$\left(\sum_{i=1}^{n}b_i\right)^2=(n-1)\left(\sum_{i=1}^{n}b_i^2+c\right).$$

Then $2b_1b_2\geq c$, with equality holding if and only if $b_1+b_2=b_3=\ldots=\ b_n$.

Let M be an n-dimensional submanifold of a Riemannian manifold $\overline{M}$. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi\subset T_xM, x\in M$, and ∇ the Riemannian connection of M. Also, let σ be the second fundamental form and R the Riemannian curvature tensor of M, then the Gauss equation is given by

$$\overline{R}(X,Y,Z,W)=R(X,Y,Z,W)\quad+g\left(\sigma(X,W),\sigma(Y,Z)\right)-g\left(\sigma(X,Z),\sigma(Y,W)\right),$$

for any vectors X,Y,Z,W tangent to M.

Let $x\in M$ and $\{E_1,\ldots,E_n\}$ an orthonormal basis of the tangent space T_xM. We denote by H the mean curvature vector i.e.,

$$H(x)=\frac{1}{n}\sum_{i=1}^{n}\sigma(E_i,E_i)$$

Also, we set

$$\sigma_{ij}^{r}=g\left(\sigma\left(E_i,E_j\right),E_r\right),$$

and

$$\|\sigma\|^2=\sum_{i,j=1}^{n}g\left(\sigma\left(E_i,E_j\right),\sigma\left(E_i,E_j\right)\right)$$

3. Slant submanifolds in Confomal Sasakian manifolds.

Definition 3.1. A $(2m+1)$-dimensional manifold $\overline{M}^{2m+1}$ is said to be a Sasakian manifold if it admits an endomorphism ϕ of its tangent bundle $T\overline{M}^{2m+1}$, a vector field ξ and a one form η, satisfying:

$$\phi^2=-Id+\eta\otimes\xi, \qquad \eta(\xi)=1, \qquad \eta\circ\phi=0$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \qquad \eta(X)=g(X,\xi)$$

$$\left(\tilde{\nabla}_X\phi\right)Y=-g(X,Y)\xi+\eta(Y)X, \qquad \left(\tilde{\nabla}_X\xi\right)=\phi X,$$

for any vector field X,Y on $\overline{M}^{2m+1}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g.

Definition 3.2. A $(2m+1)$-dimensional Riemannian manifold $\overline{M}$ endowed with the almost contact metric structure $\left(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g}\right)$ is called a conformal Sasakian manifold if for a C^∞ function $f:\overline{M}\to R$ there are

$$\tilde{g}=\left(\exp(f)\right)\tilde{g},\ \phi=\overline{\phi},\ \eta=\exp(f)^{\frac{1}{2}}\eta,\ \xi=\exp(-f)^{\frac{1}{2}}\overline{\xi}$$

such that $\left(\overline{M},\ \phi,\xi,\eta,\tilde{g}\right)$ is a Sasakian manifold (see [1]).

A plane section π in $T_x\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by X and $\overline{\phi}X$, where X is a unit tangent vector orthogonal to $\overline{\xi}$. The sectional curvature of a $\overline{\phi}$-section is called a $\overline{\phi}$-sectional curvature. A conformal Sasakian manifold with constant $\overline{\phi}$-sectional curvature c is said to be a conformal Sasakian space form and is denoted by $\overline{M}$ *(c).*

The curvature tensor of $\overline{M}$ *(c)* of a conformal Sasakian space form $\overline{M}$ *(c)* is given by [1]

$$\begin{aligned}\overline{R}(X,Y,Z,W)&=\frac{c+3}{4}\exp(f)\left(\overline{g}(Y,Z)\overline{g}(X,W)-\overline{g}(X,Z)\overline{g}(Y,W)\right)\\&+\frac{c-1}{4}\exp(f)\{\overline{\eta}(X)\overline{\eta}(Z)\overline{g}(Y,W)-\overline{\eta}(Y)\overline{\eta}(Z)\overline{g}(X,W)\\&+\overline{g}(X,Z)\overline{g}\left(\overline{\xi},W\right)\overline{\eta}(Y)-\overline{g}(Y,Z)\overline{g}\left(\overline{\xi},W\right)\overline{\eta}(X)\}\\&+\frac{c-1}{4}\{\overline{g}\left(\overline{\phi}Y,Z\right)\overline{g}\left(\overline{\phi}X,W\right)-\overline{g}\left(\overline{\phi}X,Z\right)\overline{g}\left(\overline{\phi}Y,W\right)\\&-2\overline{g}\left(\overline{\phi}X,Y\right)\overline{g}\left(\overline{\phi}Z,W\right)\}-\frac{1}{2}\{B(X,Z)\overline{g}(Y,W)-B(Y,Z)\overline{g}(X,W)\\&+B(Y,W))\overline{g}(X,Z)-B(X,W)\overline{g}(Y,Z)\}\end{aligned}$$

