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# Linear deformations of closed non-singular 1-forms into even contact and Engel defining forms

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## Abstract
An Engel structure is a maximally non-integrable plane field on a 4-dimensional smooth manifold. Line fields, contact structures, even-contact structures (maximally non-integrable hyperplane fields on even-dimensional manifolds) and Engel structures are the only distributions which are stable under $C^2$-small perturbations. In this paper we investigate linear deformations of closed non-singular one forms into even and engel defining froms. For both we prove a necessary and sufficient condition of deformability and provide several examples.

**Keywords:** Codimension one foliations, Linear deformations, Even contact structures, Engel structures.

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## 1. Introduction\

Contact structures, even contact structures, and Engel structures are geometric structures on differentiable manifolds, mainly studied in the field of differential geometry especially in the study of dynamical systems and control theory. They all involve the study of distributions of tangent spaces and their behavior. All these structures are defined by non-integrable distributions, meaning that they describe systems where the tangent planes ”twist” in a non-trivial way and cannot be integrated into submanifolds.

Contact structures are defined in odd-dimensional geometry, arising in classical mechanics and dynamical systems. Although even contact structures extend these ideas to even-dimensional manifolds, bridging contact and symplectic geometry. Contact and even contact structures have a well-defined Reeb vector field that is transverse to the distribution and is integrable in the sens of Frobenuis as a distribution spanned by a one vector field. Engel structures are uniquely defined on 4-dimensional manifolds and are important in the study of nonholonomic systems and control theory. It is characterized by a rank-2 distribution $\mathcal{D}$ of the tangent bundle that satisfies a certain maximal non-integrability conditions. Engel structures derive sometimes from Cartan prolongations of contact 3-manifold and have a hierarchly of derived distributions crucial in understanding the structure's dynamics. The distribution $\mathcal{D}$ is part of a flag of distributions: $\mathcal{D} \subset \mathcal{E} \subset \mathrm{T}M$ , where $\mathcal{D}$ is a rank-2 distribution, and $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is a rank-3 distribution, which is an even contact structure, that is a hyperplane field defined as the kernel of a 1 -form $\alpha$ such that $\alpha \wedge (d\alpha)^n$ never vanishes on a $2n$ -dimensional manifold, in this case we have $[\mathcal{E}, \mathcal{E}] = \mathrm{TM}$ .

We can introduce so-called Engel defining forms $\alpha$ and $\beta$ , which were first studied in [1] and used as a technical tool in [11,[15,16] . With any choice of Engel defining forms, it is possible to associate a rank- 2 distribution $\mathcal{R} = \mathrm{span}(T, R)$ , where $T$ and $R$ are vector fields. The distribution $\mathcal{R}$ is tranverse to $\mathcal{D}(\mathrm{TM} = \mathcal{D} \oplus \mathcal{R})$ , which is not usually integrable as in the case of Reeb field of contact or even contact structure. Futhermore in [13], a necessary and sufficient condition for the integrability of $\mathcal{R}$ is proven by N. Pia. The aim of this paper is to prove the following two Theorems:

**Theorem 1.1.** Let $(M, \omega)$ be a $(2m+2)$ -dimensional oriented even contact manifold admitting an orientable characteristic foliation $\mathcal{L} = \ker(\omega \wedge (d\omega)^n)$ with volume-preserving holonomy, such that $M$ is foliated by closed hypersurfaces transverses to $\mathcal{L}$ . Then for any nonvanishing closed 1-form $\eta$ on $M$ , the following are equivalents.

1. The family of 1-forms $(\eta_t)_{t\geq 0}$ defined by $\eta_t = \eta + t\omega$ in a linear deformation of $\eta$ is a family of even contact structures for any $t > 0$.
2. $\eta \wedge (d\omega)^m = 0$.

**Theorem 1.2.** Let $M$ be a closed and oriented 4-manifold endowed with a Engel structure $\mathcal{D} = \ker(\omega_1 \wedge \omega_2)$ with Engel defining forms $\omega_1$ and $\omega_2$ and Reeb distributionv $\mathcal{R} = \langle Z_1, Z_2 \rangle = \ker(d\omega_2 + c_{Z_1 Z_2}\omega_1 \wedge \omega_2)$. Suppose there exist a pair of closed non-singular 1-forms $(\eta_1, \eta_2)$ satisfying

$$\begin{cases} \eta_1 \wedge d\omega_1 = 0 \\ \eta_2 \wedge d\omega_1 = 0 \end{cases}$$

then for all $t > 0$, the pair $(\eta_1^t, \eta_2^t)$ in a Engel deformation of $(\eta_1, \eta_2)$ in the way of $(\omega_1, \omega_2)$ are Engel defining forms if and only if

$$\begin{cases} \eta_1(Z_1) + \eta_2(Z_2) \geq 0 \\ \eta_1 \wedge \eta_2, Z_1 \wedge Z_2 = 0. \end{cases}$$

The paper is organized as follow, in section 2 we give some preliminaries about contact, even contact and Engel structures, in section 3 we recall linear deformations and old results obtained by different authors. Lastly we prove in section 3 our announced Theorems.

## 2. Preliminaries

Contact, even contact and Engel structures are geometric structures on differentiable manifolds, bringing together nonintegrability conditions. Here's an overview of each:

### 2.1. Contact and even contact structure

A contact structure is a specific type of hyperplane field on an odd-dimensional manifold. It is defined by a 1 -form that satisfies a certain non-degeneracy condition, that captures the idea of non-integrable distributions. It provides a framework for understanding various physical and geometric systems, especially those involving constraints, like in classical mechanics.

**Definition 2.1.** Let $M$ be a smooth manifold of dimension $2n+1$. A contact structure on $M$ is a maximally non-integrable hyperplane distribution $\xi \subset TM$, locally given as the kernel of a 1-form $\alpha$ (called the contact form), such that the non-degeneracy condition holds:

$$\alpha \wedge (d\alpha)^n \text{ is a volume form on M.}$$

This means that $\alpha$ is «maximally non-integrable,» ensuring that the distribution of tangent spaces described by $\alpha$ does not come from a foliation of the manifold by submanifolds.

The contact structure is determined by the kernel of the contact form, i.e., $\xi = ker(\alpha)$. While the 1-form $\alpha$ is not unique (it can be scaled by a non-vanishing smooth function), the structure it induces is the same. Moreover we have :

**Proposition 2.2** Let $M$ be $a(2n+1)$ -dimensional manifold, equipped with a 1-form $\alpha$. Then the followings are equivalents.

1. The distribution $\xi = \ker(\alpha)$ is a contact structure on $M$.
2. $(d\alpha)^n_{|\xi}$ does not vanish anywhere.
3. $d\alpha_{|\xi}$ is nondegenerate.

