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Hypersurface of Para Sasakian Manifold

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Abstract

In this paper, we have studied Hypersurface of Para Sasakian Manifold. Basic informations are given in the first section. Hypersurface immersed in an almost paracontact Riemannian manifold is investigated in the second section.

Keywords and Phrases : Hypersurface, Paracontact, Riemannian metric.

1. Introduction

Let M be an $m-$dimensional differentiable manifold endowed with a tensor field F of type (1, 1), a vector field T and a $1-$form A such that

(1.1)(a) $F^2X = X - A(X)T,$

(b) $A(T) = 1,$

(c) $FT = 0,$

(d) $AoF = 0,$

(e) rank $F = m - 1,$

then M is said to have an almost paracontact structure. [7], [8].

If there exists a Riemannian metric G such that

(1.2)(a) $A(X) = G(X, T),$

(b) $G(FX, FY) = G(X, Y) - A(X)A(Y).$

Then M is said to have an almost paracontact metric structure.[1]

We say that the almost paracontact structure is normal if

(1.3) $$[F, F] - T \otimes dA = 0.$$

where $[F, F]$ is the Nijenhuis tensor of F. [4], [6].

An almost paracontact metric structure is said to be para-Sasakian if

$$(D_X F)(Y) = A(Y)X - 2A(X)A(Y)T + g(X, Y)T, \tag{1.4}$$

where D denotes the Riemannian connexion of G. [5], [9]

An almost paracontact metric manifold is said to be a closed almost paracontact metric manifold, if A is closed.

$$D_X T = -FX. \tag{1.5}$$

Let M be an almost paracontact manifold and $\overline{M}$ be an orientable Hypersurface of M. If there exists in $\overline{M}$, a tensor field ϕ of type (1, 1), a vector field ξ and a 1−form η satisfying

(1.6)(a) $\eta(\xi) = 1$,

(b) $\phi^2 X = X - \eta(X)\xi$.

Then $\overline{M}$ is said to have an almost paracontact structure (ϕ, η, ξ) and $\overline{M}$ is called an almost parcontact manifold. [1], [8]

In an almost paracontact manifold, there exists a positive definite Riemannian metric 'g' such that

(1.7)(a) $\eta(X) = g(\xi, X)$,

(b) $g(fX, fY) = g(X, Y) - \eta(X)\eta(Y)$,

for all $X, Y \in \overline{M}$. The set $\{f, \xi, \eta, g\}$ is called an almost metric Riemannian structure.

In an almost paracontact Riemannian manifold, the following relations also hold good. [2]

(1.8)(a) $g(\phi X, Y) = g(X, \phi Y)$,

(b) $\phi(\xi) = 0, \qquad \eta o \phi = 0, \qquad \text{rank}\,(\phi) = n - 1$.

Let B be the differential of immersion b of $\overline{M}$ into M, and X, Y, Z be the tangents to $\overline{M}$. Consider C be a unit normal vector. Then we have

$$FBX = B\phi X + \eta(X)C, \tag{1.9}$$

where ϕ is a (1, 1) tensor field and η is a 1−form on $\overline{M}$.

If $\eta \neq 0$, then $\overline{M}$ is called a non-invariant Hypersurface of M and if η is identically zero, then $\overline{M}$ is said to be an invariant Hypersurface, i.e., the tangent space of $\overline{M}$ is invariant under F.

The metric 'G' of an almost paracontact metric manifold induces a Riemannian metric g on the submanifold $\overline{M}$ given by [3]

$$g(X,Y) = G(BX,BY). \tag{1.10}$$

Further the symmetric affine connexion D on M induces a symmetric affine connexion $\overline{D}$ on submanifold $\overline{M}$ such that

$$D_{BX}BY = B(\overline{D}_X Y) + h(X,Y)C, \tag{1.11}$$

where h is a symmetric tensor of type $(0,2)$ called the second fundamental form of the sub-manifold $\overline{M}$. We also have

$$D_{BX}\, C = -BHX + W(X)\, C, \tag{1.12}$$

where W is a $1-$form on $\overline{M}$ defining a normal bundle and H is (1, 1) tensor field on $\overline{M}$ such that,

$$g(HX,Y) = h(X,Y).$$

2. Hypersurface immersed in an almost paracontact Riemannian manifold

Let M be an $m-$dimensional almost paracontact Riemannian manifold with structure (F,T,A,G) and $\overline{M}$ be a hypersurface imbedded in M by the imbedding $b : \overline{M} \to M$ and B be the Jacobian of b, i.e. $p \in \overline{M} \Rightarrow b(p) \in M$. $B : T_b(\overline{M}) \to T_{b(p)}(M)$, which yields $X \in T_b(\overline{M}) \Rightarrow BX \in T_{b(p)}(M)$.