for any tangent vector fields X,Y,Z,W on $\overline{M}^{2m+1}(c), \overline{g}\left(\omega^{*},X\right)= \omega(X)$ where $\omega(X)=Xf$ and $B=\left(\overline{\nabla}\omega-\frac{1}{2}\omega\otimes\omega\right)$.

Definition 3.3. A submanifold M tangent to $\overline{\xi}$ is said to be slant if for any $x\in M$ and any $X\in T_xM$, linearly independent of $\overline{\xi}$, the angle between $\overline{\phi}X$ and T_xM is a constant $\theta\in\left(0,\frac{\pi}{2}\right)$ called the slant angle of M in $\overline{M}$. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta=0$ and $\theta=\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

For any tangent vector field X to M, we put $\overline{\phi}X=TX+QX$, where TX and QX are the tangential and normal components of $\overline{\phi}X$ respectively. We denote by

$$\|T\|^2=\sum_{i,j=1}^{n}\overline{g}^2\left(TE_i,E_j\right)\#(3.2)$$

Let $\overline{M}(c)$ be a $(2m+1)$-dimensional conformal Sasakian space form and $M\subset\overline{M}(c)$ an $(n=2k+1)$-dimensional slant submanifold tangent to $\overline{\xi}$. Let $L\subset T_xM$ be a subspace of $T_xM, \dim(L)=r$, we put

$$\Psi(L)=\sum_{1\le i<j\le r}\overline{g}^2\left(Te_i,e_j\right),$$

where $\{e_1,\ldots,e_r\}$ is an orthonormal basis of L.

It is well known the curavture tensor R of a submanifold M of a conformal Sasakian space $\overline{M}$ (c) form satisfies

$$\overline{R}(X,Y,Z,W)=\frac{c+3}{4}\exp(f)\left(\overline{g}(Y,Z)\overline{g}(X,W)-\overline{g}(X,Z)\overline{g}(Y,W)\right)$$
$$+\overline{g}(X,Z)\overline{g}\left(\overline{\xi},W\right)\overline{\eta}(Y)-\overline{g}(Y,Z)\overline{g}\left(\overline{\xi},W\right)\overline{\eta}(X)\}+\frac{c-1}{4}\{\overline{g}\left(\overline{\phi}Y,Z\right)\overline{g}\left(\overline{\phi}X,W\right)-\overline{g}\left(\overline{\phi}X,Z\right)\overline{g}\left(\overline{\phi}Y,W\right)$$
$$+\frac{c-1}{4}\exp(f)\{\overline{\eta}(X)\overline{\eta}(Z)\overline{g}(Y,W)-\overline{\eta}(Y)\overline{\eta}(Z)\overline{g}(X,W)$$
$$-2\overline{g}\left(\overline{\phi}X,Y\right)\overline{g}\left(\overline{\phi}Z,W\right)\}-\frac{1}{2}\{B(X,Z)\overline{g}(Y,W)-B(Y,Z)\overline{g}(X,W)$$
$$+B(Y,W))\overline{g}(X,Z)-B(X,W)\overline{g}(Y,Z)\}-\frac{1}{4}\omega^{*2}\left\{\overline{g}(X,Z)\overline{g}(Y,W)-\overline{g}(Y,Z)\overline{g}(X,W)\right\}$$

From (2.1) and (3.1) we obtain the following relation between scalar and mean curvature of M.

$$2\tau=\frac{c+3}{4}\exp(f)\left[n(n-1)\right]+\frac{c-1}{4}\exp(f)\left[-2n+2+3\|T\|^2\right]$$

4. Chen inequalities on Conformal sasakian space form

Let M^n be submanifold of $\overline{M}^{2m+1}$ (c), tangent to structural vector field $\overline{\xi}$, and $\pi=\operatorname{span}\left(E_1,E_2\right)$ a plane section at $x\in M$, orthogonal to $\overline{\xi}_x$, then

$$\Phi^2(\pi)=\overline{g}^2\left(\overline{\phi}E_1,E_2\right)\#(4.1)$$

is a real number in $[0,1]$ which is independent of the choice of the orthonormal basis $\{E_1, E_2\}$ of π.