**Proof.** $1 \Rightarrow 2$ **:** By definition we have $\alpha \wedge (d\alpha)^n \neq 0$, so $\forall p \in M$, we have $\alpha_p \neq 0$. Indeed if there exists $p \in M$ such that $\alpha_p = 0$, then $\forall v_1, \ldots, v_{2n+1} \in T_p M$, we have:

$$\left(\left(\alpha \wedge (d\alpha)^n\right)_p\right)(v_1, \ldots, v_{2n+1}) = \sum_{i=1}^{2n+1} (-1)^i \alpha_p(v_i)(d\alpha)^n_p\left(v_1, \ldots, \hat{v}_i, \ldots, v_{2n+1}\right) = 0.$$

This is a contradiction since is a contact form. Therefore $\alpha_p : T_p M \to \mathbb{R}$ is surjective, the dimension of $\xi_p = \ker(\alpha_p)$ is $2n$ and that imply $\xi \subset TM$ is subbundle of rank $2n$. Since

$$T_p M = \mathrm{Vec}\left(E_p, u_1, \ldots, u_{2n}\right)$$

with the condition $\alpha_p\left(E_p\right) \neq 0$ and $\alpha_p\left(u_i\right)=0,\ i=1,\ldots,2n$, we have

$$\left(\left(\alpha \wedge (d\alpha)^n\right)\right)_p\left(E_p,u_1,\ldots,u_{2n}\right)=\alpha_p\left(E_p\right)\left((d\alpha)^n\right)_p\left(u_1,\ldots,\ldots,u_{2n}\right)$$

$$+\sum_{i=1}^{2n}(-1)^i\alpha_p\left(u_i\right)\left((d\alpha)^n\right)_p\left(E_p,u_1,\ldots,\tilde{u}_i\ldots,u_{2n}\right)$$

Then for all $p\in M$ and for the frame $\{u_1,\ldots,\ldots,u_{2n}\}$ of $\xi_p$ we have

$$\left((d\alpha)^n\right)_p\left(u_1,\ldots,\ldots,u_{2n}\right)\neq 0.$$

Then we have $\left((d\alpha)^n\right)_{|\xi}$ is zero.

$2\Rightarrow 3$: If $(d\alpha)_{|\xi}$ is degenerate in $\xi$, then there exists $V\in\xi$ such that $i_V d\alpha=0$ that imply $i_V(d\alpha)^n=0$ and this is a contradiction with 2 .

$3\Rightarrow 1$ Let $X,Y\in\xi$ be two vector field such that $d\alpha\left(X,Y\right)\neq 0$ that is $\alpha\left(\left[X,Y\right]\right)\neq 0$. So $\left[X,Y\right]\in\xi$ which is equivalent to the non integrability maximal of $\xi$. Then $\xi$ is a contact structure.

It's well know tha given a contact form $\alpha$ on $M$, there exists a uniquely determined global vector field $R_\alpha$ on $M$, associated to $\alpha$, called the Reeb vector field of $\alpha$, such that:

$$i_{R_\alpha}\alpha=1 \text{ and } i_{R_\alpha}d\alpha=0.$$

This vector field is crucial in the study of contact dynamics, as it generates flows that preserve the contact structure.

**Example 2.3.**

1. Standard Contact Structure on $\mathbb{R}^{2n+1}$ : In $\mathbb{R}^{2n+1}$ with coordinates $\left(x_1,\ldots,x_n,y_1,\ldots,y_n,z\right)$, the standard contact form is given by:

$$\alpha=dz-\sum_{i=1}^{n}y_i dx_i$$

2. The 3 -sphere $\mathbb{S}^3$ can be given a contact structure by considering it as the unit sphere in $\mathbb{C}^2$.
The standard contact form on $\mathbb{S}^3$ is :

$$\alpha=\frac{1}{2}\left(x_1dy_1-y_1dx_1+x_2dy_2-y_2dx_2\right)$$

where $\left(x_1,y_1,x_2,y_2\right)$ are coordinates on $\mathbb{R}^4$.

One of the key properties of contact structures is Darboux's theorem, which states that all contact structures are locally equivalent. Specifically, any contact structure in a neighborhood around a point can be transformed into the standard contact structure on $\mathbb{R}^{2n+1}$. This implies that, unlike symplectic structures, there are no local invariants for contact structures, making them locally "trivial". Namely we have the following well know Theorem :

**Theorem 2.4** (Darboux's theorem). About each point $p$ of a contact manifold ( $M^{2n+1},\alpha\Big)$ there exists local coordinates $\left(x_1,\cdots,x_n,y_1,\cdots,y_n,z\right)$ around $p$ such that the 1 -forms can be expressed as follows:

$$\alpha=dz-\sum_{i=1}^{n}y_i dx_i.$$

Even contact structure is a generalization of contact structures to even-dimensional manifolds. Like contact structures, even contact structures describe non-integrable distributions, but they exist on even-dimensional manifolds and exhibit certain symplectic-like properties when restricted to subspaces of the tangent bundle.

**Definition 2.5.** Let $M$ be a $\left(2n+2\right)$-dimensional manifold and $\mathcal{E}$ a codimension one distribution of $M$. The distribution $\mathcal{E}$ is an even contact structure on $M$ if for every local 1-form $\eta$, with $\mathcal{E}=\ker\eta$, the $\left(2n+1\right)$-form $\eta\wedge d\eta^n\neq 0$.

The pair $\left(M,\eta\right)$ is called an even contact manifold and the above 1-form is called an even contact form or even contact structure.

To each even contact structure one can associate a line field $\mathcal{L} \subset \mathcal{E}$, called its isotropic foliation or kernel foliation, which is the kernel of $d\alpha$ restricted to $\xi$. When $n = 2$, the even contact condition is equivalent to $[\mathcal{E}, \mathcal{E}] = TM$.

**Proposition 2.6.** Let $M$ be a $2n+2$-dimensional manifold, and let $\alpha$ be a 1-form. Define a distribution $\mathcal{E}$ as $\mathcal{E} = \ker(\alpha)$. The followings are equivalent.
1. The distribution $\mathcal{E}$ is an even contact structure.
2. A $2n$-form $(d\alpha)^n_{|\mathcal{E}}$ on a distribution $\mathcal{E}$ does not vanish anywhere.
3. The rank of a linear map $\mathcal{E} \to \mathcal{E}^\star, X \mapsto d\alpha$ *(X,.),* is $2n$.

*Proof. Same as Proposition 2.2 .*

We note that $\mathcal{E}$ has dimension $2n+1$, then the rank of $d\eta_{|\mathcal{E}}$ is $2n$. Hence $d\eta_{|\mathcal{E}}$ has a kernel $Z$ that belongs to $\mathcal{E}$. Furthermore for any $f \in \mathcal{C}^\infty$, $\ker d(f\eta)_{|\mathcal{E}} = \ker f(d\eta)_{|\mathcal{E}}$ thus the line fiels $W$ defined by the kernel of $(2n+1)$-forms $\eta \wedge (d\eta)^n$ do not depend on the choice of a local defining form $\eta$ for $\mathcal{E}$. Hence, we have the following definition inspired in [?].

**Definition 2.7.** [16] The line field $W$ is the characteristic line field for $\mathcal{E}$. The foliation

$$\mathcal{F} = \ker\left(\eta \wedge d\eta^n\right) = \langle W \rangle$$

induced by this line field is called the characteristic foliation.

For an explicit construction, we refer the reader to see ([4]).

**Proposition 2.8.** [10] Let $\left(M^{2m+2}, \mathcal{E} = \ker\eta\right)$ be an even contact structure with orientable characteristic foliation $\mathcal{F} = \ker\left(\eta \wedge (d\eta)^m\right)$, then the following are equivalent:

1. $\mathcal{F}$ has volume-preserving holonomy;
2. $\mathcal{F}$ is the kernel of a closed $(2n+1)$-form;
3. $\eta$ can be chosen so that d $\eta$ has constant rank $2m$;
4. there exists a vector field $Z \in \hat{\mathrm{I}}\ (M)$ transverse to $\mathcal{E}$ whose flow preserves $\mathcal{E}$.

**Proposition 2.9.** [14] If $\mathcal{E}$ is an even contact structure in $M$ and if $H$ is a hypersurface in $M$ transverse to $\mathcal{E}$, then the plane field $\zeta$ given by $\zeta = TH \cap \mathcal{E}$ is a contact structure on $H$.

**Example 2.10.** Let $(N, \zeta)$ be a contact manifold and $\phi$ the coordinate on $[0,1]$. The hyperplane field $\mathcal{E} = \zeta \oplus \mathbb{R}\left(\partial_\phi + Z\right)$ in $N \times [0,1]$ is then an even contact structure whose isotropic foliation is spanned by $\partial_\phi + Z$ if and only if $Z$ is a contact vector field (Reeb vector field) in $(N, \zeta)$.

