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RESEARCH ARTICLE

Killing Vector Fields on A Family of Four-Dimensional Walker Manifolds

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Abstract
In this paper, we study Killing vector fields on four-dimensional Walker manifold and we obtain conditions for a vector field on a family of four-dimensional Walker manifold to be Killing.

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1. Introduction
A pseudo-Riemannian metric on a four dimensional manifold is said to be a Walker metric if there exists a two dimensional null distribution on , which is parallel with respect to the Levi-Civita connection of . This type of metrics has been introduced by Walker [12] who has shown that they have a local canonical form depending on three smooth functions. Various curvature properties of some special classes of Walker metrics have been studied in [2] where several examples of neutral metrics with interesting geometric properties have been given. Conditions for a restricted four dimensional Walker manifold to be Einstein, locally symmetric, Einstein and locally conformally flat are given in [3]. Examples of Walker Osserman metrics of signature which admits a field of parallel null 3-planes are given in [5, 6]. A lot of examples of Walker structures have appeared, which proved to be important in differential geometry and general relativity as well .

In this paper, we focus on Killing vector fields for Walker manifolds. We study questions of existence and characterization of Killing vector fields on a family of four dimensional Walker manifold. The problem of the existence of Killing vector fields in pseudo-Riemannian manifolds has been analyzed by many authors (physicists and mathematicians) with different points of view and by using several techniques. Sanchez [9] reviews the geometric consequences of the existence of a (non-trivial) Killing vector field on a Lorentzian manifold ( ). In [10] Sanchez studied the structure of Killing vector fields on a generalized Robertson-Walker space-time. He

obtained necessary and sufficient conditions for a vector field to be Killing on generalized Robertson-Walker space-times and gave a characterization of them as well as an explicit list for the globally hyperbolic case. In [7], the authors provide a global characterization of the Killing vector fields of a standard static space-time by a system of partial differential equations.

In [1], the authors study magnetic curves on Walker manifolds to classify the corresponding Killing magnetic curves. By imposing some conditions for the existence of Killing vector fields on a 3-dimensional Walker manifold, the authors obtain their classifications. Consequently, they prove that the coordinate vector field is Killing on the 3-dimensional Walker manifold if and only if the Walker manifold is strict. The authors [1] establish that if a 3 -dimensional Walker manifold admits a unit space-like vector field that is normal to a light-like curve, then the curve is a reparametrization of a geodesic. They also obtain magnetic curves corresponding to a Killing vector field admitted by the 3 -dimensional Walker manifold. Furthermore, some characterization of the normal magnetic trajectories associated to the Killing vector field on 3-dimensional Walker manifold are obtained and some examples of Killing magnetic curves on such manifolds are provided.

Motivated by the papers [1, 4] , we study Killing vector vector fields on a family of Walker manifolds of dimension four. As a consequence, we obtain some restrictions for the existence of such vector fields on these spaces. The outline of the paper is as follows:. In section 2, we shall describe the Walker metric which consider in this work. In section 3, a characterization of Killing vector fields on a family of four dimensional Walker manifold will be given with examples.

Throughout, all geometric objects on the manifold are assumed to be smooth.

2. Description of The Metric

A four dimensional pseudo-Riemannian manifold (M,g) of signature $(2,2)$ is said to be a Walker manifold if it admits a parallel totally isotropic two plane field [2]. Let consider the following family of Walker metrics g_a on $O\subset\mathbb{R}^4$ given by :

$$g_a = 2(dx_1\circ dx_3+dx_2\circ dx_4)+a(x_1,x_2,x_3,x_4)dx_3\circ dx_3 \#(2.1)$$

where a is a function of the following form :

$$a(x_1,x_2,x_3,x_4)=x_1b(x_3,x_4)+x_2c(x_3,x_4)+d(x_3,x_4).$$

For any function $h(x_1,x_2,x_3,x_4)$, we denote by $h_i:=\frac{\partial h(x_1,x_2,x_3,x_4)}{\partial x_i}$ and $h_{ij}:=\frac{\partial^2 h(x_1,x_2,x_3,x_4)}{\partial x_i\partial x_j}$. From a straightforward calculation shows that the non-zero components of the Levi-Civita connection of the metric (2.1) are given by :

$$\nabla_{\partial_1}\partial_3=\frac{1}{2}a_1\partial_1,\nabla_{\partial_1}\partial_4=\frac{1}{2}a_1\partial_2,\nabla_{\partial_2}\partial_3=\frac{1}{2}a_2\partial_1,\nabla_{\partial_2}\partial_4=\frac{1}{2}a_2\partial_2,$$