Theorem 4.1. Let M be an n-dimensional submanifold in a conformal Sasakian space form $\overline{M}^{2m+1}$ (c), such that $\bar{\xi} \in TM$. Then for any point $x \in M$ and $\pi = \text{span}\{E_1, E_2\}$, we have

$$\delta_M \leq \frac{n-2}{2}\left[\frac{n^2}{n-1}\| \quad H^2 + \frac{c+3}{4}\exp(f)(n+1) + \frac{1}{4}\omega\|^2 (n+1)\right]$$

$$+\frac{1}{2}(n-1)\text{Trace}(B) - \frac{1}{2}\text{Trace}(B)_{|\pi}$$

The equality case of (4.2) holds at a point $x \in M$ if and only if there exists orthonormal basis $\left\{E_1, \ldots, E_n = \bar{\xi}\right\}$ of $T_x M$ and an orthonormal basis $\{E_{n+1}, \ldots, E_{2m+1}\}$ of $T_x^{\perp} M$ such that the shape operator of M in $\overline{M}(c)$ at x takes the form:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdot & . & \cdot & 0 \\ 0 & b & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & & & \mu_{I_{n-2}} & & \end{pmatrix}, a+b=\mu \#(4.3)$$

$$A_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \cdot & \cdot & \cdot & 0 \\ \sigma_{12}^r & \sigma_{11}^r & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & & & 0_{n-2} & & \end{pmatrix}, r = \{n+2, \ldots, 2m+1\}.\#(4.4)$$

Proof. Putting

$$\Delta = 2\tau - \frac{n^2(n-2)}{n-1}\| H \|^2 - \frac{c+3}{4}\exp(f)[n(n-1)]$$

$$-(n-1)\text{Trace}(B) - \frac{1}{4}\omega^{*2}[n(n-1)]$$

From (3.2) and (4.5) we obtain

$$n^2 \| H \|^2 = (n-1)\left(\Delta + \| \sigma \|^2\right).\#(4.6)$$

Let $x \in M, \pi \subset T_x M, \dim\pi = 2, \pi = \text{span}\{E_1, E_2\}$, we put $E_{n+1} = \dfrac{H}{\| H \|}$.

The relation (4.6) becomes

$$\left(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\right)^2 = (n-1)\left(\Delta + \sum_{i,j=1}^{n}\sum_{r=n+1}^{2m+1}\left(\sigma_{ij}^r\right)^2\right)\#(4.7)$$

or, equivalently

$$\left(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\right)^2 = (n-1)\left(\sum_{i=1}^{n}\left(\sigma_{ii}^{n+1}\right)^2 + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}\left(\sigma_{ij}^r\right)^2 + \Delta\right).\#(4.8)$$

Using lemma (2.1), we have from previous relation

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geq \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}\left(\sigma_{ij}^r\right)^2 + \Delta.\#(4.9)$$

For $X = W = E_1$ and $Y = Z = E_2$, from (3.1), , we obtain

$$K(\pi)=\frac{c+3}{4}\exp(f)+\frac{c-1}{4}\exp(f)\left[3\overline{g}^{2}\left(\overline{\phi}E_1,E_2\right)\right]+\frac{1}{2}\text{Trace}(B)_{|\pi}+\frac{1}{4}\omega^{*2}+\sum_{r=n+1}^{2m+1}\left[\sigma_{11}^{r}\sigma_{22}^{r}-\left(\sigma_{12}^{r}\right)^{2}\right],$$

$$\geq\frac{c+3}{4}\exp(f)\left[3\overline{g}^{2}\left(\overline{\phi}E_1,E_2\right)\right]+\frac{1}{2}\text{Trace}(B)_{|\pi}+\frac{1}{4}\omega^{*2}+\frac{1}{2}\sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^{2}$$

$$+\frac{1}{2}\sum_{i,j=1}^{m}\sum_{r=n+2}^{2m+1}\left(\sigma_{ij}^{r}\right)^{2}+\sum_{r=n+2}^{2m+1}\left[\sigma_{11}^{r}\sigma_{22}^{r}-\sum_{r=n+1}^{2m+1}\left(\sigma_{12}^{r}\right)^{2}\right]+\frac{\Delta}{2}.$$

which implies that

$$K(\pi)\geq\frac{c+3}{4}\exp(f)+\frac{c-1}{4}\exp(f)\left[3\overline{g}^{2}\left(\overline{\phi}E_1,E_2\right)\right]\quad+\frac{1}{2}\text{Trace}(B)_{|\pi}$$