**Example 2.11.** Let $Z$ be a contact vector field in the contact manifold ( $N, \zeta = X_1, X_2$ ). For a sufficiently big positive integer *n*, consider the plane field $\mathcal{D}$ spanned by

$$W = \partial_\phi + Z \text{ and } X_n = \cos(n\phi) X_1 + \sin(n\phi) X_2$$

The distribution $\mathcal{E}$ given by

$$\mathcal{E}_n = [\mathcal{D}_n, \mathcal{D}_n]$$

is an even contact structure on $N \times [0,1]$. In particular, the isotropic foliation is spanned by $W$.

## 2.2. Engel manifolds

**Definition 2.12.** Let $M$ be a 4-dimensional manifold. A distribution of rank 2 or a 2-plane field $\mathcal{D}$ on $M$ is a distribution of 2-dimensional tangent planes $\mathcal{D}_p \subset T_p M$ for any $p \in M$.

It is considered as a rank 2 sub bundle of the tangent bundle $TM$. We can think $\mathcal{D}$ of as a locally free sheaf of smooth vector fields on $M$. Let $[\mathcal{D}, \mathcal{D}]$ denote the sheaf generated by all Lie brackets [ $X, Y$ ] of vector fields $X, Y$ on $M$, which are cross sections of $\mathcal{D}$. We set

$$\mathcal{D}^2 = D + [D, D] \text{ and } \mathcal{D}^3 := \mathcal{D}^2 + \left[\mathcal{D}^2, \mathcal{D}\right] \#(2.1)$$

The Engel structure is defined as follows.

**Definition 2.13.** A distribution $\mathcal{D}$ of rank 2 on a 4 -dimensional manifold $M$ is called an Engel structure if it satisfies,

$$\dim \mathcal{D}_p^2 = 3 \quad \text{and} \quad \dim \mathcal{D}_p^3 = 4 \ \#(2.2)$$

at any point $p \in M$. The pair $(M, \mathcal{D})$ is called Engel manifold.

**Example 2.14.** Consider $\mathbb{R}^4$ with coordinates $(\omega, x, y, z)$ and the following distribution $\mathcal{D}$ givent by

$$\mathcal{D} = \langle \partial_\omega, \omega\partial_x + x\partial_z + \partial_y \rangle = \langle X, Y \rangle \quad \text{where} \quad X = \partial_\omega \quad \text{and} \quad Y = \omega\partial_x + x\partial_z + \partial_y.$$

Using 2.1 one can get easily

$$\mathcal{D}^2 = \langle X, Y, \partial_x \rangle, \qquad \text{and} \qquad \mathcal{D}^3 = \langle X, Y, \partial_x, \partial_z \rangle.$$

$\mathcal{D}$ is an Engel structure. It is the standard Engel distribution on $\mathbb{R}^4$.

The following definition gives a link between Engel structre and Even contact structure.

**Definition 2.15.** [5] An Engel structure $\mathcal{D}$ is a smooth plane field on a 4-manifold $M$ such that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ is an even contact structure.

Let $(M, \mathcal{D})$ be an Engel manifold. By the definition and the equation in 2.1, we see that, the distribution $\mathcal{E} = \mathcal{D}^2$ is an even contact structure on $M$. See [16] for more details.

Notice that if an even contact structure $\mathcal{E}$ is induced by an Engel structure, that is $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$, then $\mathcal{L}$ is tangent to $\mathcal{D}$. Thus, an Engel structure $\mathcal{D}$ induces a flag of distributions

$$\mathcal{L} \subset \mathcal{D} \subset \mathcal{E} \subset TM.$$

Moreover, if $\mathcal{E}$ comes from an Engel structure, then it is canonically oriented, and an orientation of $TM$ induces an orientation of $\mathcal{L}$ and vice versa. If we assume that both $\mathcal{D}$ and $M$ areoriented, then this shows that $\mathcal{D}$ induces a framing of $TM$ which is well defined up to homotopy. In particular, two Engel structures cannot be homotopic through Engel structures if the associated framings are not homotopic.

The fact that there is a line field associated to an even contact structure/Engel structure means that Gray's stability theorem cannot hold for these distributions: a varying family of even contact structures or Engel structures will induce a varying family of isotropic foliations. Hence, the even contact structures/Engel structures in the family are not diffeomorphic to each other since dynamical properties of the isotropic foliation can change. See for details [5].

## 2.3. Engel defining forms and Reeb distribution

**Definition 2.16.** [16] Two non-singular 1-forms $\alpha, \beta$ are called Engel defining forms if they verify the following properties:

$$\alpha \wedge d\alpha \neq 0$$

$$\alpha \wedge \beta \wedge d\alpha = 0 \#(2.5)$$

$$\alpha \wedge \beta \wedge d\beta \neq 0$$

The distribution $\mathcal{D} = \ker(\alpha \wedge \beta)$ is an Engel distribution on $M$.

The condition $\alpha \wedge d\alpha \neq 0$ ensures that $\mathcal{E} = \ker\alpha$ is an even contact structure; denote with $\mathcal{L} = \ker(\alpha \wedge d\alpha)$ its characteristic foliation. The condition $\alpha \wedge \beta \wedge d\beta \neq 0$ implies that ker $\beta$ is an even contact structure, and it ensures that its characteristic foliation is transverse to $\mathcal{E}$. Finally the equation $\alpha \wedge \beta \wedge d\beta = 0$ implies that $\mathcal{L} \subset \ker\beta$. Namely we have :

**Proposition 2.17.** Let $M$ be a 4-manifold and $(\alpha, \beta)$ be a pair of Engel form. Then we have:

a) $\mathcal{E} = \ker(\alpha)$ is an even contact structure with characteristic foliation $\mathcal{L} = \ker(\alpha \wedge d\alpha)$.

b) $\mathcal{E}' = \ker(\beta)$ is an even contact structure with caracteristic foliation $\mathcal{L}'$. $\mathcal{L} \subset \mathcal{E}'$ and $\mathcal{L}'$ transverse to $\mathcal{E}$.

Proof. The first part of *a*) is trivial by (2.3). Additionnaly we have

$$\mathcal{L} = \ker\left[\mathcal{E} \to \mathcal{E}^\star \;\; v \mapsto d\alpha(v, .)\right]$$

If $X_0 \in \mathcal{L}$ then $i_{X_0}(\alpha \wedge d\alpha) = \alpha(X_0)d\alpha - \alpha \wedge i_{X_0}d\alpha$ or $X_0 \in \mathcal{E}$ et $i_{X_0}d\alpha = 0$ then

$$i_{X_0}(\alpha \wedge d\alpha) = 0 \Rightarrow X_0 \in \ker(\alpha \wedge d\alpha)$$

Conversely if $i_{X_0}(\alpha \wedge d\alpha) = 0$ for $X_0 \in \mathcal{E}$. Then we have $i_{X_0}d\alpha \wedge \alpha = 0$ that is $i_{X_0}d\alpha$ is a multiple of $\alpha$. There exists $h : M \to \mathbb{R}$ such that $i_{X_0}d\alpha = h\alpha$. Therefore

$$\forall X \in \mathcal{E}, i_{X_0}d\alpha(X) = 0 \Rightarrow X_0 \in \mathcal{L}.$$

b) since (2.5) we have $\beta \wedge d\beta \neq 0$ . If $\mathcal{L}' = \ker(\beta \wedge d\beta)$ then for $W \in \mathcal{L}'$ , we have

$$
\begin{aligned}
i_W\left(\alpha \wedge \beta \wedge d\beta\right) \neq 0, \text{ by5)} \quad & \Rightarrow \alpha(W)\beta \wedge d\beta - \alpha \wedge i_W\left(\beta \wedge d\beta\right) \neq 0 \\
& \Rightarrow \alpha(W)(\beta \wedge d\beta) \neq 0 \\
& \Rightarrow \alpha(W) \neq 0 \\
& \Rightarrow W \notin \mathcal{E}.
\end{aligned}
$$

So $\mathcal{L}'$ doesn't intersect $\mathcal{E}$ . We use the same proof by the condition of (2.5) for showing that $\mathcal{L} \subset \mathcal{E}'$ .

