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On Some Type of Curvature Tensors on Trans-Sasakian Manifolds Satisfying a Condition with $\xi \in N(k)$

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Abstract

In this paper conharmonically flat, pseudo projectively flat, pseudo quasi conformally flat, projectively flat, quasi conformally flat and conformally flat trans-Sasakian manifold satisfying $\phi(\text{grad } \alpha) = (2n - 1) \text{ grad } \beta$ with vector ξ belonging to k-nullity distribution have been studied.

Keywords : k-nullity distribution in trans-Sasakian manifold, cosymplectic and trans-Sasakian structures.

Mathematics Subject Classification 2000 : 53C25.

1. Introduction

The notation of locally ϕ-symmetric Sasakian manifold was introduced by T. Takahashi [16] in 1977. ϕ-recurrent Sasakian manifold and pseudo projective curvature tensor on a Riemannian manifold were studied by the author [4] and [11] respectively.

Also J. A. Oubina in 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [1], α-Sasakian [6], Kenmotsu [6], β-Kenmotsu [6] and cosymplectic [1] manifolds, which was called trans-Sasakian manifold [10]. In [18], it is proved that trans-Sasakian structures are generalized quasi-Sasakian. Many authors [3], [9], [8], [7], [13], [15], [14], have studied various type of properties in trans-Sasakian manifold.

2. Preliminaries

Let M be an almost contact metric manifold [1] with an almost contact structure (ϕ, ξ, η, g), where ϕ is a (1, 1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in TM$.

An almost contact structure (ϕ, ξ, η, g), on M is called trans-Sasakian structure [10] if $(M \times R, J, G)$ belongs to the class W_4 [5], where J is the almost complex structure on $M \times R$ defined by

$$J(X, f\, d/dt) = (\phi X - \phi\xi, \eta(X)\, d/dt)$$

for all vector fields X on M and smooth functions f on $M \times R$, and G is the product metric on $M \times R$. This may be expressed by the condition [15]

$$(\nabla_X \phi) Y = \alpha (g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.4}$$

for some smooth functions α and β on M. From (2.4) it follows that

$$\nabla_X \xi = -\alpha \phi X + \beta (X - \eta(X)\xi) \tag{2.5}$$

$$(\nabla_X \eta) Y = -\alpha\, g(\phi X, Y) + \beta\, g(\phi X, \phi Y). \tag{2.6}$$

In [15], U. C. De and M. M. Tripathi obtained some results which shall be useful for next sections. They are

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y)$$
$$+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y \tag{2.7}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X) \tag{2.8}$$

$$2\alpha\beta + \xi\alpha = 0 \tag{2.9}$$

$$S(X, \xi) = (2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\phi X)\alpha \tag{2.10}$$

$$Q\xi = (2n(\alpha^2 - \beta^2) - \xi\beta)\,\xi - (2n-1)\,\mathrm{grad}\,\beta + \phi\,\mathrm{grad}\,\alpha \tag{2.11}$$

where R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature. Also

$$g(QX, Y) = S(X, Y) \tag{2.12}$$

Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S.

When

$$\phi\,(\mathrm{grad}\,\alpha) = (2n-1)\,\mathrm{grad}\,\beta, \tag{2.13}$$

then (2.10) and (2.11) reduces to

$$S(X, \xi) = 2n(\alpha^2 - \beta^2)\,\eta(X), \tag{2.14}$$

$$Q\xi = 2n(\alpha^2 - \beta^2)\,\xi. \tag{2.15}$$

The k-nullity distribution [17] of a Riemannian manifold (M, g), for a real number k, is a distribution

$$N(k) : p \to N_p(k) = [Z \in T_pM : R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y)\}] \tag{2.16}$$

for all $X, Y \in T_pM$. Hence if the characteristic vector field ξ of the contact metric manifold M^{2n+1} belongs to the k-nullity distribution then we have

$$R(X, Y)\,\xi = k\{\eta(Y)X - \eta(X)Y\}, \tag{2.17}$$

$$S(X, \xi) = 2nk\eta(X). \tag{2.18}$$

We can define k-nullity distribution in a trans-Sasakian manifold by

$$\begin{aligned} N(k) : p \to N_p(k) = [Z \in T_pM : R(X, Y)Z = k[(\alpha^2 - \beta^2)(g(Y, Z)X - g(X, Z)Y) \\ + 2\alpha\beta\,(g(Y, Z)\,\phi X - g(X, Z)\,\phi Y) + (Y\alpha)\,\phi X - (X\alpha)\,\phi Y \\ + (Y\beta)\phi^2 X - (X\beta)\,\phi^2 Y]]. \end{aligned} \tag{2.19}$$