Operating F to BX and to the unit normal vector C of $\overline{M}$ respectively, we obtain vector fields FBX and FC which can be written in the form.

$$FBX = B\phi X + \eta(X)C. \tag{2.1}$$

$$FC = B\xi + \lambda C. \tag{2.2}$$

where ϕ, ξ, η and λ define respectively a linear transformation field, a vector field, $1-$form and a scalar function λ on $\overline{M}$.

Let g be induced Riemannian metric on $\overline{M}$, [3]

$$g(X,Y) = G(BX,BY). \tag{2.3}$$

Operating F on both sides in (2.1), we get,

$$F^2BX = FB\phi X + F\eta(X)C.$$

$$BX - A(BX)T = B\phi^2X + \eta(\phi X)\,C + \eta(X)FC. \tag{2.4}$$

Using (2.2), we get

$$BX - A(BX)T = B\phi^2X + \eta(\phi X)C + \eta(X)\{B\xi + \lambda C\}. \tag{2.5}$$

Let us put $A(BX) = B(\eta' X)$, where η' is 1−form on $\overline{M}$ and $T = B\xi'$, then (2.5) can be written as

$$B\{X - \eta'(X)\xi'\} = B\{\phi^2X + \eta(X)\xi\} + \{\eta(\phi X) + \lambda\eta(X)\}C.$$

Which yields

$$\phi^2X = X - \eta(X)\xi - \eta'(X)\,\xi'. \tag{2.6}$$

$$\eta(\phi X) = -\lambda\eta(X). \tag{2.7}$$

Operating F on both sides of (2.2), we obtain

$$F^2C = F(B\xi) + F(\lambda C),$$

$$C - A(C)T = B\phi\xi + \eta(\xi)C + \lambda\{B\xi + \lambda C\}.$$

$$C - A(C)T = B\{\phi\xi + \lambda\xi\} + (\eta(\xi) + \lambda^2)C. \tag{2.8}$$

From which we obtain

$$\eta(\xi) = 1 - \lambda^2. \tag{2.9}$$

$$\phi(\xi) = -\lambda\xi. \tag{2.10}$$

From (1.2)(b)

$$G(FX', FY') = G(X', Y') - A(X')\,A(Y'),$$

where X', Y' stand for vector fields on M.

Now $G(FBX, FBY) = G(BX, BY) - A(BX)A(BY)$, using (2.1), we get

$$G\{B\phi X + \eta(X)C, B\Phi Y + \eta(Y)C\} = g(X, Y) - \eta'(X)\eta'(Y),$$

$$G\{B\phi X, B\phi Y\} + \eta(X)\eta(Y)G(C, C) = g(X, Y) - \eta'(X)\eta'(Y).$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - \eta'(X)\eta'(Y). \tag{2.11}$$

Replacing X by ϕX in (2.7), we find

$$\eta(\phi^2 X) = -\lambda\eta(\phi X) = -\lambda\{-\lambda\eta(X)\}.$$

$$\eta(\phi^2 X) = \lambda^2\eta(X). \tag{2.12}$$

From (2.6), we get

$$\eta(\Phi^2 X) = \eta(X) - \eta(X)\eta(\xi) - \eta'(X)\eta(\xi').$$

Using (2.12) and (2.9), we get

$$\lambda^2\eta(X) = \eta(X) - \eta(X)(1-\lambda^2) - \eta'(X)\eta(\xi'),$$

which gives

$$\eta(\xi') = 0. \tag{2.13}$$

From (1.1)(c), $FT = 0$, using $(T = B\xi')$ then we get, $F(B\xi') = 0$, and using (2.1), $B(\phi\xi') + \eta(\xi')C = 0$. Using (2.13), we get