$$\nabla_{\partial_3}\partial_3=\frac{1}{2}(aa_1+a_3)\partial_1+\frac{1}{2}(aa_2-a_4)\partial_2-\frac{1}{2}a_1\partial_3-\frac{1}{2}a_2\partial_4,$$

$$\nabla_{\partial_3}\partial_4=\frac{1}{2}a_4\partial_1+\frac{1}{2}a_3\partial_2$$

$$\nabla_{\partial_3}\partial_4=\frac{1}{2}(aa_1-a_3)\partial_1+\frac{1}{2}(aa_2-a_4)\partial_2-\frac{1}{2}a_1\partial_3-\frac{1}{2}a_2\partial 4\#(2.2)$$

Since Walker metrics are of indefinite signature, deciding on their geodesic completeness is not an easy task. However, defining a Walker metric locally on $\mathbb{R}^4$ with Christoffel symbols satisfying $\Gamma_{ij}^k=0$ for all $i,j\leqslant k$, it is geodesically complete. Also, in [8], the authors shown that four dimensional strict Walker manifolds are geodesically complete.

In general, to describe the geometry of a pseudo-Riemannian manifold, one must first understand the curvature of the manifold. We shall investigate a wide variety of curvature properties and we shall derive geometrical results. The influence of the curvature on a manifold can be measured in many different ways and it appears in many different contexts. An example of such influence is the existence of additional structures on the manifold under certain curvature conditions. This is specially clear in dimension 4 when we study Einstein or locally conformally flat structures. Using (2.2), we can completely determine the curvature tensor of the metric (2.1) by the following formula: $\mathcal{R}(\partial_i,\partial_j)\partial_k=([\nabla_{\partial_i},\nabla_{\partial_j}]-\nabla_{[\partial_i,\partial_i]})\partial_k$. Then, taking into account (2.1), we can determine all components of the (0,4)-curvature tensor $R_{ijkl}=g_a(\mathcal{R}(\partial_i,\partial_j)\partial_k,\partial_l)$. We obtain that, the non-zero component of the (0,4)-curvature tensor of the metric (2.1) are given by:

$$R_{1313}=\frac{1}{2}a_{11},R_{1323}=\frac{1}{2}a_{12},R_{1424}=\frac{1}{2}a_{12},R_{1334}=\frac{1}{4}(a_1a_2-2a_{14})$$

$$R_{1414}=\frac{1}{2}a_{11},R_{1434}=\frac{1}{4}(2a_{13}-a_1^2),R_{2323}=\frac{1}{2}a_{22}$$

$$R_{2334}=\frac{1}{4}\left(a_2^2-2a_{24}\right), R_{2424}=\frac{1}{2}a_{22},\ R_{2434}=\frac{1}{4}\left(2a_{23}-a_1a_2\right)$$

$$R_{3434}=\frac{1}{4}\left(2a_{33}+2a_{44}-aa_1^2-aa_2^2\right).\#(2.3)$$

Using this, we can calculate the compoments ρ_{ij} with respect to ∂_i of the Ricci tensor of the metric (2.1):

$$\rho_{13}=\frac{1}{2}a_{11}, \rho_{14}=\frac{1}{2}a_{12}, \rho_{23}=\frac{1}{2}a_{12}, \rho_{24}=\frac{1}{2}a_{22},$$

$$\rho_{33}=\frac{1}{2}\left(a_2^2+aa_{11}+aa_{22}-2a_{24}\right),$$

$$\rho_{34}=\frac{1}{2}\left(-a_1a_2+a_{14}+a_{23}\right),$$

$$\rho_{44}=\frac{1}{2}\left(a_2^2+aa_{11}-2a_{13}+aa_{22}\right).\#(2.4)$$

The scalar curvature defined by $\tau=$ trace ρ of the metric (2.1) is:

$$\tau=a_{11}+a_{22}$$

3. Killing vector fields on four-dimensional Walker MANIFOLDS

In this section, we find the conditions for the existence of Killing vector fields on the four dimensional Walker manifold (2.1).

