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RESEARCH ARTICLE

Proposed theorems on an almost complex golden structure and its frame bundle

Anowar Hussain Sadiyal
Department of Computer Engineering, College of Computer, Qassim University, Buraydah, Saudi Arabia
Email: sadieal@qu.edu.sa

Abstract
The goal of this research is to ascertain the connection between CRstructure and an almost complex golden structure and to identify some fundamental findings. A few theorems on CR-structure and an almost complex golden structure are proved, and integrability criteria are discussed.

Keywords: CR-structures, Almost complex structure, Nijenhuis tensor, Integrability.
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1. Introduction

In differential geometry, the theory of the tangent bundle of submanifolds is an intriguing subject. Regarding the almost complex structure of the ambient manifold, there are three different kinds of submanifolds: holomorphic submanifolds, fully real submanifolds, and CR-(Cauchy-Riemannian) submanifolds. A novel class of submanifolds of the complex manifold was begun by Bejancu [1], who researched the CR-submanifold of a Kahlerian manifold. Bejancu presented the idea of the CR-submanifold and described its fundamental characteristics. Many researchers, such as Bejancu [2], Blair and Chen [3], Chen [4], Dragomir at el [5], and Yao and Kon [8], made significant contributions to CR-submanifolds. For the recent studies on tangent bundle and geometric structures, we refer to ([10]-[17]) and many more. In this paper, we study the integrability conditions and Nijenhuis tensor on CR-structures and an almost complex golden structure.

The equation $x^2 - x + \frac{3}{2}I = 0$ will be examined. The solutions of the equation are represented by the equation $x = \frac{1}{2}\left(1 \pm \sqrt{5}i\right)$. Let F be a nonzero tensor field on a n-dimensional manifold M of type $(1,1)$ and class C^∞ such that

$$F^2 - F + \frac{3}{2}I = 0\#(1.1) \tag{1.1}$$

such structure on M is called an almost complex golden structure of rank r. If the rank of F is constant and $r = r(F)$, then M is called an almost complex
golden manifold. Let us introduce the operators as follows

$$l = -\frac{2\left(F^2 - F\right)}{3}, m = I + \frac{2\left(F^2 - F\right)}{3} \tag{1.2}$$

where I denotes the identity operator on M.

Theorem 1.1 Let M be an almost complex golden manifold. Then

$$l + m = I, l^2 = l, \text{ and } m^2 = m \tag{1.3}$$

Proof: In the view of equation (1.1), the proof is trivial.

For $F \neq 0$ satisfying equation (1.1), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively. If the $\operatorname{rank}(F)=$ constant and $r=r(F)$ on M, then $\dim D_l = r$ and $\dim D_m = (n-r)[17]$.

Theorem 1.2 Let M be an almost complex golden manifold. Then

$$Fl = lF = F, Fm = mF = 0 \tag{1.4}$$

$$\frac{2(F^2-F)}{3} = -l, \frac{2(F^2-F)}{3} l = -l, \frac{2(F^2-F)}{3} m = 0. \tag{1.5}$$

Thus $\left(\frac{2(F^2-F)}{3}\right)^{\frac{1}{2}}$ acts on D_l as an almost complex structure and on D_m as a null operator.

Proof: In the view of equation (1.1), the proof is trivial.

2. Nijenhuis tensor

The Nijenhuis tensor $N(X,Y)$ of F satisfying (1.1) in M is expressed as follows for every vector field X,Y on M.

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y] \tag{2.1}$$

We state the following proposition [6].

Proposition 2.1 A necessary and sufficient condition for an almost complex golden structure F to be integrable is that $N(X,Y)=0$ for any two vector fields X and Y on M.