**Definition 2.18.** Let $(M, \mathcal{D} = \ker(\omega_1 \wedge \omega_2))$ be an Engel manifold. The Reeb distribution associated to the forms $\omega_1$ and $\omega_2$ is $\mathcal{R} = Z_1$ and $Z_2$ satisfies the following conditions:

$$
\begin{aligned}
& i_{Z_2}\left(\omega_1 \wedge d\omega_2\right) = 0, \quad \omega_2\left(Z_2\right) = 1, \quad \omega_1\left(Z_2\right) = 0 \\
& i_{Z_1}\left(\omega_2 \wedge d\omega_2\right) = 0, \quad \omega_2\left(Z_1\right) = 0, \quad \omega_1\left(Z_1\right) = 1
\end{aligned} \#(2.7)
$$

The two following propositions will link some propreties between Engel structures and Engel defining forms.

**Proposition 2.19.** [14], [15] Let $M^4$ be a parallelizable smooth manifold and $\omega_1, \omega_2$ two 1-forms.

If $\omega_1$ and $\omega_2$ are Engel defining forms then the plane field $\mathcal{D} = \ker\omega_1 \wedge \omega_2$ is an orientable Engel structure. Conversely if $\mathcal{D}$ is an orientable Engel structure there exist Engel defining forms $\eta$ and $\omega$ such that $\mathcal{D} = \ker(\omega_1 \wedge \omega_2)$ .

**Proposition 2.20.** [13] Let $(M, \mathcal{D} = \ker\omega_1 \wedge \omega_2)$ be a 4-manifold with Engel structure $\mathcal{D} = \ker\omega_1 \wedge \omega_2$ of Engel defining forms $\omega_1$ and $\omega_2$ whose Reeb distribution $\mathcal{R} = \langle Z_1, Z_2 \rangle$. Then the Reeb distribution $\mathcal{R}$ is given by

$$
R = ker\left\{d\omega_2 + C_{Z_1.Z_2}\omega_1 \wedge \omega_2\right\} \text{ with} C_{Z_1.Z_2} = d\omega_2\left(Z_1, Z_2\right) \#(2.8)
$$

and it is integrable if and only if

$$
d\left(C_{Z_1.Z_2}\omega_1\right) \wedge \omega_2 = 0 \#(2.9)
$$

For more details about Even contact structure and Engel structure, see ([11], [13],[15],[15]).

## 3. Old results about Linear deformations

In this section, we recall the notion of deformation of codimension 1 foliation into contact structure.

**Definition 3.1.** Let $M$ be a smooth manifold. A foliation $\mathcal{F}$ of codimension 1 on $M$ is deformable into contact structures if there exist a one parameter family $\{\mathcal{F}_t\}_{t \geq 0}$ of hyperplane fields satisfying $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_t$ is a contact distribution, for all $t > 0$ .

Now, suppose $\mathcal{F}$ is defined by a closed 1-form $\alpha_0$ . Following [7], we say that a deformation $\mathcal{F}_t$ of $\mathcal{F}$ is linear if there is a 1-parameter family of 1-forms of the form: $\alpha_t = \alpha_0 + t\alpha$ , where $\alpha$ is a fixed 1-form on $M$ with $\mathcal{F}_t = \ker(\alpha_t)$ . We will say that $\alpha_t$ is a linear deformation of $\alpha_0$ via $\alpha$ .

The following result was proven by Dathe and Rukimbira in [7.]

**Theorem 3.2.** 77 Let $M$ be a closed $(2n+1)$-dimensional manifold, let $\alpha_0$ be a closed 1 -form and $\alpha$ a 1 -form on $M$ . Then, the following conditions are equivalent.

(i) The linear deformation $\alpha_t = \alpha_0 + t\alpha$ of $\alpha_0$ is a contact 1-form for all $t > 0$ .

(ii) The 1 -form $\alpha$ is contact and $\alpha_0(Z_\alpha) = 0$ , where $Z_\alpha$ is the Reeb vector field of $\alpha$ .

Now let us given the notion of contact pair structure in the sens of Hadjar and Bande in sens of G. Bande and A. Hadjar as defined in [2].

**Definition 3.3.** [2] Let $M$ be an even-dimensional manifold. A pair ( $\alpha, \beta$ ) of 1-forms is called contact pair on $M$ of type $(k, l)$ if $\alpha \wedge (d\alpha)^k \wedge \beta \wedge (d\beta)^l$ is a volum form on $M$ and

$$
(d\alpha)^{k+1} = (d\beta)^{l+1} = 0.
$$

For any contact pair forms $(\alpha, \beta)$ on $M$ , it corresponds a pair $(Z_\alpha, Z_\beta)$ of commuting vector fiels uniquely determined by the relations

$$
\begin{cases}
\alpha\left(Z_\alpha\right) = \beta\left(Z_\beta\right) = 1 \text{ and } \alpha\left(Z_\beta\right) = \beta\left(Z_\alpha\right) = 0 \\
i_{Z_\alpha}d\alpha = i_{Z_\alpha}d\beta = 0 \text{ and } i_{Z_\beta}d\alpha = i_{Z_\beta}d\beta = 0
\end{cases} \#(3.1)
$$

Following H. Dathe and P. Rukimbira in [7], we have the following definition in [12].

**Definition 3.4.** Let $\alpha_0$ and $\beta_0$ be two closed 1-forms on $M$ that are linearly independent at every point $m \in M$. A linear deformation of $(\alpha_0, \beta_0)$ is a pair $(\alpha_t, \beta_t)$ of 1-forms of the type:

$$\alpha_t = \alpha_0 + t\alpha \text{ and } \beta_t = \beta_0 + t\beta \ \forall t \geq 0 \#(3.2)$$

for some $\alpha$ and $\beta \in \Omega^1(M)$.

The following result is proved by the authors of ([12]).

**Theorem 3.5.** [12] Let $M$ be a closed and oriented even-dimensional manifold and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations in $M$ defined by two linearly independent closed 1-forms $\alpha_0$ and $\beta_0$, respectively. Consider the linear deformation $(\alpha_t, \beta_t)$ given by:

$$\alpha_t = \alpha_0 + t\alpha \quad and \quad \beta_t = \beta_0 + t\beta \quad \forall t \geq 0. \#(3.2)$$

for some $\alpha$ and $\beta \in \Omega^1(M)$. Then the couple $(\alpha_t, \beta_t)$ is a contact pair of type $(k, l)$ for $t > 0$ whose associated pair of Reeb vector fields has the form $\left(\frac{1}{t}X, \frac{1}{t}Y\right)$ (where $X$ and $Y$ are vector fields on $M$) if and only if $(\alpha, \beta)$ is a contact pair of type $(k, l)$ whose pair of Reeb vector fields $(Z_\alpha, Z_\beta)$ satisfies the compatibility conditions

$$\alpha_0(Z_\alpha) = \alpha_0(Z_\beta) = \beta_0(Z_\alpha) = \beta_0(Z_\beta) \#(3.3)$$

## 4. Main results

### 4.1. Linear deformations of closed 1-forms into even contact forms

**Definition 4.1.** Let $M$ be a $(2m+2)$-dimensional smooth manifold endowed with a non-singular closed 1-form $\eta$. A linear deformation of $\eta$ into an even contact structure is given by a one parameter family $\eta_t$ of the type

$$\eta_t = \eta + t\omega \quad \forall\, t \geq 0, \quad \text{where } \omega \text{ is an even contact form on } M \ \#(4.1)$$

and for all $t > 0, \eta_t$ is an Even contact structure on $M$.

