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The Study of Decomposition of Curvature Tensor Field in a Kaehlerian Recurrent Space of First Order

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Abstract

Takano [2] have studied decomposition of curvature tensor in a recurrent space. Sinha and Singh [3] have been studied and defined decomposition of recurrent curvature tensor field in a Finsler space. Singh and Negi studied decomposition of recurrent curvature tensor field in a Kaehlerian space. Negi and Rawat [6] have studied decomposition of recurrent curvature tensor field in Kaehlerian space. Rawat and Silswal [11] studied and defined decomposition of recurrent curvature tensor fields in a Tachibana space. In the present paper, we have studied the decomposition of curvature tensor fields R^h_{ijk} in terms of two non-zero vectors and a tensor field in a Kaehlerian recurrent space of first order and several theorems have been established and proved. The relation between projective curvature tensor P^h_{ijk} and Riemannian curvature tensor R^h_{ijk} is established therein.

Keywords : Käehlerian recurrent space, curvature tensor, projective curvature tensor.

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1. Introduction

When in a 2n-dimensional real space X_{2n} of class C^r $(r \geq 2)$, there is given a mixed tensor field F_i^h ; $R_{, i, j, \ldots = 1, 2, 3, \ldots, 2n}$, satisfying

$$F_i^l F_l^h = -A_i^h . \tag{1.1}$$

We say that the space admits an almost complex structure and we call such a space an almost complex space. If an almost complex space has a positive definite Riemannian metric $ds^2 = g_{ij}\, d\xi^j\, d\xi^i$ which satisfies

$$F_j^l F_l^k g_{lk} = g_{ji}, \tag{1.2}$$

then the space is called an Almost-Hermitian space.

In this case the tensor $F_{ih} \overset{\text{def}}{=} F_i^l g_{lh}$ is anti symmetric (or skew-symmetric) in i and h. If an almost-Hermitian space satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \tag{1.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g_{ji} of the Riemannian space then it is called an almost-Kaehlerian space and if it satisfies

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0, \tag{1.4}$$

then it is called a K-space.

In an almost-Hermitian space, if

$$\nabla_j F_{ih} = 0, \quad \text{or} \quad \nabla_i F_{ih,j} = 0. \tag{1.5}$$

Then it is called a Kaehlerian space.

The Riemannian curvature tensor field is defined by

$$R^h_{ijk} = \partial_i \begin{Bmatrix} h \\ jk \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} + \begin{Bmatrix} h \\ il \end{Bmatrix} \begin{Bmatrix} l \\ jk \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix} \begin{Bmatrix} l \\ ik \end{Bmatrix}, \tag{1.6}$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes real local coordinates.

The Ricci tensor and the scalar curvature are given by

$$R_{ij} = R^a_{aij} \quad \text{and} \quad R = R_{ij}\, g^{ij} \text{ respectively.}$$

It is well known that these tensors satisfy the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j}, \tag{1.7}$$

$$R_{,i} = 2R^a_{i,a}, \tag{1.8}$$

$$F^a_i R_{aj} = - R_{ia} F^a_j, \tag{1.9}$$

and
$$F^a_i R^j_a = R^a_i F^j_a. \tag{1.10}$$

The holomorphically projective curvature tensor is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)} (R_{ik} \delta^h_j - R_{jk} \delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2 S_{ij} F^h_k), \tag{1.11}$$

where $S_{ij} = F^a_i R_{aj}$.

The Bianchi identities in K^n are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \tag{1.12}$$

and

$$R^h_{ijk,a} + R^h_{kia,j} + R^h_{iaj,k} = 0. \tag{1.13}$$

The commulative formulae for the curvature tensor fields are given as follows

$$T^i_{,jk} - T^i_{,kj} = T^a R^i_{ajk} \tag{1.14}$$

and
$$T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml}. \tag{1.15}$$

A kaehlerian space Kn is said to be Kaehlerian recurrent space of first order if its curvature tensor field satisfies the condition

$$\nabla_a R^h_{ijk} = \lambda_a R^h_{ijk},$$

i.e.,
$$R^h_{ijk,a} = \lambda_a R^h_{ijk}, \tag{1.16}$$

where λ_a is a non-zero vector and is known as recurrent vector field. The space is said to be Ricci-recurrent space of first order, if is satisfies the condition

$$R_{ij,a} = \lambda_a R_{ij}. \tag{1.17}$$

Multiplying the above equation by g^{ij}, we have

$$R_{,a} = \lambda_a R. \tag{1.18}$$

2. Decomposition of Curvature Tensor Field R^h_{ijk}

We consider the decomposition of curvature tensor field R^h_{ijk} in the following form

$$R^h_{ijk} = v^h X_i \phi_{jk} \tag{2.1}$$

where two vectors v^h, X_i and the tensor field ϕ_{jk} are such that

$$\lambda_h v^h = 1. \tag{2.2}$$

Theorem 2.1. Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} takes the forms

$$X_i \phi_{jk} + X_j \phi_{ki} + X_k \phi_{ij} = 0, \tag{2.3}$$

and

$$\lambda_a \phi_{jk} + \lambda_j \phi_{ka} + \lambda_k \phi_{aj} = 0. \tag{2.4}$$

Proof. From equation (1.12) and (2.1), we have

$$v^h [X_i \phi_{jk} + X_j \phi_{ki} + X_k \phi_{ij}] = 0.$$

Since $v^h \neq 0$

$$X_i \phi_{jk} + X_j \phi_{ki} + X_k \phi_{ij} = 0. \tag{2.5}$$

From equations (1.13), (1.16) and (2.1), we have

$$v^h X_i [\lambda_a \phi_{jk} + \lambda_j \phi_{ka} + \lambda_k \phi_{aj}] = 0. \tag{2.6}$$