Definition 2.1 If X,Y are two vector fields in M, then their Lie bracket $[X,Y]$ is defined by

$$[X,Y] = XY - YX \tag{2.2}$$

3. CR-structure

Let M be a differentiable manifold and T_cM be its complexified tangent bundle. A CR-structure on M is a complex subbundle H of T_cM such that $H_p \cap H_p = 0$ and H is involutive, i.e., for complex vector fields X and Y in $H, [X,Y]$ is in H. In this case, we say M is a CR-manifold. Let F be an almost complex golden integrable structure satisfying equation (1.1) of rank $r=2m$ on M. We define complex subbundle H of T_cM by $H_p = \{X - \sqrt{-1}FX, X \in X(D_l)\}$, where $\chi(D_l)$ is the $\eta(D_m)$ module of all differentiable sections of D_l. Then $\operatorname{Re}(H) = D_l$ and $H \cap Hp = 0$, where H_p denotes the complex conjugate of $H[7]$.

Theorem 3.1 If P and Q are two elements of H, then the following relations hold

$$[P,Q] = [X,Y] - [FX,FY] - \sqrt{-1}([X,FY] + FX,Y]). \tag{3.1}$$

Proof: Let us define $P = X - \sqrt{-1}FX$ and $Q = Y - \sqrt{-1}FY$. Then by direct calculations and simplification, we obtain

$$\begin{aligned}[P,Q] &= \left[X - \sqrt{-1}FX, Y - \sqrt{-1}FY\right] \\ &= [X,Y] - [FX,FY] - \sqrt{-1}([X,FY] + FX,Y]).\end{aligned}$$

Theorem 3.2 If an almost complex golden structure satisfying (1.1) is integrable, then we have

$$(F-I)[FX,FY] + F^2[X,Y] = -\frac{3}{2}l([FX,Y] + [X,FY]) \tag{3.2}$$

Proof: From equation (2.1) we have

$$N(X,Y)=[FX,FY]-F[FX,Y]-F[X,FY]+F^2[X,Y].$$

Since $N(X,Y)=0$, we obtain

$$[FX,FY]+F^2[X,Y]=F([FX,Y]+[X,FY]) \tag{3.3}$$

operating (3.3) by $-\frac{2(F-I)}{3}$, we get

$$-\frac{2(F-I)}{3}[F \mid X,FY]+F^2[X,Y]=-\frac{2(F^2-F)}{3}([FX,Y]+[X,FY] \tag{3.4}$$

$$(F-I)[FX,FY]+F^2[X,Y]=-\frac{3}{2}l([FX,Y]+[X,FY]) \tag{3.5}$$

on making use of equation (1.2) we obtain (4.7)), which proves Theorem (3.2).

Theorem 3.3 The following identities hold

$$mN(X,Y)=m[FX,FY] \tag{3.6}$$

$$mN\left(\frac{2(F-I)}{3}X,Y\right)=m\left[\frac{2(F^2-F)}{3}X,FY\right] \tag{3.7}$$

Proof: The proof of (3.6) and (3.7) follows by virtue of Theorems (1.1), (1.2) and equations (1.2) and (2.1).

Theorem 3.4 For any two vector fields X and Y the following conditions are equivalent.

i. $mN(X,Y) \quad = 0$

ii. $m[FX,FY] \quad = 0$

iii. $mN\left(\frac{2(F-I)}{3}X,Y\right) \quad = 0$

iv. $m\left[\frac{2(F^2-F)}{3}X,FY\right] \quad = 0$

v. $m\left[\frac{2(F^2-F)}{3}lX,FY\right] \quad = 0$

Proof: Using equations (1.1), (1.2), (2.1) and Theorems (1.2) and (3.3). The above conditions are equivalent.

Theorem 3.5 If $\left(\frac{2(F^2-F)}{3}\right)^{\frac{1}{2}}$ acts on l as an almost complex structure, then

$$m\left[\frac{2(F^2-F)}{3}lX,FY\right]=m[-X,FY]=0 \tag{3.8}$$

Proof: In view of equation (1.4), we see that $\left(\frac{2\left(F^2-F\right)}{3}\right)^{\frac{1}{2}}$ acts on l as an almost complex structure then equation (3.8) follows in an obvious manner. To show that $m\left[\frac{2\left(F^2-F\right)}{3}lX, FY\right]$ we use Definition (2.1), i.e., $\left[X,Y\right]=XY-YX$ where X,Y are C^{∞} vector fields and in view of equation (1.4), the result follows directly.