Using (4.5) in 4.10 we have

$$\inf\Bigg(K(\pi)\geq\tau-\frac{c+3}{8}\exp(f)\left[n(n-1)\right]-\frac{c-1}{8}\exp(f)\left[-2n+2+3\|T\|^{2}\right]$$

$$-\frac{(n-1)}{2}\text{Trace}(B)-\frac{1}{8}\omega^{*2}\left[n(n-1)\right]+\frac{c+3}{4}\exp(f)$$

$$+\frac{c-1}{4}\exp(f)\left[3\overline{g}^{2}\left(\overline{\phi}E_1,E_2\right)\right]+\frac{1}{2}\text{Trace}(B)_{|\pi}$$

$$+\frac{1}{4}\omega^{*2}-\frac{n^{2}(n-2)}{2(n-1)}\|H\|^{2}$$

On solving the above equation and using (4.1), we obtained the inequality (4.2). The case of equality at a point $x\in M$ holds if and only if it achieves the equality in the previous inequality and we have the equality in the Lemma 2.1

$$\begin{cases}\sigma_{ij}^{n+1}=0,\forall i\neq j \text{ and } i,j>2\\ \sigma_{ij}^{r}=0, i\neq j, i,j>2, r=n+1,\dots,2m+1\\ \sigma_{11}^{r}+\sigma_{22}^{r}=0,\forall r=n+2,\dots,2m+1,\\ \sigma_{1j}^{n+1}=\sigma_{2j}^{n+1}=0,\forall j>2\\ \sigma_{11}^{n+1}+\sigma_{22}^{n+1}=\sigma_{33}^{n+1}=\dots=\sigma_{nn}^{n+1}\end{cases}$$

We may chose $\{E_1,E_2\}$ such that $\sigma_{12}^{n+1}=0$ and we denote by $a=\sigma_{11}^{r}$, $b=\sigma_{22}^{r},\mu=\sigma_{33}^{n+1}=\dots=\sigma_{nn}^{n+1}$.

So with respect to the choosen orthonormal basis, the shape operator of M takes the desired forms (4.12) and (4.13).

Theorem 4.2. Let M be an $(n=2k+1)$ dimensional slant submanifold of a conformal Sasakian space form $\overline{M}^{2m+1}$ (c) then:

$$\delta_M\leq\frac{n-2}{2}\left[\frac{n^{2}}{n-1}\|H\|^{2}+\frac{c+3}{4}\exp(f)(n+1)+\frac{1}{4}\|\omega\|^{2}(n+1)\right]$$

$$+\frac{c-1}{4}\exp(f)\left[\frac{3}{2}(n-3)\cos^{2}(\theta)-(n-1)\right]\#(4.11)$$

$$+\frac{1}{2}(n-1)\text{Trace}(B)-\frac{1}{2}\text{Trace}(B)_{|\pi}$$

The equality case of (4.11) holds at a point $x\in M$ if and only if there exists orthonormal basis $\{E_1,\dots,E_n=\overline{\xi}\}$ of T_xM and an orthonormal basis $\{E_{n+1},\dots,E_{2m+1}\}$ of $T_x^{\perp}M$ such that the shape operator of M in $\overline{M}(c)$ at x takes the form:

$$A_{n+1}=\begin{pmatrix} a & 0 & 0 & \cdot & . & . & 0 \\ 0 & b & 0 & \cdot & \cdot & . & 0 \\ 0 & 0 & & & \mu_{I_{n-2}} & & \end{pmatrix}, a+b=\mu \#(4.12)$$

$$A_r=\begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & . & . & . & 0 \\ \sigma_{12}^r & \sigma_{11}^r & 0 & \cdot & . & . & 0 \\ 0 & 0 & & & 0_{n-2} & & \end{pmatrix}, r=\{n+2,\dots,2m+1\}.\#(4.13)$$