We will say that $\eta$ is a linear deformable into even contact structure via $\omega$.

**Theorem 4.2.** Let $(M, \omega)$ be a $(2m+2)$-dimensional oriented even contact manifold admitting an orientable characteristic foliation $\mathcal{L} = \ker\left(\Omega \wedge (d\Omega)^m\right)$ with volume-preserving holonomy, such that $M$ is foliated by closed hypersurfaces transverses to $\mathcal{L}$. Then for any nonvanishing closed 1 -form $\eta$ on $M$, the following are equivalents.

1. The family of 1-forms $(\eta_t)_{t\geq 0}$ defined by $\eta_t = \eta + t\omega$ in a linear deformation of $\eta$ is a family of even contact structures for any $t > 0$.
2. $\eta \wedge (d\omega)^m = 0$.

Proof. Under the assumptions of Theorem 4.2 we have

$$\eta_t \wedge (d\eta_t)^m = t^m \left[\eta \wedge (d\omega)^m + t\omega \wedge (d\omega)^m\right] .\#(4.2)$$

If $\eta \wedge (d\omega)^m = 0$ then for all $t > 0, \eta_t$ is an even contact structure on $M$.

Conversely suppose, suppose for all $t > 0, \eta_t$ is an even contact structure on $M(\eta_t \wedge d\eta_t \neq 0)$. Then $\eta_t$ is a contact structure on any closed oriented transverse hypersurface $H$ to the characteristic foliation $\mathcal{F}$. Therefore $\eta_t \wedge (d\eta_t)^m_{|H} > 0$ and this implies that $\left[\eta \wedge (d\omega)^m + t\omega \wedge (d\omega)^m\right]_{|H} > 0$. Hence, from Dathe and Rukimbira (see [7] ), $\eta_t$ is a linear deformation of $\eta$ if $\eta_{|H}(Z) = 0$ where $Z$ is the Reeb vector field of $\omega_{|H}$ moreover $\eta \wedge (d\omega)^m_{|H} = 0$ on every such hypersurface $H$. Since these hypersurfaces foliated $M$, then $\eta \wedge (d\omega)^m = 0$ on $M$.

Let us given an application of Theorem 4.2.

**Example 4.3.** Consider the 6-dimensional Lie algebra $n_6^{12}$ given by

$$[X_1, X_6] = X_5, \qquad [X_1, X_5] = X_4 \qquad \text{and} \qquad [X_2, X_3] = X_4. \#(4.3)$$

From the equation (4.3), the structural equations are given by:

$$d\lambda_5 = \lambda_1 \wedge \lambda_6, \quad d\lambda_4 = \lambda_1 \wedge \lambda_5 = \lambda_2 \wedge \lambda_3, \quad d\lambda_1 = d\lambda_2 = d\lambda_3 = d\lambda_6 = 0. \#(4.4)$$

By easy cumputation, we see that the pair $(\lambda_4,\lambda_6)$ defines a contact pair structure of type $(2,0)$ and in this case $\lambda_4$ is an even contact structure on $n_6^{12}$ since $\lambda_4\wedge d\lambda_4^2\neq 0$ and the characteristic foliation is given by $\mathcal{F}=X_6$.

The 1-forms $\lambda_1,\lambda_2$ and $\lambda_3$ define the foliations $\mathcal{F}_{\lambda_1},\mathcal{F}_{\lambda_2}$ and $\mathcal{F}_{\lambda_3}$ respectively and morever

$$\lambda_1\wedge d\lambda_4=\lambda_2\wedge d\lambda_4=\lambda_3\wedge d\lambda_4=0 \text{ and } \lambda_1(X_6)=\lambda_2(X_6)=\lambda_3(X_6)=0.$$

Hence each foliations $\mathcal{F}_{\lambda_1},\mathcal{F}_{\lambda_2}$ and $\mathcal{F}_{\lambda_3}$ is linearly deformable into even contact structures via $\lambda_4$ and these characteristic foliation spanned by $\langle \frac{1}{t}X_6\rangle\ \forall t>0.$

## 4.2. Linear deformations of pair of closed non-singular 1-forms into Engel defining forms

**Definition 4.4.** Let $M$ be a 4-dimensional smooth manifold endowed with an Engel distribution $\mathcal{D}=\ker(\omega_1\wedge\omega_2)$ with Engel defining form $\omega_1$ and $\omega_2$. A linear deformation of a pair of codimension 1 foliations $(\mathcal{F}_1,\mathcal{F}_2)$ defining by a pair $(\eta_1,\eta_2)$ of non-singular closed 1-forms is a pair of 1-parameter family $\{(\eta_1^t,\eta_2^t)\}_{t\geq 0}$ of the type:

$$\eta_1^t=\eta_1+t\omega_1 \text{ and } \eta_2^t=\eta_2+t\omega_2 \#(4.5)$$

and for all $t>0,(\eta_1^t,\eta_2^t)$ is couple of Engel defining forms.

We will say that the pair of foliations $(\mathcal{F}_1,\mathcal{F}_2)$ defined by the pair of integrable 1-forms $(\eta_1,\eta_2)$ is linear deformable via the pair of Engel defining forms $(\omega_1,\omega_2)$.

Using the notations of the Definition 4.4 and we get, by simple calculations, the following equalities:

$$\eta_1^t\wedge d\eta_1^t=t[\eta_1\wedge d\omega_1+t\omega_1\wedge d\omega_1]\#(4.6)$$

$$\eta_1^t\wedge\eta_2^t\wedge d\eta_1^t=t\left[\eta_1\wedge\eta_2\wedge d\omega_1+t(\omega_1\wedge\eta_2\wedge d\omega_1+\omega_2\wedge\eta_1\wedge d\omega_1)+t^2\omega_1\wedge\omega_2\wedge d\omega_1\right]\#(4.7)$$

$$\eta_1^t\wedge\eta_2^t\wedge d\eta_1^t=t\left[\eta_1\wedge\eta_2\wedge d\omega_2+t(\omega_1\wedge\eta_2\wedge d\omega_2+\omega_2\wedge\eta_1\wedge d\omega_2)+t^2\omega_1\wedge\omega_2\wedge d\omega_2\right]\#(4.8)$$

For simplifying these expressions, we need the following three lemmas.

**Lemma 4.5.** By the above notations and assumptions we have:

$$i_{Z_2}d\omega_2=-C_{Z_1\cdot Z_2}\omega_1 \text{ and } i_{Z_1}d\omega_2=C_{Z_1.Z_2}\omega_2 \text{ with } C_{Z_1.Z_2}=d\omega_2(Z_1,Z_2)\#(4.9)$$

Proof. From the Definition 2.18, we have $i_{Z_2}(\omega_1\wedge d\omega_2)=0\Rightarrow\omega_1\wedge i_{Z_2}d\omega_2=0$. Hence we have $i_{Z_1}(\omega_1\wedge i_{Z_2}d\omega_2)=0\Rightarrow i_{Z_2}d\omega_2-\omega_1 d\omega_2(Z_2,Z_1)=0$ then $i_{Z_2}d\omega_2=-C_{Z_1.Z_2}\omega_1$. By the same way we show that $i_{Z_1}d\omega_2=C_{Z_1.Z_2}\omega_2$.