Theorem 3.6 For $X,Y\in\chi\left(D_l\right)$, we have

$$l\left(\left[X,FY\right]+\left[FX,Y\right]\right)=\left[X,FY\right]+\left[FX,Y\right]$$

Proof: Since $\left[X,FY\right]$ and $\left[FX,Y\right]\in\chi-\left(D_l\right)$. On making use of (1.4) and Definition (2.1), we obtain the result.

Theorem 3.7 The integrable an almost complex golden structure satisfying (1.1) on M defines a CR-structure H on it such that $ReH\equiv D_l$.

Proof: After applying equations (3.1), (4.7), and Theorem (3.6), we obtain $\left[P,Q\right]\in\chi\left(D_l\right)$, given that $\left[X,FY\right]$ and $\left[FX,Y\right]\in\chi\left(D_l\right)$. An almost complex golden structure satisfying (1.1) on M defines a CR-structure H.

Definition 3.1 Let $\tilde{K}$ be the complementary distribution of $\operatorname{Re}\left(H\right)$ to TM. We define a morphism of vector bundles $F:TM\rightarrow TM$ given by $F\left(X\right)=0$ for all $X\in\chi\left(\tilde{K}\right)$, such that

$$F\left(X\right)=\frac{1}{2}\sqrt{-1}\left(P-\overline{P}\right) \tag{3.9}$$

where $X+\sqrt{-1}Y\in\chi\left(H_p\right)$ and $\overline{P}$ is a complex conjugate of P.

Corollary 3.1 If $P=X+\sqrt{-1}Y$ and $\overline{P}=X-\sqrt{-1}Y$ belong to H_p and $F\left(X\right)=\frac{1}{2}\sqrt{-1}\left(P-\overline{P}\right), F\left(Y\right)=\frac{1}{2}\sqrt{-1}\left(P+\overline{P}\right)$ and $F\left(-Y\right)=\frac{1}{2}\sqrt{-1}\left(P+\overline{P}\right)$, then $F\left(X\right)=-Y, F^2\left(X\right)=-X$ and $F\left(-Y\right)=-X$.

Proof: On using Definition (3.1), we have

$$\begin{aligned} F\left(X\right) &= \frac{1}{2}\sqrt{-1}\left(X+\sqrt{-1}Y-X-\sqrt{-1}Y\right) \\ &= \frac{1}{2}\sqrt{-1}\left(2\sqrt{-1}Y\right)=-1 \end{aligned}$$

Thus, $F\left(X\right)=-Y$, which on operating by F yields

$$F\left(F\left(X\right)\right)=F\left(-Y\right)$$

But

$$F\left(Y\right)=\frac{1}{2}\left(X+\sqrt{-1}Y+X-\sqrt{-1}Y\right) \tag{3.10}$$

which on simplifying gives

$$F\left(Y\right)=X$$

Also,

$$F(-Y) = -\frac{1}{2}\left(X + \sqrt{-1}Y + X - \sqrt{-1}Y\right)$$
$$= -X \tag{3.11}$$

Combining equations (3.10) and (3.12), we get

$$F^2(X) = -X$$

Theorem 3.8 If M has a CR-structure H, then we have $F^2 - F + \frac{3}{2}I = 0$ and consequently an almost complex golden structure is defined on M such that the distributions D_l and D_m coincide with $\mathrm{Re}(H)$ and K respectively.