Definition 3.1. A Killing vector field ξ on a pseudo-Riemannian manifold (M,g) is a smooth vector field that preserves the metric tensor, i.e., the Lie derivative with respect to ξ of the metric g vanishes:

$$\mathcal{L}_\xi g=0.\#(3.1)$$

The above definition is equivalent to

$$g\left(\nabla_X\xi,Y\right)+g\left(X,\nabla_Y\xi\right)=0,\#(3.2)$$

for any vector fields $X,Y\in\mathfrak{X}(M)$ and where ∇ is the Levi-Civita connection of g. Note that, existence and characterization of Killing vector fields on pseudo-Riemannian metrics have been studied by several authors. For instance, the author [9] reviews in depth the geometric consequences of the existence of a (non-trivial) Killing vector field K on a Lorentzian manifold (M,g). He mainly considers the case where K is of timelike type at every point of M. A Lorentzian manifold that admits such a vector field is called stationary. The author first explains how a compact stationary Lorentzian manifold must be geodesically complete; then he continues with other topics concerning Killing vector fields and geodesics. Also, the authors [7] propose a system of partial differential equations to characterize Killing vector fields of a standard static spacetime. By studying this system, they determine all Killing vector fields in the same domain when the Riemannian part is compact. In addition, they deal with the characterization of Killing vector fields on a standard static spacetime.

Next, we find the conditions for the existence of Killing vector fields on the family of four dimensional Walker manifold (2.1) and we obtain general expressions. Let

$$\xi=\xi^1\partial_1+\xi^2\partial_2+\xi^3\partial_3+\xi^4\partial_4$$

$$X=X^1\partial_1+X^2\partial_2+X^3\partial_3+X^4\partial_4$$

$$Y=Y^1\partial_1+Y^2\partial_2+Y^3\partial_3+Y^4\partial_4$$

where $\xi^i, X^i, Y^i, i=1,2,3,4$ are functions on x_1,x_2,x_3,x_4.

Localement, the components of the Lie derivative with respect to ξ of the metric (2.1) $\left(L_\xi g_a\right)\left(\partial_i,\partial_j\right)$ are given by :

$$\left(L_\xi g_a\right)\left(\partial_i,\partial_j\right)=\sum_k^4\left[\xi^k\partial_k g_{ij}+\left(\partial_i\xi^k\right)g_{kj}+\left(\partial_j\xi^k\right)g_{ik}\right].\#(3.3)$$

From (3.3), we have the following lemma:

Lemma 3.2. The non-zero components of the Lie derivative in the direction of $\xi = \sum_1^4 \xi^k \partial_k$ of the Walker metric (2.1) are given by :

$$\left(L_\xi g_a\right)(\partial_1,\partial_1) = 2\partial_1\xi^3, \left(L_\xi g_a\right)(\partial_1,\partial_2) = \left(L_\xi g_a\right)(\partial_2,\partial_1) = \partial_1\xi^4 + \partial_2\xi^3,$$

$$\left(L_\xi g_a\right)(\partial_1,\partial_3) = \left(L_\xi g_a\right)(\partial_3,\partial_1) = \partial_1\xi^1 + a\left(\partial_1\xi^3\right) + \partial_3\xi^3,$$

$$\left(L_\xi g_a\right)(\partial_1,\partial_4) = \left(L_\xi g_a\right)(\partial_4,\partial_1) = \partial_1\xi^2 + a\left(\partial_1\xi^4\right) + \partial_4\xi^3,$$

$$\left(L_\xi g_a\right)(\partial_2,\partial_2) = 2\partial_2\xi^4,$$

$$\left(L_\xi g_a\right)(\partial_2,\partial_3) = \left(L_\xi g_a\right)(\partial_3,\partial_2) = \partial_2\xi^1 + a\left(\partial_2\xi^3\right) + \partial_3\xi^4,$$

$$\left(L_\xi g_a\right)(\partial_2,\partial_4) = \left(L_\xi g_a\right)(\partial_4,\partial_2) = \partial_2\xi^2 + a\left(\partial_2\xi^4\right) + \partial_4\xi^4,$$

$$\left(L_\xi g_a\right)(\partial_3,\partial_3) = \xi^k a_k + 2\partial_3\xi^1 + 2a\partial_3\xi^3$$

$$\left(L_\xi g_a\right)(\partial_3,\partial_4) = \left(L_\xi g_a\right)(\partial_4,\partial_3) = \partial_3\xi^2 + a\partial_3\xi^4 + \partial_4\xi^1 + a\partial_4\xi^3$$

$$\left(L_\xi g_a\right)(\partial_4,\partial_4) = \xi^k a_k + 2\partial_4\xi^2 + 2a\partial_4\xi^4$$

From the Lemma 3.2, we find the conditions for the vector field ξ to be Killing.

Firstly, we give the form of ξ^3 and ξ^4.