$$(\phi\xi') = 0. \tag{2.14}$$

Again using (2.6) and replacing X by ξ

$$\phi^2\xi = \xi - \eta(\xi)\xi - \eta'(\xi)\xi'$$

$$\lambda^2\xi = \xi - (1-\lambda^2)\xi - \eta'(\xi)\xi',$$

we get

$$\eta'(\xi) = 0. \tag{2.15}$$

Again replacing X and Y by ξ' in (2.11)

$$g(\phi\xi', \phi\xi') = g(\xi', \xi') - \eta(\xi')\eta(\xi') - \eta'(\xi')\eta'(\xi'),$$

Using (2.13) and (2.14), we get

$$\eta'(\xi') = 1, \quad (\text{since}\eta'(X) = A(BX) \geq 0). \tag{2.16}$$

Summing up, we have

(2.17)(i) $\phi^2 X = X - \eta(X)\xi - \eta'(X)\,\xi'$,

(ii) $\eta(\phi X) = -\,\eta(X)\,\lambda$,

(iii) $\eta(\xi) = 1 - \lambda^2$,

(iv) $\eta(\xi') = 0$,

(v) $\eta'(\xi) = 0,$

(vi) $\eta'(\xi') = 1,$

(vii) $\phi(\xi) = -\lambda\xi,$

(viii) $\phi(\xi') = 0,$

(ix) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - \eta'(X)\eta'(Y).$

Using (2.17)(ix) and replacing X and Y by ξ, we get

$$g(\phi\xi, \phi\xi) = g(\xi, \xi) - \eta(\xi)\eta(\xi) - \eta'(\xi)\eta'(\xi)$$

$$g(-\lambda\xi, -\lambda\xi) = 1 - (1 - \lambda^2)^2 - 0$$

$$\lambda^2 = 1 - (1 + \lambda^4 - 2\lambda^2),$$

on solving

$$\lambda = 0, 1, -1.$$

Theorem (2.1) If a hypersurface is immersed in an almost paracontact metric structure manifold then in the hypersurface structure $\{\Phi, \eta, \xi, \eta', \xi', g\}$ is induced which is given by (2.17)(i) to (ix), where scalar function λ becomes either 0 or 1 or -1.

Case I. If $\lambda = 0$, then (2.17) becomes

(2.18)(i) $\phi^2 X = X - \eta(X)\,\xi - \eta'(X)\xi',$

(ii) $\eta(\phi X) = 0,$

(iii) $\eta(\xi) = 0,$

(iv) $\eta(\xi') = 0,$

(v) $\eta'(\xi) = 0,$

(vi) $\eta'(\xi') = 1,$

(vii) $\phi(\xi) = 0,$

(viii) $\phi(\xi') = 0,$

(ix) $g(\phi X,\, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) - \eta'(X)\eta'(Y).$

Operating ϕ on both sides of (2.18)(i), we get

(2.19) $$\phi^3 X = \phi X - \eta(X)\phi(\xi) - \eta'(X)\phi\xi'.$$

Using (2.18)(vii), (viii) becomes

$$\phi^3 X = \phi X \tag{2.20}$$

i.e.

$$\phi^3 - \phi = 0.$$

Case II. If $\lambda = \pm 1$, then (2.17), we get

(2.21)(i) $\phi^2 X = X - \eta(X)\,\xi - \eta'(X)\,\xi'$,

(ii) $\eta(\phi X) = \mp\eta(X)$,

(iii) $\eta(\xi) = 0$,

(iv) $\eta(\xi') = 0$,

(v) $\eta'(\xi) = 0$,

(vi) $\eta'(\xi') = 1$,

(vii) $\phi(\xi) = \mp\xi$,

(viii) $\phi(\xi') = 0$,

(ix) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - \eta'(X)\eta'(Y)$.