Proof: Suppose M has a CR-structure on M. Then in view of Definition (3.1) and Corollary (3.1) we can write

$$F(X) = -Y \tag{3.12}$$

operating (3.12) by $\frac{2(F-I)}{3}$ we get

$$\frac{2(F-I)}{3}F(X) = \frac{2(F-I)}{3}(-Y) \tag{3.13}$$

On simplifying the above equation we get

$$F^2 - F + \frac{3}{2}I = 0$$

4. An almost complex golden structure on the frame bundle

Let M be an n-dimensional differentiable manifold of class C^∞ and FM its frame bundle over the manifold M. Suppose the base space M is covered by a system of coordinate neighborhoods (U, x^i) such that $F(U) = \pi^{-1}(U)$ where (x^i) is a system of local coordinates defined in the neighborhood U and $\pi : FM \to M$ the projection map. The local components of the vector X_α of the frame $p_x \in U$ are given by $X_\alpha = X_\alpha^i \left(\frac{\partial}{\partial x^i}\right)_x$. Thus $\left\{FU, \left(x^i, X_\alpha^i\right)\right\}$ is a coordinate system in FM.

Let ∇ be a linear connection and X a vector field on M with local components $\tilde{A}_{ij}^h$ and X^i, respectively. Let vector fields

X^H and $X^\alpha, \alpha = 1, 2, \ldots m$. be the horizontal lift and the α^{th}-vertical lift of X on FM and defined by [?]

$$X^H = X^i \frac{\partial}{\partial x^i} - X^i \Gamma_{ik}^h X_\alpha^k \frac{\partial}{\partial x^h} \tag{4.1}$$

$$X^{(\alpha)} = X^i \frac{\partial}{\partial X_\alpha^i} \tag{4.2}$$

Let f be a differentiable function on M, we write f^V for function i.e. vertical lift in FM and $f^H = 0$ its horizontal lift

[?].

If F is a tensor field on M of type $(1,1)$ with components F_j^h in U, then

$$F^H = F_j^h \frac{\partial}{\partial X^h} \otimes dx^j + X_\alpha^k \left(\tilde{A}_{jk}^i F_i^h - \tilde{A}_{ik}^h F_j^i \right) \frac{\partial}{\partial X_\alpha^h} \otimes dx^j + \delta_\alpha^\beta F_j^h \frac{\partial}{\partial X_\alpha^h} \otimes dX_\beta^j \tag{4.3}$$

is local components of F^H in FU.

Let τ be a 1-form on M with local components τ_i in U, then

$$\begin{aligned} \tau^V &= \tau_i dx^i \\ \tau^{H_\alpha} &= X_\alpha^j \Gamma_{ij}^h \tau_h dx^i + \tau_i dX_\alpha^i \\ X^H &= \sum_{\alpha=1}^m \left(X_\alpha^j \Gamma_{ij}^h \tau_h dx^i + \tau_i dX_\alpha^i \right) \end{aligned} \tag{4.4}$$

are local components of τ^V, τ^{H_α} and X^H in FU.
The following formulas of horizontal and vertical lifts are given by

$$\begin{aligned} X^H\left(f^V\right) &= (X(f)^v) \\ X^{(\alpha)}\left(f^V\right) &= 0 \\ F^H\left(X^{(\alpha)}\right) &= (F(X))^\alpha \\ F^H\left(X^H\right) &= (F(X))^H \\ F^H(\lambda A) &= F^C(\lambda A) = \lambda\left(F^\circ A\right) \\ \tau^V\left(X^H\right) &= (F(X))^V \\ \tau^V\left(X^{(\alpha)}\right) &= 0 \\ \tau^{H_\alpha}\left(X^H\right) &= 0 \\ \tau^{H_\alpha}\left(X^{(\beta)}\right) &= \delta_\alpha^\beta (\tau(X))^V \end{aligned} \tag{4.5}$$

for all vector fields X,Y on M and λA is fundamental vector field associated to A where $A \in gl(n,R), gl(n,R)$ is general linear group and R is Euclidean space.
The brackets of vertical and horizontal lifts are expressed by the following formulas

$$\begin{aligned} \left[X^{(\alpha)}, Y^{(\beta)}\right] &= 0 \\ \left[X^H, Y^{(\alpha)}\right] &= (\nabla_X Y)^{(\alpha)} \\ \left[X^H, Y^H\right] &= \left[X^H, Y^H\right] - \gamma R(X,Y) \end{aligned} \tag{4.6}$$

where $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Let (M,g) be an n-dimensional Riemannian manifold and FM its frame bundle. Let X^H and $X^{(\alpha)}, \alpha = 1,2,\ldots,n$, be horizontal and vertical lifts of a vector field X on FM with respect to the Levi-Civita connection ∇ of a Riemannian metric g.