Proof. : Let $x \in M$ and $\{E_1,\dots,E_n=\bar{\xi}\}$ be orthonormal basis of $T_x M$, with

$$E_1, E_2=\frac{1}{\cos(\theta)}TE_1,\dots,E_{2K}=\frac{1}{\cos(\theta)}TE_{2k-1}, E_{2k+1}=\bar{\xi}$$

$$\bar{g}\left(\bar{\phi}E_1,E_2\right)=\bar{g}\left(\bar{\phi}E_1,\frac{1}{\cos(\theta)}TE_1\right)=\frac{1}{\cos(\theta)}\bar{g}\left(\bar{\phi}E_1,TE_1\right)$$

and

$$\bar{g}^2\left(\bar{\phi}E_i,E_{i+1}\right)=\cos^2(\theta),$$

then

$$\sum_{i,j=1}^{n}\bar{g}^2\left(\bar{\phi}E_i,E_j\right)=(n-1)\cos^2(\theta)$$

From (3.2) and previous equation we obtained

$$2\tau=\frac{c+3}{4}\exp(f)\left[n(n-1)\right]+\frac{c-1}{4}\exp(f)\left[3(n-1)\cos^2(\theta)-2n+2\right]$$

Putting

$$\Delta=2\tau-\frac{n^2(n-2)}{n-1}\|H\|^2-\frac{c+3}{4}\exp(f)\left[n(n-1)\right]-\frac{1}{4}\omega^{*2}\left[n(n-1)\right]$$

and following the same steps as above we get the desired Theorem.

Corollary 4.3. Let M be an ($n=2k+1$)-dimensional invariant submanifold of conformal Sasakian space form $\bar{M}(c)$ then we have:

$$\delta_M\le\frac{(n-2)(n+1)}{2}\left(\frac{(c+3)}{4}\exp(f)+\frac{1}{4}\omega^{*2}\right)+\frac{(c-1)(n-7)}{8}\exp(f)$$
$$+\frac{1}{2}\left((n-1)\operatorname{Trace}(B)-\operatorname{Trace}(B)_{|\pi}\right)$$

Corollary 4.4. Let M be an ($n=2k+1$)-dimensional anti-invariant submanifold of conformal Sasakian space form $\bar{M}(c)$ then we have:

$$\delta_M\le\frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2+\frac{c+3}{4}\exp(f)(n+1)+\frac{1}{4}\|\omega\|^2(n+1)\right]$$
$$+\frac{1}{2}\left[(n-1)\operatorname{Trace}(B)-\operatorname{Trace}(B)_{|\pi}\right]-\frac{(c-1)(n-1)}{4}\exp(f).$$

In order to prove a new theorem, we will use the following lemma, which is the contact version of Lemma from [9].

Lemma 4.5. Let M be an $(n=2k+1)$-dimensional slant submanifold in a $(2m+1)$-dimensional conformal Sasakian space form $\overline{M}(c)$. Let $n_1,\ldots,n_k$ be integers ≥ 2 satisfying $n_1<n$ and $n_1+\ldots+n_k\leq n$. For $x\in M$, let $L_j\subset T_xM$ be a subspace of $T_xM, \dim L_j=n_j, \forall j\in \{1,\ldots,k\}$. Then we have

$$\tau-\sum_{j=1}^{k}\tau(L_j)\leq d(n_1,\ldots, \quad n_k)\|H\|^2+\frac{b(n_1,\ldots,n_k)}{4}\left[(c+3)\exp(f)+\omega^{*2}\right]$$

$$+\frac{1}{2}\left[(n-1)\mathrm{Trace}(B)-\sum_{j=1}^{k}(n_j-1)\mathrm{Trace}(B)\right]$$

Proof. Let $x\in M$ and $\left\{E_1,\ldots,E_n=\overline{\xi}\right\}$ an orthonormal basis of T_xM. Let $L_1,\ldots,L_k$ be k mutually orthogonal subspaces of $T_xM,\dim(L_j)= \ n_j$, defined by

$$L_k=sp\left\{E_{n_1+\ldots+n_{k-1}+1},\ldots,E_{n_1+\ldots+n_{k-1}+n_k}\right\},\ j=1,\ldots,k.$$