**Lemma 4.6.** Let $(M^4,\ker(\omega_1\wedge\omega_2))$ be an Engel manifold with a pair $(\omega_1,\omega_2)$ of defining Engel forms and Reeb distribution $\mathcal{R}=\langle Z_1,Z_2\rangle$, then for any 1-forms $\delta\in\Omega^1(M)$ we have:

$$\delta\wedge\omega_1\wedge d\omega_2=-\delta(Z_2)\Omega \text{ and } \delta\wedge\omega_2\wedge d\omega_2=\delta(Z_1)\Omega \text{ with } \Omega=\omega_1\wedge\omega_2\wedge d\omega_2\#(4.10)$$

Proof. The technique is to take the interior produit of a 5-forms by the vector field $Z_1$ or $Z_2$ and using the equations in the Definition (2.18).

Let $\delta\in\Omega^1(M)$. The pair of Engel forms satisfies the equations (2.3), (2.4) and (2.5). We have $\delta\wedge\Omega=0$.

$$\begin{aligned}0=i_{Z_1}(\delta\wedge\Omega\} &=(i_{Z_1}\delta)\Omega-\delta\wedge i_{Z_1}\Omega\\ &=\delta(Z_1)\Omega-\delta\wedge\omega_2\wedge d\omega_2+\delta\wedge\omega_1\wedge i_{Z_1}(\omega_2\wedge d\omega_2)\\ &=\delta(Z_1)\Omega-\delta\wedge\omega_2\wedge d\omega_2\\ 0=i_{Z_2}(\delta\wedge\Omega\} &=(i_{Z_2}\delta)\Omega-\delta\wedge i_{Z_2}\Omega\\ &=\delta(Z_2)\Omega-\delta\wedge\omega_1\wedge d\omega_2+\delta\wedge\omega_2\wedge i_{Z_2}(\omega_1\wedge d\omega_2)\\ &=\delta(Z_2)\Omega+\delta\wedge\omega_1\wedge d\omega_2.2\end{aligned}$$

Then we get the result.

**Proposition 4.7.** With assumption of Lemma 4.6, we have:

$$\begin{cases} \omega_2 \wedge \eta_1 \wedge d\omega_2 + \eta_1 \wedge \omega_2 \wedge d\omega_2 = (\eta_1(Z_1) + \eta_2(Z_2))\Omega \\ \eta_1 \wedge \eta_2 \wedge d\omega_2 = \eta_1 \wedge \eta_2, Z_1 \wedge Z_2 \Omega + C_{Z_1.Z_2} \eta_1 \wedge \eta_1 \wedge \omega_1 \wedge \omega_2 \end{cases} \#(4.11)$$

Proof. The first point of the above system follows from the first equation of Lemma 4.6, $\delta = \eta_1$ then $\delta = \eta_2$. For the second one, observe that $\omega_1 \wedge \eta_1 \wedge \eta_2 \wedge d\omega_2 = 0$ and then by contracting with $Z_1$, we have:

$$i_{Z_1}(\omega_1 \wedge \eta_1 \wedge \eta_2 \wedge d\omega_2) = 0$$

By developping ang using the Lemma 4.5, one has the following equation:

$$\omega_1(Z_1)\eta_1 \wedge \eta_2 \wedge d\omega_2 - \eta_1(Z_1)\omega_1 \wedge \eta_2 \wedge d\omega_2 + \eta_2(Z_1)\omega_1 \wedge \eta_1 \wedge d\omega_2 + C_{Z_1.Z_2}\eta_1 \wedge \eta_2 \wedge \omega_1 \wedge \omega_2 = 0 . \#(4.12)$$

Using the equations of Lemma 4.6, the equation (4.12) becomes

$$\eta_1 \wedge \eta_2 \wedge d\omega_2 - \eta_1(Z_1)\eta_2(Z_2)\Omega + \eta_2(Z_1)\eta_1(Z_2)\Omega - C_{Z_1.Z_2}\eta_1 \wedge \eta_2 \wedge \omega_1 \wedge \omega_2 = 0 \#(4.13)$$

Hence it follows that

$$\eta_1 \wedge \eta_2 \wedge d\omega_2 = \eta_1 \wedge \eta_2, Z_1 \wedge Z_2 \Omega + C_{Z_1.Z_2}\eta_1 \wedge \eta_1 \wedge \omega_1 \wedge \omega_2 \#(4.14)$$

Corollary 4.8. Let $(M^4, \mathcal{D} = (\omega_1 \wedge \omega_2))$ be an Engel manifold with Engel defining forms $\omega_1$, $\omega_2$ and Reeb distribution $\mathcal{R} = Z_1, Z_2$. Then any pair of closed 1-formes $(\eta_1, \eta_2)$ on $M$, and for all $t > 0$, the linear deformation $(\eta_1^t, \eta_2^t)$ of $(\eta_1, \eta_2)$ via $(\omega_1, \omega_2)$ satisfies the following system:

$$\begin{cases} \eta_1^t \wedge d\eta_1^t = t(\eta_1 \wedge d\omega_1 + t\omega_1 \wedge d\omega_1) \\ \eta_1^t \wedge \eta_2^t \wedge d\eta_1^t = t[\eta_1 \wedge \eta_2 \wedge d\omega_1 + t(\omega_1 \wedge \eta_2 \wedge d\omega_1 + \omega_2 \wedge \eta_1 \wedge d\omega_1)] \\ \eta_1^t \wedge \eta_2^t \wedge d\eta_2^t = t[(g + t(h+t))\Omega + C_{Z_1.Z_2}\eta_1 \wedge \eta_2 \wedge \omega_1 \wedge \omega_2] \end{cases} \#(4.15)$$

*with* $g = \langle \eta_1 \wedge \eta_2, Z_1 \wedge Z_2 \rangle = \eta_1(Z_1)\eta_2(Z_2) - \eta_1(Z_2)\eta_2(Z_1)$ *and* $h = \eta_1(Z_1) + \eta_2(Z_2)$.

If we fix the generic condition

$$\begin{cases} \eta_1 \wedge d\omega_1 = 0 \\ \eta_2 \wedge d\omega_1 = 0 \end{cases}$$

then system 4.15 becomes:

$$\begin{cases} \eta_1^t \wedge d\eta_1^t = t^2 \omega_1 \wedge d\omega_1 \\ \eta_1^t \wedge \eta_2^t \wedge d\eta_1^t = 0 \\ \eta_1^t \wedge \eta_2^t \wedge d\eta_2^t = t[(g + t(h+t))\Omega + C_{Z_1.Z_2}\eta_1 \wedge \eta_2 \wedge \omega_1 \wedge \omega_2] \end{cases} \cdot \#(4.16)$$

with $g = \langle \eta_1 \wedge \eta_2, Z_1 \wedge Z_2 \rangle = \eta_1(Z_1)\eta_2(Z_2) - \eta_1(Z_2)\eta_2(Z_1)$ and $h = \eta_1(Z_1) + \eta_2(Z_2)$.

Therefore, we get the following theorem.