So when $\xi, Z \in N(k)$, we have

$$\eta(R(X,Y)Z) = k\,(\alpha^2 - \beta^2)\{(g(Y, Z)\,\eta(X) - g(X, Z)\,\eta(Y)\}, \tag{2.20}$$

$$S(X, Z) = (2nk(\alpha^2 - \beta^2) - \xi\beta)\, g(X, Z) - (2n-1)\, X\beta - (\phi X)\, \alpha \tag{2.21}$$

whenever $Z \in N_p(k)$. Putting $Z = \xi \in N_p(k)$, we obtain

$$S(X, \xi) = (2nk(\alpha^2 - \beta^2) - \xi\beta)\, \eta(X) - (2n-1)\, X\beta - (\phi X)\, \alpha \tag{2.22}$$

Now from (2.13), we get

$$S(X, \xi) = 2nk(\alpha^2 - \beta^2)\, \eta(X) \tag{2.23}$$

and

$$Q\xi = 2nk(\alpha^2 - \beta^2)\, \xi, \qquad \text{when} \quad \xi \in N_p(k). \tag{2.24}$$

A conharmonic curvature tensor in a Riemannian manifold is defined as [2]

$$H(X, Y) Z = R(X, Y) Z - \frac{1}{2n-1} [S(Y, Z) X - S(X, Z) Y + g(Y, Z) QX - g(X, Z) QY] \tag{2.25}$$

Q is defined in (2.12).

A pseudo projective curvature tensor given by [14]

$$\overline{P}(X, Y) Z = \alpha R(X, Y) Z + b[S(Y, Z) X - S(X, Z) Y] - \frac{r}{2n+1}\left[\frac{a}{2n} + b\right] [g(Y, Z) X - g(X, Z) Y] \tag{2.26}$$

where a, b are constants such that a, b $\neq 0$, R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature.

Recently the first two authors of [12] introduced the notion of pseudo quasi-conformal curvature tensor $\tilde{T}$ of type (0, 4) on a Riemannian manifold of dimension greater than three which includes the projective, quasi conformal, conformal and concircular curvature tensor as special cases and defined as

$$\tilde{T}(X, Y, Z, W) = (p + d) R(X, Y, Z, W) + \left(q - \frac{d}{n-1}\right) [S(Y, Z) g(X, W) - S(X, Z) g(Y, W)] + q[S(X, W) g(Y, Z) - S(Y, W) g(X, Z)] - \frac{r}{n(n-1)} [p + 2(n-1)q] [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)] \tag{2.27}$$

where R is the curvature tensor of type (0, 4), S is the Ricci-tensor of type (0, 2), r is the scalar curvature of the manifold and p, q, d are the arbitrary constants non-zero simultaneously. In particular, if (i) $p = q = 0, d = 1$; (ii) $p \neq 0, q \neq 0$ and $d = 0$; (iii) $p = 1, q = -\frac{1}{n-2}, d = 0$ and (iv) $p = 1, q = d = 0$, then $\widetilde{T}$ reduces to the projective curvature tensor, quasi conformal curvature tensor, conformal curvature tensor and concircular curvature tensor respectively.

The above results will be useful in the next sections.

3. Conharmonically flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying ϕ (grad α) = (2n – 1) grad β

Let us consider $H(X, Y)Z = 0$, then from (2.25), we have

$$R(X, Y)Z = \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \quad (3.1)$$

(3.1) can be written as

$$'R(X, Y, Z, W) = \frac{1}{2n-1}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)g(QX, W) - g(X, Z)g(QY, W)]. \quad (3.2)$$

where $g(R(X, Y)Z, W) = {}'R(X, Y, Z, W)$.

Using (2.12) in (3.2), we have

$$'R(X, Y, Z, W) = \frac{1}{2n-1}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)]. \quad (3.3)$$

Putting $W = \xi$ and using (2.3), (2.20), (2.23) in (3.3), we have

$$S(Y, Z)\eta(X) - S(X, Z)\eta(Y) + k(\alpha^2 - \beta^2)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} = 0 \quad (3.4)$$

Again putting $X = \xi$ and using (2.1), (2.3), (2.23) in (3.4), we get

$$S(Y, Z) = -k(\alpha^2 - \beta^2)g(Y, Z) + (2n-1)k(\alpha^2 - \beta^2)\eta(Y)\eta(Z)$$

i.e. the manifold is an η-Einstein manifold.

Thus we can state the following theorem :

Theorem 3.1. In a conharmonically flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$, is an η-Einstein manifold.