Let $\{E_{n+1}\}=\frac{H}{\|H\|}, E_{n+1}\in T_x^{\perp}M$. We set

$$a_i=\sigma_{ii}^{n+1}=g\left(\sigma(E_i,E_i),E_{n+1}\right)$$
$$b_1=a_1$$
$$b_2=a_2+\ldots+a_{n_1},$$
$$b_3=a_{n_1+1}+\ldots+a_{n_1+n_2},$$
$$b_{k+1}=a_{n_1+\ldots+n_{k-1}+1}+\ldots+a_{n_1+n_2+\ldots+n_{k-1}+n_k},$$
$$b_{k+2}=a_{n_1+\ldots+n_{k+1}+1},$$
$$b_{\Lambda+1}=a_n.$$
$$\Gamma_1=\{1,\ldots,n_1\},$$
$$\Gamma_2=\{n_1+1,\ldots,n_1+n_2\},$$
$$\Gamma_k=\{n_1+\ldots+n_{k-1}+1,\ldots,n_1+\ldots+n_{k-1}+n_k\}.$$

From (3.3), we have

$$\sum_{j=1}^{k}\tau(L_j)=\sum_{j=1}^{k}[\frac{c+3}{8}$$

$$\exp(f)n_j(n_j-1)+\frac{c-1}{4}\exp(f)3\Psi(L_j)\Big]$$

$$+\sum_{j=1}^{k}\sum_{r=n+1}^{2m+1}\sum_{\alpha_j<\beta_j}\left[\sigma_{\alpha_j\alpha_j}^{r}\sigma_{\beta_j\beta_j}^{r}-\left(\sigma_{\alpha_j\beta_j}^{r}\right)^2\right]$$

then

$$2\tau= \quad \frac{c+3}{4}\exp(f)\left[n(n-1)\right]+\frac{c-1}{4}\exp(f)\left[-2n+2+3\|T\|^2\right]$$

$$+(n-1)\mathrm{Trace}(B)+\frac{1}{4}\omega^{*2}\left[n(n-1)\right]+n^2\|H\|^2-\|\sigma\|^2.$$

we can rewrite the foregoing equation as

$$n^2 \| H \|^2 = \left(\varepsilon + \| \sigma \|^2\right)\Lambda,$$

where

$$\varepsilon = 2\tau - 2d(n_1,\ldots,n_k)\| H \|^2 - \frac{c+3}{4}\exp(f)\left[n(n-1)\right]$$
$$-\frac{c-1}{4}\exp(f)\left[-2n+2+3\| T \|^2\right]-(n-1)\mathrm{Trace}(B) \#(4.19)$$
$$-\frac{1}{4}\omega^{*2}\left[n(n-1)\right]-n^2\| H \|^2+\| \sigma \|^2,$$

and

$$\Lambda = n+k-\sum_{j=1}^{k} n_j .$$

From (4) we have

$$\left(\sum_{i=1}^{\Lambda+1} b_i\right)^2 = \Lambda\left[\varepsilon + \sum_{i=1}^{\Lambda+1} b_i^2 + \sum_{i\neq j}\left(\sigma_{ij}^{n+1}\right)^2 + \sum_{i=1}^{n}\left(\sigma_{ii}^{n+1}\right)^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2\right] - 2\Lambda\left[\sum_{2\le\alpha_1<\beta_1\le n_1} a_{\alpha_1}a_{\beta_1} + \ldots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\right],$$

with $\alpha_j, \beta_j \in \Gamma_j, \forall j \in \{1,\ldots,k\}$. Applying lemma (2.1) we obtained

$$\left[\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \ldots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\right] \ge \frac{1}{2}\left[\varepsilon + \sum_{i\neq j}\left(\sigma_{ii}^{n+1}\right) + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2\right]$$

It follows that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{2m+1}\sum_{\alpha_j<\beta_j}\left[\sigma_{\alpha_j\alpha_j}^{r}\sigma_{\beta_j\beta_j}^{r} - \left(\sigma_{\alpha_j\beta_j}^{r}\right)^2\right] \ge \frac{\varepsilon}{2} + \frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{(\alpha,\beta)\notin\Gamma^2}\left(\sigma_{\alpha\beta}^{r}\right)^2 + \sum_{r=n+2}^{2m+1}\sum_{\alpha_j\in\Gamma_j}\left(\sigma_{\alpha_j\alpha_j}^{r}\right)^2 \ge \frac{\varepsilon}{2} \#(4.20)$$