Lemma 3.3. The functions ξ^3 and ξ^4 have the following forms:

$$\xi^3 = x_2 A(x_3,x_4) + B(x_3,x_4) \# (3.4)$$

and

$$\xi^4 = -x_1 A(x_3,x_4) + C(x_3,x_4) \# (3.5)$$

where A, B, C are some functions.

Proof. From the expressions of $\left(L_\xi g_a\right)(\partial_1,\partial_1)$ and $\left(L_\xi g_a\right)(\partial_2,\partial_2)$ of the above Lemma 3.2, we have:

$$\xi^3 = f(x_2,x_3,x_4) \text{ and } \xi^4 = f(x_1,x_3,x_4) \# (3.6)$$

From the expression $\left(L_\xi g_a\right)(\partial_1,\partial_2)$, we find (3.4) and (3.5).

Secondly, we give the form of ξ^1

Lemma 3.4. The function ξ^1 has the following form:

$$\xi^1 = -x_1 x_2 A_3(x_3,x_4) - x_1 B_3(x_3,x_4) + D(x_2,x_3,x_4) \# (3.7)$$

where A, B, D are some functions.

Proof. From the expression of $\left(L_\xi g_a\right)(\partial_1,\partial_3)$, we get (3.7).

Thirdly, we give the form of ξ^2.

Lemma 3.5. The function ξ^2 as the following form:

$$\xi^2 = x_1x_2A_4(x_3,x_4) - x_2C_4(x_3,x_4) + E(x_1,x_3,x_4) \# (3.8)$$

where A,C,E are some functions.

Proof. From the expression of $(L_\xi g_a)(\partial_2,\partial_4)$, we obtain (3.8).

The functions A,B,C,D,E of the above Lemmas sould satisfying the following partial differential equations system :

$$\begin{cases} 2x_2A_4(x_3,x_4) + E_1(x_1,x_3,x_4) - aA(x_3,x_4) + B_4(x_3,x_4) & = 0 \\ -2x_1A_3(x_3,x_4) + D_2(x_2,x_3,x_4) + aA(x_3,x_4) + C_3(x_3,x_4) & = 0 \\ \xi^k a_k + 2\partial_3\xi^1 + 2a\partial_3\xi^3 & = 0, (3.9) \\ \partial_3\xi^2 + a\partial_3\xi^4 + \partial_4\xi^1 + a\partial_4\xi^3 & = 0 \\ \xi^k a_k + 2\partial_4\xi^2 + 2a\partial_4\xi^4 & = 0 \end{cases}$$

obtaining form $(L_\xi g_a)(\partial_1,\partial_4), (L_\xi g_a)(\partial_2,\partial_3), (L_\xi g_a)(\partial_3,\partial_3), (L_\xi g_a)(\partial_3,\partial_4)$ and $(L_\xi g_a)(\partial_4,\partial_4)$.

Now we can prove the following characterization of Killing vector fields on (M, g_a).

Proposition 3.6. Let (M,g_a) be a Walker manifold of dimension 4, where g_a is the metric given by (2.1). If a vector field $\xi = \xi^1\partial_1 + \xi^2\partial_2 + \xi^3\partial_3 + \xi^4\partial_4$ on (M,g_a) is Killing, then there exist some functions A,B,C,D,E, such that one of the following cases occurs

1. Assume that $A(x_3,x_4) = 0$. Then we have :

$$\xi^1 = -x_1B_3(x_3,x_4) + D(x_2,x_3,x_4),$$
$$\xi^2 = -x_2C_4(x_3,x_4) + E(x_1,x_3,x_4),$$
$$\xi^3 = B(x_3,x_4), \xi^4 = C(x_3,x_4)$$

with

$$E_1(x_1,x_3,x_4) + B_4(x_3,x_4) = 0, D_2(x_2,x_3,x_4) + C_3(x_3,x_4) = 0,$$
$$b(C_3+B_4) - 2B_{34} = 0, c(C_3+B_4) - 2C_{34} = 0,$$
$$b(B_3-C_4) + B_{44} - B_{33} = 0, c(B_3-C_4) + C_{44} - C_{33} = 0.$$

Recall that the functions b and c come from of (2.1) and are functions of x_3, x_4

2. Now we assume that $A(x_3,x_4) = A \in \mathbb{R}^*$. Then, we have :

$$\xi^1 = -x_1B_3(x_3,x_4) + D(x_2,x_3,x_4),$$
$$\xi^2 = -x_2C_4(x_3,x_4) + E(x_1,x_3,x_4),$$
$$\xi^3 = Ax_2 + B(x_3,x_4), \xi^4 = -Ax_1 + C(x_3,x_4)$$

with

$$E_1(x_1,x_3,x_4) + B_4(x_3,x_4) + D_2(x_2,x_3,x_4) + C_3(x_3,x_4) = 0,$$

$$-x_2C_{34}+E_3+aC_3-x_1B_{34}+D_4+aB_4=0,$$
$$-x_1B_{33}+D_3+x_2C_{44}-E_4+a\left(B_3-C_4\right)=0.$$

Proof. The proof come from the partial differential equations system (3.9).