Operating ϕ on both sides of (2.21)(i), we get

$$\phi^3 X = \phi X - \eta(X)\phi(\xi) - \eta'(X)\phi(\xi').$$

Using (2.21)(vii), (viii), we get

$$\phi^3 X = \phi X \pm \eta(X)\,\xi. \tag{2.22}$$

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On Weakly Symmetric and Weakly Ricci-Symmetric Almost r-Para Contact Manifolds of LP-Sasakian and Kenmotsu Type

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Abstract

The present paper deals with weakly symmetric and weakly Ricci-symmetric almost r-para contact manifolds of LP-Sasakian type and Kenmotsu type. We obtain necessary conditions in order that an almost r-para contact manifolds of LP-Sasakian and of Kenmotsu type be weakly symmetric and weakly Ricci-symmetric, respectively .

Keywords and Phrases : Almost r-para contact manifold , weakly symmetric manifold, weakly Ricci-symmetric manifold.
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1. Introduction

The notions of weakly symmetric and weakly Ricci-symmetric Riemannian manifolds were introduced by L. Tamassy and T. Q. Binh in 1992 and 1993 (see [9], [8]). In 2000, U. C. De, T. Q. Binh and A. A. Shaikh gave necessary conditions for the compatibility of several k-contact structures with weak symmetry and weak Ricci-symmetry [4]. In 2002, C. Özgür studied on weak symmetries of Lorentzian para-Sasakian manifolds [10] and also the author considered weakly symmetric Kenmotsu manifolds in [11]. Then N. Aktan and A. Görgülü studied in 2007 on weak symmetries of almost r-para contact Riemannian manifold of P-Sasakian type [1]. Here we study weakly symmetric and weakly Ricci-symmetric almost r-para contact manifolds of LP-Sasakian type and Kenmotsu type.

2. Preliminaries

A non-flat differentiable manifold (M^n, g) $(n > 2)$ is called weakly symmetric if there exist 1-forms $\alpha, \beta, \gamma, \delta$ and σ on M such that

$$\begin{aligned}(\nabla_X \acute{R})(Y,Z,U,V) &= \alpha(X)\acute{R}(Y,Z,U,V) + \beta(Y)\acute{R}(X,Z,U,V) \\ &\quad + \gamma(Z)\acute{R}(Y,X,U,V) + \delta(U)\acute{R}(Y,Z,X,V) \\ &\quad + \sigma(V)\acute{R}(Y,Z,U,X) \qquad (2.1)\end{aligned}$$

holds for vector fields X, Y, Z, U, V on M;
where $\acute{R}(X,Y,Z,U) = g(R(X,Y)Z,U)$.

A differentiable manifold $(M^n, g)\,(n > 2)$ is called weakly Ricci symmetric if there exist 1-forms ρ, μ, ν such that

$$(\nabla_X S)(Y,Z) = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \nu(Z)S(X,Y) \qquad (2.2)$$

holds for all vector fields X, Y, Z; where $S(X,Y) = g(QX,Y)$,
Q be the symmetric endomorphism of the tangent space of M.

If M is weakly symmetric, then from (2.1), we obtain (see [8], [9])

$$\begin{aligned}(\nabla_X S)(Z,U) &= \alpha(X)S(Z,U) + \beta(Z)S(X,U) + \delta(U)S(Z,X) \\ &\quad + \beta(R(X,Z)U) + \delta(R(X,U)Z) \qquad (2.3)\end{aligned}$$

An n-dimensional differentiable manifold M is called a Lorentzian Para-Sasakian (briefly LP-Sasakian) manifold ([6], [7]) if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \qquad (2.4)$$

$$\phi^2 = I + \eta(X)\xi, \qquad (2.5)$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \qquad (2.6)$$

$$g(X,\xi) = \eta(X), \nabla_X \xi = \phi X, \qquad (2.7)$$

$$(\nabla_X \phi)(Y) = [g(X,Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y), \qquad (2.8)$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

In a LP-Sasakian manifold , the following relations hold

$$\phi\xi = 0, \eta(\phi X) = 0 \qquad (2.9)$$

$$\text{rank}\phi = n - 1. \qquad (2.10)$$

Let (M, ϕ, ξ, η, g) be an n-dimensional almost contact Riemannian manifold, where ϕ is a (1,1) tensor field, ξ is the structure vector field, η is a 1-form and g is a Riemannian metric. It is well known (ϕ, ξ, η, g) satisfy the following [2]:

$$\eta(\xi) = 1, \tag{2.11}$$

$$g(X, \xi) = \eta(X), \tag{2.12}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.13}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.14}$$

$$\phi(\xi) = 0, \tag{2.15}$$

$$\eta(\phi X) = 0, \tag{2.16}$$

$\forall$ vector fields X, Y on M.