From the relation (4.18), (4.19) and (4.20), we get the desired inequality

Theorem 4.6. Let M be an $(n=2k+1)$ dimensional slant submanifold in a $(2m+1)$-dimensional conformal Sasakian space form $\overline{M}(c)$, then we have

$$\delta(n_1,\ldots,n_k) \le d(n_1,\ldots,n_k)\| H \|^2 + \frac{b(n_1,\ldots,n_k)}{4}\left[(c+3)\exp(f)+\omega^{*2}\right]$$
$$+\frac{c-1}{8}\exp(f)\left[3(n-1)\cos^2(\theta) - 6\sum_{j=1}^{k} m_j\cos^2(\theta) - 2n+2\right]$$
$$+\frac{1}{2}\left[(n-1)\mathrm{Trace}(B) + \sum_{j=1}^{k}(n_j-1)\mathrm{Trace}(B)\right] \#(4.21)$$

where $n_j = 2m_j + \vartheta_j, \vartheta_j \in \{0,1\}, \forall j \in \{1,\ldots,k\}$. If P is a tangent vector field on M , the equality case holds at $x \in M$ if and only if there exists an orthonormal basis $\{E_1,\ldots,E_n\}$ of T_xM and an orthonormal basis $\{E_{m+1},\ldots,E_{2m+1}\}$ of $T_x^{\perp}M$ such that the shape operators of M in $\overline{M}$ at x have the following forms:

$$A_{e_{m+1}}=\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix},$$

$$A_{e_r}=\begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & rI \end{pmatrix},\ r=n+2,\ldots,2m+1,$$

where $a_1,\ldots,a_n$ satisfy $a_1+\cdots+a_{n_1}=\cdots=a_{n_1+\cdots+n_k}=a_n$, and each A_j^r is a symmetric $n_j\times n_j$ submatrix satisfying $\operatorname{tr}\left(A_1^r\right)=\cdots=\operatorname{tr}\left(A_k^r\right)=r$, and I is an identity matrix.

Proof. Let $x\in M$ and $\left\{E_1,\ldots,E_n=\overline{\xi}\right\}$ an orthonormal basis of T_xM, Let $L_1,\ldots,L_k$ be k mutually orthogonal subspaces of $T_xM,\dim\left(L_j\right)=n_j$, defined by

$$L_k=\operatorname{sp}\left\{E_{n_1+\ldots+n_{k-1}+1},\ldots,E_{n_1+\ldots+n_{k-1}+n_k}\right\},\ j=1,2,\ldots,k,$$

it follows that $\Psi\left(L_j\right)=m_j\cos^2(\theta)$, where $n_j=2m_j+\vartheta_j,\vartheta_j\in\{0,1\}\ \forall\ \ j\in\{1,\ldots,k\}$. Also from (4.14) It follows that $\|T\|^2=(n-1)\cos^2(\theta)$. Thus from (4.17), we get the desired Theorem.

Corollary 4.7. Let M be an $(n=2k+1)$-dimensional invariant submanifold of conformal Sasakian space form $\overline{M}(c)$, then we have

$$\delta(n_1,\ldots,n_k)\leq\frac{b(n_1,\ldots,n_k)}{4}\left[(c+3)\exp(f)+\omega^{*2}\right]$$
$$+\frac{c-1}{8}\exp(f)\left[(n-1)-6\sum_{j=1}^{k}m_j\right]+\frac{1}{2}\left[(n-1)\operatorname{Trace}(B)-\sum_{j=1}^{k}\left(n_j-1\right)\operatorname{Trace}(B)\right]$$

where $n_j=2m_j+\vartheta_j,\vartheta_j\in\{0,1\}\ \forall j\in\{1,\ldots,k\}$.

Corollary 4.8. Let M be an $(n=2k+1)$-dimensional anti-invariant submanifold of conformal Sasakian space form $\overline{M}(c)$, then we have

$$\delta(n_1,\ldots,n_k)\leq d(n_1,\ldots,n_k)\|H\|^2+\frac{b(n_1,\ldots,n_k)}{4}\left[(c+3)\exp(f)+\omega^{*2}\right]$$
$$+\frac{1}{2}\left[(n-1)\operatorname{Trace}(B)-\sum_{j=1}^{k}\left(n_j-1\right)\operatorname{Trace}(B)\right]-\frac{(c-1)(n-1)}{4}\exp(f)$$

where $n_j=2m_j+\vartheta_j,\vartheta_j\in\{0,1\}\ \forall j\in\{1,\ldots,k\}$.

5. Data Availability Statement

The authors declare that this research is purely theoretical and does not associate with any datas.

6. Conflict of interests

The authors declare that they have no conflict of interest, regarding the publication of this paper.

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