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On Kenmostu Manifolds Satisfying Certain Conditions

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Abstract

In this paper, we study 3-dimensional Kenmostu manifolds, weakly Ricci symmetric Kenmostu manifolds and generalized Ricci recurrent Kenmostu manifolds and prove that conformably flat Kenmostu manifold is η-Einstein manifolds, deduced that the square length of Ricci tensor. Further proved that if weakly Ricci-symmetric Kenmostu manifolds satisfies Ricci symmetric condition then manifolds is Einstein manifold. In last we prove that if generalized Ricci recurrent Kenmostu manifolds satisfies the condition $(\nabla_X \eta)(Y) = 0$ then $\alpha(X) = \beta(Y)$.

Keywords : Kenmostu manifold, weakly Ricci symmetric manifold, generalized Ricci recurrent Kenmostu manifold, η-Einstein manifold.

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1. Introduction

Let (M, g) be a n-dimension $n \geq 3$, differentiable manifolds of class C^∞ we denoted by ∇ its Levi-Civita connection. We define endomorphism R(X, Y) Z and $X \wedge Y$ by

$$R(X, Y) Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X, Y]} Z \quad \text{and} \quad X \wedge Y = g(Y, Z) X - g(X, Z) Y \tag{1.1}$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie-algebra of the vector fields on (M, g). The Riemannian Christoffel curvature tensor R is defined as

$$R(X, Y) Z = g(R(X, Y) Z, W), \qquad W \in \chi(M).$$

Let S and r denote the Ricci tensor and the scalar curvature of (M, g) respectively then Weyl conformal curvature tensor C is defined as

$$C(X, Y) Z = R(X, Y) Z - \frac{1}{(n-1)}[S(Y, Z) X - S(X, Z) Y + g(Y, Z) QX - g(X, Z) QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z) X - g(X, Z)Y] \quad (1.2)$$

where Q is the Ricci operator defined by $g(\phi X, Y) = S(X, Y)$ [5].

L. Mamassy and T. Q. Binh [7] [8], introduced the notion of weakly symmetric and weakly Ricci symmetric Sasakian manifolds and M. Kon [3] introduced the notion of Ricci η-parallelity for Sasakian manifolds. A Riemannian manifolds is called weakly Ricci symmetric manifold if there exist 1-form ρ, η and ν such that relation

$$(\nabla_X S)(Y, Z) = \rho(X) S(Y, Z) + \mu(Y) S(X, Z) + \nu(Z) S(X, Z)$$

holds for any vector fields X, Y and Z, where S is the Ricci tensor of type (0, 2) of the manifold. A weakly Ricci symmetric manifold is said to be proper if $\rho = \mu = \nu = 0$ is not the case.

In the present paper we study Kenmostu manifolds with certain conditions. This paper is organised as follows section-2 contain the necessary details about the Kenmostu manifolds and basic results. In section-3 we study 3-dimensional Kenmostu manifolds and proved that manifolds is η-Einstein manifolds and also find the square length of Ricci tensor. Further we shall prove that in 3-dimensional Kenmostu manifolds the relation $Q\xi = \left(1 - \frac{r}{2}\right)\xi$ always holds.

In section - 4, 5 we study weakly Ricci symmetric, 3-dimensional generalized recurrent Kenmostu manifolds and obtained several results.

2. Kenmostu Manifold

Let (M, g) be a almost contact manifolds [9] with a almost contact structure (ϕ, ξ, η, g) consisting a (1, 1) tensor field ϕ, a vector fields ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0 \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \quad (2.2)$$

$$g(X, \phi Y) = -g(\phi X, Y) \quad (2.3)$$

for all vector fields $X, Y \in \xi(M)$.

An almost contact metric manifolds is called Kenmostu manifolds if it satisfies [1]

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\, \xi - \eta(Y)\, \phi X, \quad \text{for all } X, Y \in \chi(M) \tag{2.4}$$

where ∇ is Levi-Civita connection of Riemannian metric g of type (0, 2). From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\, \xi \tag{2.5}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\, \eta(Y). \tag{2.6}$$

Moreover the curvature R, the Ricci tensor S and the Ricci operator Q satisfy [1]

$$S(X, \xi) = (1 - n)\, \eta(X), \qquad S(\phi X, \phi Y) = S(X, Y) = (1 - n)\eta(X)\eta(Y) \tag{2.7}$$

$$Q\xi = (1 - n)\, \xi \qquad \text{and} \qquad \text{rank}\, (\phi) = 2m \tag{2.8}$$

$$R(\xi, X)\, Y = \eta(Y)\, X - g(X, Y)\, \xi \tag{2.9}$$

where $n = 2m + 1$, Kenmostu manifolds have been studied by various authors [11], [10] and [6].