**Theorem 4.9.** Let $M$ be a closed and oriented 4-manifold endowed with an Engel structure $\mathcal{D} = \ker(\omega_1 \wedge \omega_2)$ with Engel defining forms $\omega_1, \omega_2$ and Reeb distribution $\mathcal{R} = \langle Z_1, Z_2 \rangle = \ker(d\omega_2 + c_{Z_1 Z_2}\omega_1 \wedge \omega_2)$. Suppose there exists a pair of closed non-singular 1-form $(\eta_1, \eta_2)$ satisfying

$$\begin{cases} \eta_1 \wedge d\omega_1 = 0 \\ \eta_2 \wedge d\omega_1 = 0 \end{cases}$$

then for all $t>0$, the pair $(\eta_1^t,\eta_2^t)$ in an Engel deformation of $(\eta_1,\eta_2)$ in the way of $(\omega_1,\omega_2)$ are Engel defining forms if and only if

$$\begin{cases}\eta_1(Z_1)+\eta_2(Z_2)\geq 0\\ \langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle = 0.\end{cases}$$

Proof. Let us remark that, for the 4-form $\eta_1\wedge\eta_2\wedge\omega_1\wedge\omega_2$ is a volum form on $M$, then there exists a function $K\in C^\infty(M)$ suth that $\omega_1\wedge\omega_2\wedge\eta_1\wedge\eta_2) = K\omega_1\wedge\omega_2\wedge d\omega_2$.

$$\eta_1\wedge\eta_2\wedge\omega_1\wedge\omega_2(Z_1,Z_2,Z_1,Z_2)-\eta_1\wedge\eta_2\wedge\omega_1\wedge\omega_2(Z_1,Z_2,Z_2,Z_1)$$
$$= K\omega_1\wedge\omega_2\wedge d\omega_2(Z_1,Z_2,Z_1,Z_2)-\omega_1\wedge\omega_2\wedge d\omega_2(Z_1,Z_2,Z_2,Z_1)$$
$$=\eta_1(Z_1)\eta_2(Z_2)-\eta_1(Z_2)\eta_2(Z_1)=K\left[d\omega_2(Z_1,Z_2)-d\omega_2(Z_2,Z_1)\right]=2KC_{Z_1Z_2}$$

Hence, we obtain $KC_{Z_1Z_2}=\frac{1}{2}\langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle$ and therefore the system 4.16 becomes

$$\begin{cases}\eta_1^t\wedge d\eta_1^t &= t^2\omega_1\wedge d\omega_1\\ \eta_1^t\wedge\eta_2^t\wedge d\eta_1^t &= 0\\ \eta_1^t\wedge\eta_2^t\wedge d\eta_2^t &= t\left[\frac{3}{2}\langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle + t(\eta_1(Z_1)+\eta_2(Z_2))+t^2\right]\Omega.\end{cases} \#(4.17)$$

It suffises to prove that $\eta_1^t\wedge\eta_2^t\wedge d\eta_2^t>0 \Leftrightarrow \begin{cases}\eta_1(Z_1)+\eta_2(Z_2)\geq 0\\ \eta_1\wedge\eta_2, Z_1\wedge Z_2 = 0.\end{cases}$

If we suppose that $\eta_1^t\wedge\eta_2^t\wedge d\eta_2^t>0$ then for all $t>0$, one has

$$\frac{3}{2}\langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle + t(\eta_1(Z_1)+\eta_2(Z_2))+t^2>0 \quad \#(4.18)$$

and when $t$ goes to 0, we have by continuity also that $\langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle\geq 0$. Furthermore $\eta_1\wedge\eta_2\wedge d\omega_2 = \langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle\Omega = d(\eta_1\wedge\eta_2\wedge\omega_2)$. By Stock's theorem, since $M$ is closed we have

$$\int_M \eta_1\wedge\eta_2\wedge d\omega_2 = \int_M \langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle\Omega = \int_M d(\eta_1\wedge\eta_2\wedge\omega_2) = \int_{\partial M}\eta_1\wedge\eta_2\wedge\omega_2 = 0.$$

It follows that $\langle \eta_1\wedge\eta_2, Z_1\wedge Z_2\rangle = 0$. Returning in the equation 4.18, we have for all $t>0$, $t(\eta_1(Z_1)+\eta_2(Z_2))+t^2>0$ which implies that $\eta_1(Z_1)+\eta_2(Z_2)+t>0$ and when again $t$ goes to zero, we finally obtain $\eta_1(Z_1)+\eta_2(Z_2)\geq 0$. The converse is trivial.

**Corollary 4.10.** Let $M$ be a closed and orientable 4-manifold endowed with a Engel structure $\mathcal{D}=\ker(\omega_1\wedge\omega_2)$ with Engel defining forms $\omega_1$ and $\omega_2$ and Reeb distribution

$$\mathcal{R} = \langle Z_1, Z_2\rangle = \ker(d\omega_2 + C_{Z_1Z_2}\omega_1\wedge\omega_2)$$

Suppose there exists a pair of closed non-singular 1-forms $(\eta_1,\eta_2)$ such that

$$\eta_1\wedge d\omega_1 = 0 \text{ and } \eta_2\wedge d\omega_1 = 0$$

If $\eta_1$ and $\eta_2$ are proportionals, then for all $t>0$, the pair $(\eta_1^t,\eta_2^t)$ is a pair of Engel defining forms if and only if

$$\eta_1(Z_1)+\eta_2(Z_2)\geq 0.$$

Proof. If the assumptions of the Corollary 4.10 is true then the 4 -form $\eta_1 \wedge \eta_2 \wedge \omega_1 \wedge \omega_2$ vanish and therefore the system (4.16) becomes:

$$\begin{cases} \eta_1^t \wedge d\eta_1^t = t^2 \omega_1 \wedge d\omega_1 \\ \eta_1^t \wedge \eta_2^t \wedge d\eta_1^t = 0 \\ \eta_1^t \wedge \eta_2^t \wedge d\eta_2^t = t^2\left[\left(\eta_1(Z_1)+\eta_2(Z_2)+t\right)\Omega\right] \end{cases}$$

So, it suffices to prove that $\eta_1^t \wedge \eta_2^t \wedge d\eta_2^t > 0 \Leftrightarrow \eta_1(Z_1)+\eta_2(Z_2)\geq 0$.

Suppose that $\eta_1^t \wedge \eta_2^t \wedge d\eta_2^t > 0$ then for all $t > 0$, $(\eta_1(Z_1)+\eta_2(Z_2)+t) > 0$ and when $t$ goes to 0, we have by continuity that $\eta_1(Z_1)+\eta_2(Z_2)\geq 0$. The converse is trivial.

**Corollary 4.11.** Let $M$ be a closed and orientable 4-manifold endowed with a Engel structure $\mathcal{D}=\ker(\omega_1 \wedge \omega_2)$ with Engel defining forms $\omega_1$ and $\omega_2$ and Reeb distribution $\mathcal{R}=\langle Z_1, Z_2\rangle$. Suppose there exist a pair of closed non-singular 1 -forms $(\eta_1,\eta_2)$ such that

$$\eta_1 \wedge d\omega_1 = \eta_2 \wedge d\omega_1 = 0$$

Then we obtain the following properties.

1. If the Reeb distribution is given by $\mathcal{R}=\ker(\eta_1 \wedge \eta_2)$ the pair of 1 -forms $(\eta_1,\eta_2)$ is deformable to the pair ( $\eta_1^t,\eta_2^t$ ) of Engel defining forms in the way of Definition 4.4
2. If for $i,j\in\{1,2\},\eta_i(Z_j)=0$, then for all $t>0$, the pair of 1 -forms $(\eta_1,\eta_2)$ is deformable to the pair $(\eta_1^t,\eta_2^t)$ of Engel defining forms in the way of Definition 4.4 and the Reeb distribution associted to $\eta_1^t$ and $\eta_2^t$ is given by $\mathcal{R}^t = \langle \frac{1}{t}Z_1, \frac{1}{t}Z_2\rangle$

Proof. 1. For the first part, let us remark that if $M$ is a closed and orientable 4-manifold endowed with an Engel structure $\mathcal{D}=\ker(\omega_1 \wedge \omega_2)$ with Engel defining forms $\omega_1,\omega_2$ and Reeb distribution $\mathcal{R} = (Z_1, Z_2) = \ker(d\omega_2 + C_{Z_1 Z_2}\omega_1 \wedge \omega_2)$ then:

$$\omega_1 \wedge \omega_2 \wedge (d\omega_2 + C_{Z_1 Z_2}\omega_1 \wedge \omega_2) = \Omega.$$

In that case, if the Reeb distribution is given by $\mathcal{R}=\ker(\eta_1 \wedge \eta_2)$, where $\eta_1,\eta_2$ are two closed non-singular 1-forms on $M$, then

$$\eta_1 \wedge \eta_2 = d\omega_2 + C_{Z_1 Z_2}\omega_1 \wedge \omega_2.$$

Therefore $C_{Z_1 Z_2} = \frac{1}{2}\langle \eta_1 \wedge \eta_2, Z_1 \wedge Z_2\rangle = 0$ and $\eta_1(Z_1)+\eta_2(Z_2)=0$.