4. Pseudo projectively flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$

Let us consider $\overline{P}(X, Y)Z = 0$, then from (2.26), we have

$$R(X, Y)Z = \frac{r}{2n+1}\left[\frac{1}{2n} + \frac{b}{a}\right][g(Y, Z)X - g(X, Z)Y] - \frac{b}{a}[S(Y, Z)X - S(X, Z)Y] \tag{4.1}$$

where a, b are constants such that $a, b \neq 0$, R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature. (4.1) can be written as

$$'R(X, Y, Z, W) = \frac{r}{2n+1}\left[\frac{1}{2n} + \frac{b}{a}\right][g(Y, Z)g(X,W) - g(X, Z)g(Y,W)] - \frac{b}{a}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] \tag{4.2}$$

where $g(R(X, Y)Z, W) = {'R}(X, Y, Z, W)$.

Putting $W = \xi$ and using (2.3), (2.20) in (4.2), we have

$$k(\alpha^2 - \beta^2)\{(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} = \frac{r}{2n+1}\left[\frac{1}{2n} + \frac{b}{a}\right][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - \frac{b}{a}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \tag{4.3}$$

Again putting $X = \xi$ and using (2.1), (2.3), (2.23) in (4.3), we get

$$S(Y, Z) = \frac{a}{b}\left[\frac{r}{2n+1}\left[\frac{1}{2n} + \frac{b}{a}\right] - k(\alpha^2 - \beta^2)\right]g(Y, Z) + \frac{a}{b}\left[\frac{2nkb}{a}(\alpha^2 - \beta^2) + k(\alpha^2 - \beta^2) - \frac{r}{2n+1}\left[\frac{1}{2n} + \frac{b}{a}\right]\right]\eta(Y)\eta(Z)$$

i.e. the manifold is an η-Einstein manifold.

Thus we can state :

Theorem 4.1. In a pseudo projectively flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$, is an η-Einstein manifold.

5. Pseudo quasi conformally flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$

Let us consider $\tilde{T}(X, Y, Z, W) = 0$, then from (2.27), we have

$$(p+d)\,R(X, Y, Z, W) = \frac{r}{n(n-1)}[p + 2(n-1)q]\,[g(Y, Z)\,g(X, W) - g(X, Z)\,g(Y, W)] - \left(q - \frac{d}{n-1}\right)[S(Y, Z)\,g(X, W) - S(X, Z)\,g(Y, W)] - q[S(X, W)\,g(Y, Z) - S(Y, W)\,g(X, Z)] \tag{5.1}$$

where R is the curvature tensor of type (0, 4), S is the Ricci-tensor of type (0, 2), r is the scalar curvature of the manifold and p, q, d are the arbitrary constants non-zero simultaneously.

Putting $W = \xi$ and using (2.3), (2.20), (2.23) in (5.1), we have

$$\{k(p+d)(\alpha^2 - \beta^2) + 2nkq\,(\alpha^2 - \beta^2) - \frac{r}{n(n-1)}[p + 2(n-1)q]\,[g(Y, Z)\,\eta(X) - g(X, Z)\,\eta(Y)]\} = \left(\frac{d}{n-1} - q\right)[S(Y, Z)\,\eta(X) - S(X, Z)\,\eta(Y)]. \tag{5.2}$$

Again putting $X = \xi$ and using (2.1), (2.3), (2.23) in (5.2), we get

$$S(Y, Z) = a_1\,g(Y, Z) + b_1\,\eta(Y)\,\eta(Z)$$

where $a_1 = \dfrac{1}{\dfrac{d}{n-1} - q}\{k(p+d)(\alpha^2 - \beta^2) + 2nkq\,(\alpha^2 - \beta^2) - \dfrac{r}{n(n-1)}[p + 2(n-1)q]$

$$b_1 = \frac{1}{\dfrac{d}{n-1} - q}\{2nk\,(\alpha^2 - \beta^2)\left(\frac{d}{n-1} - q\right) - k(p+d)(\alpha^2 - \beta^2) -$$

$$- 2nkq\,(\alpha^2 - \beta^2) + \frac{r}{n(n-1)}\,[p + 2(n-1)q]\}$$

i.e. the manifold is an η-Einstein manifold.

Thus we can state the following theorem :

Theorem 5.1. In a pseudo quasi conformally flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$, is an η-Einstein manifold.

We have the following corollaries :

Corollary 5.1. In a projectively flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$, is an Einstein manifold.

Corollary 5.2. In a quasi conformally flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$, is an η-Einstein manifold.

Corollary 5.3. In a conformally flat trans-Sasakian manifold with $\xi \in N(k)$ satisfying $\phi\,(\text{grad}\,\alpha) = (2n-1)\,\text{grad}\,\beta$, is an η-Einstein manifold.

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