A Kenmostu manifold is normal (that is Nijenhuis tensor of ϕ equals $-2d\eta \otimes \xi$) but not Sasakian manifolds. Moreover it is also not compact, from (2.5), we get $\text{div}\, \xi = n - 1$.

A Kenmostu manifold is said to be η-Einstein manifolds if its Ricci tensor S is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\, \eta(Y) \tag{2.10}$$

for any $X, Y \in \chi(M)$ and α, β are function on (M, g) [1][5].

3. Three-Dimensional Kenmostu Manifold

A Kenmostu manifold (M, g) is called conformally flat if it is conformally equivalent to Euclidean space. Let (M, g) be a 3-dimensional manifolds it is well-known [2] the conformal curvature of Weyl vanishes identically that for 3-dimensional manifolds (M, g) curvature tensor R is satisfies

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y + \frac{r}{2}[g(X, Z)Y - g(Y, Z)X] \qquad (3.1)$$

where Q is the Ricci operator, that is $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of (M, g). Putting $Z = \xi$ in (3.1) and using (2.7) and (2.8), we have

$$\eta(Y)QX = [\{\frac{r}{2} + n\}\eta(Y)X - \eta(X)Y] + \eta(X)QY. \qquad (3.2)$$

Putting $Y = \xi$ in (3.2) and using (2.1) and (2.8), we have

$$S(X, W) = [\frac{r}{2} + n]g(X, W) - [\frac{r}{2} + 2n - 1]\eta(X)\eta(W) \qquad (3.3)$$

which is η-Einstein manifolds with $\alpha = \frac{r}{2} + n$, $\beta = 1 - \frac{r}{2} - 2n$.

We state the result :

Theorem (3.1). A conformally flat 3-dimensional Kenmostu manifolds is an η-Einstein manifolds.

Corollary. In a 3-dimensional Kenmostu manifold the relation $Q\xi = \left[1 - \frac{r}{2}\right]\xi$ holds.

For conformally flat Kenmostu manifolds, we have

$$\dot{r} = \left[\frac{r}{2} + n\right]n + \left[1 - 2n - \frac{r}{2}\right] \quad \text{and} \quad S(\xi, \xi) = \left[\frac{r}{2} + n\right]n + \left[1 - \frac{r}{2} - 2n\right] \qquad (3.4)$$

where $\dot{r}$ is the scalar curvature of conformally flat Kenmostu manifold.

Let L be the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor S, then

$$g(LX, Y) = S(X, Y) \quad \text{for all } X, Y. \qquad (3.5)$$

Let l^2 be the square length of the Ricci tensor which is defined as

$$l^2 = S(Le_i, e_i) \qquad (3.6)$$

where $\{e_i\}$, i = 1, 2, 3, ..., n is orthogonal basis of the tangent space at a point of the manifold. From (3.3), we have

$$S(Le_i, e_i) = \left[\frac{r}{2} + n\right] g(Le_i, e_i) + \left[1 - \frac{r}{2} - 2n\right] \eta(Le_i)\,\eta(e_i)$$

$$= \left[\frac{r}{2} + n\right] S(e_i, e_i) + \left[1 - \frac{r}{2} - 2n\right] S(e_i, \xi)\, g(e_i, \xi). \quad (3.7)$$

Using (3.4), (3.5) and (3.6) in (3.7), we have

$$l^2 = \left[\frac{r}{2} + n\right]^2 n + \left[1 - 2n - \frac{r}{2}\right]\left[1 + \frac{r}{2}\right]. \quad (3.8)$$

We state the result.

Theorem (3.2). In a 3-dimensional conformally flat Kenmostu manifolds the length of the Ricci tensor S is given by

$$l = \sqrt{\left[\frac{r}{2} + n\right]^2 n + \left[1 - 2n - \frac{r}{2}\right]\left[1 + \frac{r}{2}\right]}.$$

4. Weakly Ricci Symmetric Kenmostu Manifold with η-Parallel Ricci Tensor

Definition. The Ricci tensor S of a weakly Ricci symmetric 3-dimensional Kenmostu manifold is said to η-parallel if it satisfies the condition

$$(\nabla_X S)(\phi Y, \phi Z) = 0 \quad \text{for all X, Y and Z.} \quad (4.1)$$

Let (M, g) be a weakly Ricci symmetric Kenmostu manifold, then from (1.3), we have

$$(\nabla_X S)(\phi Y, \phi Z) = \rho(X)\, S(\phi Y, \phi Z) + \mu(\phi Y)\, S(X, \phi Z) + \nu(Z)\, S(X, \phi Y). \quad (4.2)$$

Using (2.7), (4.1) in (4.2), we have

$$\rho(X)\,[S(Y, Z) - (1 - n)\,\eta(Y)\,\eta(Z)] + \mu(\phi Y)\,[(1 - n)\, g(\phi X, \phi Z)]$$
$$+ \nu(\phi Z)[(1 - n)\, g(X, \phi Y)] = 0.$$

Putting X = ξ in above and using (2.1), we have

$$\rho(\xi)\,[S(Y, Z) - (1 - n)\eta(Y)\eta(Z)] = 0. \tag{4.3}$$

This implies that $\rho(\xi) \neq 0$ and $S(Y, Z) = (1 - n)\eta(Y)\eta(Z)$, which is Einstein manifold, we state the results.