For the second part, by setting $Z_1^t = \frac{1}{t}Z_1$ and $Z_2^t = \frac{1}{t}Z_2$, we obtain:

$$\eta_1(Z_1)=0 \text{ and } \omega_1(Z_1)=1 \Rightarrow \eta_1^t(Z_1^t)=\eta_1^t\left(\frac{1}{t}Z_1\right)=\frac{1}{t}\eta_1^t(Z_1)=1$$

$$\eta_1(Z_2)=0 \text{ and } \omega_1(Z_2)=0 \Rightarrow \eta_1^t(Z_2^t)=\eta_1^t\left(\frac{1}{t}Z_2\right)=\frac{1}{t}\eta_1^t(Z_2)=0$$

$$\eta_2(Z_1)=0 \text{ and } \omega_2(Z_1)=0 \Rightarrow \eta_2^t(Z_1^t)=\eta_2^t\left(\frac{1}{t}Z_1\right)=\frac{1}{t}\eta_2^t(Z_1)=0$$

$$\eta_2(Z_2)=0 \text{ and } \omega_2(Z_2)=1 \Rightarrow \eta_2^t(Z_2^t)=\eta_2^t\left(\frac{1}{t}Z_2\right)=\frac{1}{t}\eta_2^t(Z_2)=1$$

By this method, we also get

$$i_{Z_2^t}\left(\eta_1^t \wedge d\eta_2^t\right)=0 \text{ and } i_{Z_1^t}\left(\eta_2^t \wedge d\eta_2^t\right)=0.$$

For illustrate the results of the Theorem 4.9, let us study the following examples.

**Example 4.12.** On $\mathbb{R}^4$, the following forms

$$\alpha = dz - ydx \text{ and } \beta = dy - \omega dx$$

are Engel forms with are the local prototype of Engel structures. The Reeb distribution associated is guiven by $\mathcal{R} = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ Consider the following pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ defined the pair of 1-forms $(\eta_1, \eta_2)$ with $\eta_1 = \eta_2 = dx$.

Since $\begin{cases} \eta_1 \wedge d\alpha = \eta_2 \wedge d\beta = 0 \\ \eta_1\left(\frac{\partial}{\partial y}\right) = \eta_1\left(\frac{\partial}{\partial z}\right) = 0 \\ \eta_2\left(\frac{\partial}{\partial y}\right) = \eta_2\left(\frac{\partial}{\partial z}\right) = 0, \end{cases}$ then for $t > 0$,

the pair of 1-forms $(\eta_1^t, \eta_2^t)$ as defined in Definition 4.4 define a pair of Engel forms.

**Example 4.13.** On $\mathbb{R}^4$ with coordinates $(x, y, z, t)$ the 1 -forms given by

$$\omega_1 = dz - \cos(2\pi t)dx - \sin(2\pi t)dy \text{ and } \omega_2 = -\sin(2\pi t)dx + \cos(2\pi t)dy$$

define an Engel struture on $\mathbb{R}^4$ with associated Reeb distribution

$$Z_1 = \frac{\partial}{\partial z} \quad \text{and} \quad Z_2 = -\sin(2\pi t)\frac{\partial}{\partial x} + \cos(2\pi t)\frac{\partial}{\partial y}.$$

Consider the following pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ defined the pair of 1-forms $(\eta_1, \eta_2)$

with $\eta_1 = \eta_2 = dt$. Since $\begin{cases} \eta_1 \wedge d\omega_1 = \eta_2 \wedge d\omega_1 = 0 \\ \eta_1(Z_1) = \eta_1(Z_2) = 0 \\ \eta_2(Z_1) = \eta_2(Z_2) = 0. \end{cases}$ then for $t > 0$,

the pair of 1-forms $(\eta_1^t, \eta_2^t)$ as defined in Definition (4.4) defined a pair of Engel forms.

Example 4.14. Conside the four dimensional Lie algebra nil[4] given by

$$[X_1, X_4] = X_3 \text{ and } [X_1, X_3] = X_2$$

Then the structures equations are given by

$$d\lambda_1 = d\lambda_4 = 0, \qquad d\lambda_2 = \lambda_1 \wedge \lambda_3 \qquad d\lambda_3 = \lambda_1 \wedge \lambda_4. \quad \# (4.21)$$

Using the equation (4.21), we have

$$\begin{aligned} \lambda_2 \wedge d\lambda_2 &= \lambda_2 \wedge \lambda_1 \wedge \lambda_3 \neq 0; \\ \lambda_2 \wedge \lambda_3 \wedge d\lambda_3 &= \lambda_2 \wedge \lambda_3 \wedge \lambda_1 \wedge \lambda_4 \neq 0; \\ \lambda_2 \wedge \lambda_3 \wedge d\lambda_2 &= \lambda_2 \wedge \lambda_3 \wedge \lambda_1 \wedge \lambda_3 = 0. \end{aligned}$$

Then the couple ( $\lambda_2, \lambda_3$ ) is a pair of Engel defining forms on the Lie algebra nil[4] with the associated Reeb distribution $\mathcal{R} = \langle X_2, X_3 \rangle$.

Since $d\lambda_1 = d\lambda_4 = 0$, the couple $(\lambda_1, \lambda_4)$ defines a pair of codimension one foliations ( $\mathcal{F}_{\lambda_1}, \mathcal{F}_{\lambda_4}$ ).

$$\text{Since} \begin{cases} \lambda_1 \wedge d\lambda_2 = \lambda_1 \wedge \lambda_1 \wedge \lambda_3 = 0 \\ \lambda_1 \wedge d\lambda_3 = \lambda_1 \wedge \lambda_1 \wedge \lambda_4 = 0 \end{cases} \text{and} \begin{cases} \lambda_1(X_2) = \lambda_1(X_3) = 0 \\ \lambda_4(X_2) = \lambda_4(X_3) = 0 \end{cases}$$

Then from the Theorem 4.9, the couple $(\mathcal{F}_{\lambda_1}, \mathcal{F}_{\lambda_4})$ is linear deformable into Engel structure ( $\lambda_2^t, \lambda_3^t$ ) via $(\lambda_2, \lambda_3)$ in the sens of definition 4.4 and of Reeb distribution given by

$$\mathcal{R}^t = \left\langle \frac{1}{t} X_2, \frac{1}{t} X_3 \right\rangle \quad \text{for all} \quad t > 0.$$

## 5. Conclusion

We have investigated the linear deformations of closed non-singular one forms into even and engel defining froms. For both we gave a necessary and sufficient condition of deformability. We have provided many examples for each deformation. In some approaches to canonical quantum gravity, the prequantization of the phase space involves an even contact structure. The hypersurfaces transverse to the characteristic foliation correspond to quantization levels, possibly relevant for the emergence of spacetime in quantum gravity. Even contact or Engel structures appear in geometric quantization, relevant for quantum gravity and loop quantum cosmology.

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