Introduce a tensor field $F_\alpha, \alpha = 1,2,\ldots,n$, of type $(1,1)$ on FM by

$$F_\alpha X^H = \frac{1}{2}\left\{X^H + (2x-1)X^{(\alpha)}\right\}$$

$$F_\alpha X^{(\beta)} = \frac{1}{2}\delta_\alpha^\beta\left\{X^{(\beta)} + (2x-1)X^H\right\} \tag{4.7}$$

where $x = \frac{1}{2}\left(1 \pm \sqrt{5}i\right)$

Theorem 4.1 Let FM be the frame bundle of M. Then a tensor field F_α, defined by equation (4.7), is an almost complex golden structure on FM.

Proof: In order to prove F_α is an almost complex golden structure, it suffices to show that $F_\alpha^2 - F_\alpha + \frac{3}{2}I = 0$.

In the view of equation (4.7), then

$$\left(F_\alpha^2 - F_\alpha + \frac{3}{2}I\right)X^H = F_\alpha\left(F_\alpha X^H\right) - F_\alpha X^H + \frac{3}{2}X^H$$

$$= F_\alpha \frac{1}{2}\left\{X^H + (2x-1)X^{(\alpha)}\right\} - \frac{1}{2}\left\{X^H + (2x-1)X^{(\alpha)}\right\} + \frac{3}{2}X^H$$

$$= \frac{1}{4}\left\{X^H + (2x-1)X^{(\alpha)}\right\} + \frac{(2x-1)}{4}\left\{X^{(\beta)} + (2x-1)X^H\right\} - \frac{1}{2}\left\{X^H + (2x-1)X^{(\alpha)}\right\} + \frac{3}{2}X^H$$

$$= 0$$

Similarly, it is easily proved that $\left(F_\alpha^2 - F_\alpha + \frac{3}{2}I\right)X^{(\alpha)} = 0$ which imply that $F_\alpha^2 - F_\alpha + \frac{3}{2}I = 0$.

Hence, F_α is an almost complex golden structure on FM.

References

[1] A. Bejancu, Geometry of CR submanifolds, D. Reidel Publishing Co., Dor- drecht (1986).

[2] A. Bejancu, CR submanifolds of a Kaehler manifold-I, Proc. Amer. Math. Soc. 69 (1978), 135-142.

[3] D.E. Blair and B.Y. Chen, On CR-submanifolds of Hermitian manifolds, Israel J. Math. 34 (1979), 353-363.

[4] B.Y. Chen, Geometry of submanifolds, Marcel Dekker, New York, (1973).

[5] S. Dragomir, M. H. Shahid and F.R. Al-Solamy, Geometry of Cauchy- Riemann Submanifolds, Springer Singapore, (2016).

[6] Lovejoy S. Das, Submanifolds of F-structure satisfying $F^K + (-)^{K+1}F = 0$, Internat. J. Math. Math. Sci. 26 (2001), 167-172.

[7] Lovejoy S. Das, On CR-structure and F-structure, satisfying $F^K + (-)^{K+1}F = 0$. Rocky Mountain Journal of Mathematics, 36(3) (2006), 885-892.

[8] K. Yano and Masahiro Kon, Differential geometry of CR-submanifolds, Geom. Dedicata 10 (1981), 369-391.

[9] M.N.I. Khan, A Haseeb, F (a0, a1,. , an)-structures on manifolds, Results in Nonlinear Analysis 7 (1), 8–13, 2024.

[10] M.N.I. Khan, S Chaubey, N Fatima, A Al Eid, Metallic Structures for Tangent Bundles over Almost Quadratic – Manifolds, Mathematics 11, 4683, 2023.