Multiplying (2.6) by λ_h and using (2.2), we get

$$X_i [\lambda_a \phi_{jk} + \lambda_j \phi_{ka} + \lambda_k \phi_{aj}] = 0.$$

Since $X_i \neq 0$

$$\lambda_a \phi_{jk} + \lambda_j \phi_{ka} + \lambda_k \phi_{aj} = 0. \tag{2.7}$$

Theorem 2.2. Under the decomposition (2.1), the tensor fields R^h_{ijk}, R_{ij} and ϕ_{jk} satisfy the relations

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = X_i \phi_{jk}. \tag{2.8}$$

Proof. With the help of equations (1.7), (1.16) and (1.17), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik}. \tag{2.9}$$

Multiplying (2.1) by λ_h and using relation (2.2), we have

$$\lambda_h R^h_{ijk} = X_i \phi_{jk}$$

or

$$\lambda_a R^a_{ijk} = X_i \phi_{jk}. \tag{2.10}$$

From equation (2.9) and (2.10), we get the required relations (2.8).

Theorem 2.3. Under the decomposition (2.1), the quantities λ_a and v^h behave as recurrence vector and contravariant vector respectively. The recurrent form of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a \tag{2.11}$$

and

$$v^h_{,m} = -\mu_m v^h. \tag{2.12}$$

Proof. Differentiating (2.9) covariantly with respect to x^m and using (2.1), we get

$$\lambda_{a,m} v^a X_i \phi_{jk} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \tag{2.13}$$

Multiplying (2.13) by λ_a and using (2.2) and (2.8), we have

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a (\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.14}$$

Now, multiplying (2.14) by λ_h on both sides, we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) \lambda_h = \lambda_h \lambda_a (\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \tag{2.15}$$

Since, the expression on the right hand side of (2.15) is symmetric in a and h, therefore

$$\lambda_{a,m} \lambda_h = \lambda_{h,m} \lambda_a \tag{2.16}$$

provided $\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$

The vector field λ_a being non-zero, we can have a proportional vector μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a . \tag{2.17}$$

Further, differentiating (2.2), w. r. to x^m covariantly and using relation (2.17), we have

$$\lambda_h v^h_{,m} + v^h \mu_m \lambda_h = 0,$$

or

$$v^h_{,m} + v^h \mu_m = 0, \quad [\text{since } \lambda_h \neq 0]$$

i.e.

$$v^h_{,m} = -\mu_m v^h . \tag{2.18}$$

This proves the theorem.

Theorem 2.4. Under the decomposition (2.1), the vector X_i and the tensor ϕ_{jk} satisfies the relation

$$\{\lambda_m + \mu_m\} X_i \phi_{jk} = X_i \phi_{jk,m} + \phi_{jk} X_{i,m}. \tag{2.19}$$

Proof. Differentiating (2.1) covariantly w. r. to x^m and using (1.16), (2.1) and (2.12), we get the required result (2.19).

Theorem 2.5. Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal if

$$\phi_{km}\{(X_i \delta^h_j - X_j \delta^h_i) + X_l (F^l_i F^h_j - F^l_j F^h_i)\} + 2X_l \phi_{jm} F^l_i F^h_k = 0. \tag{2.20}$$

Proof. The equation (1.11) may be written in the form

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk}, \tag{2.21}$$

where
$$D^h_{ijk} = \frac{1}{(n+2)} (R_{ik} \delta^h_j - R_{jk} \delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2 S_{ij} F^h_k), \tag{2.22}$$

contracting indices h and k in (2.1), we have

$$R_{ij} = v^k X_i \phi_{jk} . \tag{2.23}$$

In view of (2.23), we get

$$S_{ij} = F_i^l \, v^m X_l \phi_{jm}. \tag{2.24}$$

Making use of relations (2.23) and (2.24) in (2.22), we have

$$D_{ijk}^h = \frac{1}{(n+2)} [v^m \phi_{km}\{(X_i \delta_j^h - X_j \delta_i^h) + X_l (F_i^l F_j^h - F_j^l F_i^h)\} + 2v^m X_i \phi_{jm} F_i^l F_k^h]. \tag{2.25}$$

From (2.22), it is clear that

$$P_{ijk}^h = R_{ijk}^h \quad \text{if} \quad D_{ijk}^h = 0,$$

which in view of (2.25) becomes

$$v^m \phi_{km}\{(X_i \delta_j^h - X_j \delta_i^h) + X_l (F_i^l F_j^h - F_j^l F_i^h)\} + 2v^m X_l \phi_{jm} F_i^l F_k^h = 0. \tag{2.26}$$

Multiplying (2.26) by λ_m and using relation (2.2), we get the required condition (2.20)

Theorem 2.6. Under the decomposition (2.1), the scalar curvature R, satisfy the relation

$$\lambda_k R = R_{,k} = g^{ij} X_i \phi_{jk}.$$

Proof. Contracting indices h and k in (2.1), we have

$$R_{ij} = v^k X_i \phi_{jk}. \tag{2.27}$$

Multiplying (2.27) by g^{ij} both sides, we have

$$R = g^{ij} v^k X_i \phi_{jk}. \tag{2.28}$$

Multiplying (2.28) by λ_k and using (2.2), we have

$$\lambda_k R = g^{ij} X_i \phi_{jk}.$$

or, $$R_{,k} = g^{ij} X_i \phi_{jk} \quad \text{[by using (1.18)]}.$$

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