Theorem (4.1). If weakly Ricci symmetric 3-dimensional Kenmostu manifolds satisfies η-parallel Ricci tensor then the manifold is an Einstein manifold with scalar curvature $\tau = (n - 1)$.

Definition. A Kenmostu manifolds is said to be Ricci symmetric if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = 0 \quad \text{for all X, Y and Z.} \tag{4.5}$$

from equation (3.3), we have

$$(\nabla_X S)(Y, Z) = \left[1 - 2n - \frac{r}{2}\right][(\nabla_X \eta)(Y)\,\eta(Z) + (\nabla_X \eta)(Z)\,\eta(Y)]. \tag{4.6}$$

Using (2.6) and (4.5) in (4.6), we have

$$\left[1 - 2n - \frac{r}{2}\right][g(X, Y)\,\eta(Z) + g(X, Z)\,\eta(Y) - 2\eta(X)\,\eta(Y)\,\eta(Z)] = 0. \tag{4.7}$$

This implies that

$$r = 2(1 - 2n) \quad \text{and} \quad g(\phi X, \phi Y) = (\nabla_X \eta)(Y) = 0. \tag{4.8}$$

We state the result.

Theorem (4.2). If a 3-dimensional Kenmostu manifolds satisfies the Ricci symmetric condition then the relation holds :

(a) $r = 2(1 - 2n)$

(b) $g(\phi X, \phi Y) = (\nabla_X \eta)(Y) = 0.$

Further from (3.3), we have

$$(\nabla_X S)(Y, Z) = \left[1 - 2n - \frac{r}{2}\right][(\nabla_X \eta)(X)\,\eta(Y) + (\nabla_X \eta)(Y)\,\eta(X)]$$

and $$(\nabla_Y S)(X, Z) = \left[1 - 2n - \frac{r}{2}\right][(\nabla_Y \eta)(X)\,\eta(Z) + (\nabla_Y \eta)(Z)\,\eta(X)].$$

From above these two results and using the relation (4.8), we have

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \tag{4.9}$$

which is Ricci tensor of Codazzi type [4]. We state the result :

Theorem (4.3). If in a 3-dimensional Kenmostu manifold the relation (4.8) holds then the manifold has the Ricci tensor of Codazzi type.

5. Three-Dimensional Generalized Recurrent Kenmostu Manifold

A Riemannian manifold (M, g) is called Generalized recurrent [13] if its curvature tensor R satisfies

$$(\nabla_X R)(Y, Z) W = \alpha(X) R(Y, Z) W + \beta(X) [g(Z, W) Y - g(Y, W) Z] \tag{5.1}$$

where α, β are two 1-form, β is non-zero and are defined as $\alpha(X) = g(X, A)$, $\beta(X) = g(X, B)$, A, B are vector fields associated with 1-form α, β respectively.

A Riemannian manifold (M, g) is called Generalized Ricci recurrent [13] if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = \alpha(X) S(Y, Z) + (1 - n)\beta(X) g(Y, Z) \tag{5.2}$$

where α, β are defined as above.

Further from (3.3), we have

$$(\nabla_X S)(Y, Z) = \left[1 - 2n - \frac{r}{2}\right] [(\nabla_X \eta)(Y) \eta(Z) + (\nabla_X \eta)(Z) \eta(Y)]. \tag{5.3}$$

Let (M, g) be generalized Ricci recurrent manifolds, using (5.2) in (5.3)

$$\alpha(X) S(Y, Z) + (1 - n)\beta(X) g(Y, Z) = \left[1 - 2n - \frac{r}{2}\right] [g(X, Y)(Z) - \eta(X) \eta(Y) \eta(Z)$$

$$+ g(X, Z)(Y) - \eta(X) \eta(Y) \eta(Z)] . \tag{5.4}$$

Putting $Y = Z = \xi$ in (5.4) and using (2.1) and (4.8), we get

$$(1 - n)[\alpha(X) - \beta(X)] = 0. \tag{5.5}$$

This implies that $(1 - n) \neq 0$ and $\alpha(X) = \beta(X)$ for any vector field X.

Theorem (5.1). If a 3-dimensional generalized Ricci recurrent Kenmostu manifold satisfies the relation (4.8) then $\alpha = \beta$.

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