1. Case $A\left(x_3,x_4\right)=0$. From the two first equations of (3.9), we get:

$$E_1\left(x_1,x_3,x_4\right)+B_4\left(x_3,x_4\right)=0, D_2\left(x_2,x_3,x_4\right)+C_3\left(x_3,x_4\right)=0.$$

The quatrieme equation of (3.9) give us:

$$-x_1B_{34}-x_2C_{34}+a\left(C_3+B_4\right)+E_3+D_4=0\#\left(3.10\right)$$

Taking the derivative of (3.10) with respect to x_1 , we obtain :

$$b\left(C_3+B_4\right)-2B_{34}=0.\#\left(3.11\right)$$

Taking the derivative of (3.10) with respect to x_2 , we find :

$$c\left(C_3+B_4\right)-2C_{34}=0.\#\left(3.12\right)$$

Taking the difference between the third and the cinquieme equations of (3.9), we obtain:

$$-x_1B_{33}+x_2C_{44}+a\left(B_3-C_4\right)+D_3-E_4=0\#\left(3.13\right)$$

Taking the derivative of (3.13) with respect to x_1 , we find:

$$b\left(B_3-C_4\right)+B_{44}-B_{33}=0.\#\left(3.14\right)$$

Taking the derivative of (3.13) with respect to x_2 , we obtain:

$$c\left(B_3-C_4\right)+C_{44}-C_{33}=0\#\left(3.15\right)$$

2. Case $A\left(x_3,x_4\right)=A\in\mathbb{R}^*$. By adding the two first equations of (3.9), we obtain:

$$E_1\left(x_1,x_3,x_4\right)+B_4\left(x_3,x_4\right)+D_2\left(x_2,x_3,x_4\right)+C_3\left(x_3,x_4\right)=0\#\left(3.16\right)$$

Putting the functions ξ^1,ξ^2,ξ^3,ξ^4 in the quatrieme equation of (3.9), we find:

$$-x_2C_{34}+E_3+aC_3-x_1B_{34}+D_4+aB_4=0\#\left(3.17\right)$$

The difference between the troisieme and cinquieme equations of (3.9), give us :

$$-x_1B_{33}+D_3+x_2C_{44}-E_4+a\left(B_3-C_4\right)=0\#\left(3.18\right)$$

This complete the proof.

Example 3.7. The following vector field $\xi=\xi^1\partial_1+\xi^2\partial_2+\xi^3\partial_3+\xi^4\partial_4$, with

$$\xi^1=-2\left(x_1-x_2\right)+e^{x_3x_4}$$
$$\xi^2=-2\left(x_1+x_2\right)+e^{x_3+x_4}$$
$$\xi^3=2\left(x_3+x_4\right),\xi^4=2\left(-x_3+x_4\right)$$

is Killing on $\left(M,g_a\right)$.

Example 3.8. The following vector field $\xi=\xi^1\partial_1+\xi^2\partial_2+\xi^3\partial_3+\xi^4\partial_4$, with

$$\xi^1 = -2(x_1 + x_2) + x_3 + x_4,$$
$$\xi^2 = 2(x_1 - x_2) - (x_3 - x_4),$$
$$\xi^3 = Ax_2 + 2(x_3 + x_4), \xi^4 = -Ax_1 - 2(x_4 - x_3).$$

is Killing on (M, g_a).

Remark 3.9. The case $A(x_3, x_4) \neq 0$, forces (M, g_a) to be a strict Walker manifold. (See [8] and references therein for details on strict Walker manifolds).

The theory of Killing vector fields have a well-known geometrical and physical interpretations and have been studied on pseudo-Riemannian manifolds for a long time. The number of independent Killing vector fields measures the degree of symmetry of a pseudo-Riemannian manifold. Thus, the problems of existence and characterization of Killing vector fields are important and are widely discussed by both mathematicians and physicists (see [11] and references therein).

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