[11] U.C. De, M.N.I. Khan, Complete lifts of a semi-symmetric non-metric connection from a riemannian manifold to its tangent bundles, Commun. Korean Math. Soc. 38 (4), 1233-1247, 2023.

[12] M.N.I. Khan, F Mofarreh, A Haseeb, Tangent Bundles of P-Sasakian Manifolds Endowed with a Quarter-Symmetric Metric Connection, Sym- metry, 2023 15 (3), 753.

[13] M.N.I. Khan, Novel theorems for metallic structures on the frame bundle of the second order, Filomat 36:13 (2022), 4471–4482, 2022.

[14] M.N.I. Khan, Uday Chand De, Liftings of metallic structures to tangent bundles of order r, AIMS Mathematics, 7(5), 7888-7897, 2022.

[15] M.N.I. Khan, O Bahadır, Tangent bundles of lp-sasakian manifold en- dowed with generalized symmetric metric connection, Facta Universitatis, Series: Mathematics and Informatics 38 (1), 125-139, 2023.

[16] R. Kumar, L Colney, M.N.I. Khan, Lifts of a semi-symmetric non-metric connection (SSNMC) from statistical manifolds

to the tangent Bundle Re- sults in Nonlinear Analysis 6 (3), 50-65, 2023.
[17] R. Kumar, L Colney, M.N.I. Khan, Proposed Theorems on the Lifts of Kenmotsu Manifolds Admitting a Non-Symmetric Non-Metric Connection (NSNMC) in the Tangent Bundle, Symmetry 15(11), 2037, 2023.
[18] Yano, K.; Ishihara, S. Tangent and Cotangent Bundles. Marcel Dekker, Inc., New York, 1973.
[19] Khan, M. N. I. Complete and horizontal lifts of Metallic structures. In- ternational Journal of Mathematics and Computer Science, 2020, 15(4), 983-992.
[20] Khan, M. N. I. Tangent bundle endowed with quarter-symmetric non- metric connection on an almost Hermitian manifold. Facta Universitatis, Series: Mathematics and Informatics, 2020, 35(1), 167-178.
[21] Crasmareanu, M.; Hretcanu, C.E. Golden differential geometry. Chaos Solitons Fractals 2008, 38, 1229-1238.
[22] Spinadel, V.W. de., On characterization of the onset to chaos, Chaos Solitons Fractals 1997, 8(10), 1631–1643.
[23] Stakhov, A., The generalized golden proportions, a new theory of real numbers, and ternary mirror-symmetrical arithmetic, Chaos, Solitons and Fractals 2007, 33, 315–334.
[24] Naveira A. A classification of Riemannian almost-product manifolds. Rend Di Mat Di Roma 1983, 3, 577-592.
[25] Pitis G. On some submanifolds of a locally product manifold. Kodai Math J. 1986, 9, 327-333.
[26] Yano K, Kon M. Structures on Manifolds, Series in Pure Mathematics. Singapore: World Scientific, 1984.
[27] Goldberg, S.I.; Yano, K. Polynomial structures on manifolds, Kodai Math Sem Rep. 1970, 22, 199-218.
[28] Goldberg, S.I.; Petridis, N.C. Differentiable solutions of algebraic equa- tions on manifolds. Kodai Math. Sem. Rep. 1973, 25, 111–128.
[29] Ozkan, M.; Prolongations of golden structures to tangent bundles. Differ- ential Geometry Dynamical Systems 2014, 16, 227-239.
[30] Ozkan, M.; Peltek, B. A New Structure on Manifolds: Silver Structure, International Electronic Journal of Geometry, 2016, 9(2), 59–69.
[31] Gonul, S.; Erken, I. K.; Yazla, A.; Murathan, C. A Neutral relation between metallic structure and almost quadratic ϕ-structure. Turk J Math. 2019, 43, 268-278.
[32] Lachieze-Rey M. Connections and Frame Bundle Reductions. arXiv:2002.01410 [math-ph], 4 Feb 2020.