If moreover,

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi(X), \tag{2.17}$$

where ∇ denotes the Riemannian connection, then (M, ϕ, ξ, η, g) is called a Kenmotsu manifold [5]. In a Kenmotsu manifold, the following property holds

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.18}$$

A differentiable manifold (M, g) of dimension $(n + r)$ with tangent space $T(M)$ is said to be an almost r-para contact Riemannian manifold (by [3]) if there exist a tensor field ϕ of type $(1, 1)$ and r global vector fields $\xi_1, \ldots, \xi_r$ (called structure vector fields) such that

i) if $\eta_1, \ldots, \eta_r$ are dual 1-forms of $\xi_1, \ldots \xi_r$; then

$$\eta_i(\xi_j) = \delta^i_j;$$

$$g(\xi_i, X) = \eta_i(X);$$

$$\phi^2 = I - \sum_{i=1}^{r} \xi_i \otimes \eta_i \tag{2.19}$$

ii) $$g(\phi\, X, \phi\, Y) = g(X, Y) - \sum_{i=1}^{r} \eta_i(X)\eta_i(Y), \tag{2.20}$$

for $X, Y \in T(M)$.

We define an almost r-para contact manifold of LP-Sasakian type as follows:

Definition (2.1) : An almost r-para contact manifold M is said to be of LP-Sasakian type if

$$\nabla_X \xi_i = \phi\ X \tag{2.21}$$

$$(\nabla_X \phi)(Y) = \sum_{i=1}^{r} [g(X,Y) + \eta_i(X)\eta_i(Y)]\xi_i + \sum_{i=1}^{r} [X + \eta_i(X)\xi_i]\eta_i(Y), \tag{2.22}$$

$\forall X, Y \in T(M)$.

In an almost r-para contact manifold of LP-Sasakian type M, the following relations hold

$$S(\xi_i, X) = (n-1)\ \sum_{i=1}^{r} \eta_i(X) \tag{2.23}$$

$$R(\xi_i, X)\xi_i = X + \sum_{i=1}^{r} \eta_i(X)\xi_i \tag{2.24}$$

$$g(R(\xi_i, X)Y, \xi_i) = \sum_{i=1}^{r} [g(X,Y)\eta_i(\xi_i) - g(\xi_i, Y)\eta_i(X)] \tag{2.25}$$

for vector fields $X, Y \in T(M)$.

Again we define an almost r-para contact Riemannian manifold of Kenmotsu type as follows:

Definition (2.2) : An almost r-para contact Riemannian manifold M is said to be of Kenmotsu type if

$$\nabla_X \xi_i = X - \sum_{i=1}^{r} \eta_i(X)\xi_i \tag{2.26}$$

$$(\nabla_X \phi)(Y) = \sum_{i=1}^{r} [-g(X, \phi\ Y)\xi_i - \eta_i(Y)\phi(X)], \tag{2.27}$$

$\forall X, Y \in\ T(M)$.

In an almost r-para contact Riemannian manifold of Kenmotsu type M, the following relations hold

$$S(\xi_i, X) = -(n-1)\ \sum_{i=1}^{r} \eta_i(X) \tag{2.28}$$

$$R(\xi_i, X)\xi_i = X - \sum_{i=1}^{r} \eta_i(X)\xi_i \tag{2.29}$$

$$g(R(\xi_i, X)Y, \xi_i) = -g(X, Y) + \sum_{i=1}^{r} \eta_i(X)\eta_i(Y) \tag{2.30}$$

for vector fields $X, Y \in T(M)$.

Since ϕ is skew symmetric and the Ricci operator Q is symmetric in an almost r-para contact manifold of LP-Sasakian type (or Kenmotsu type), $Q\,\phi + \phi\,Q = 0$ and thus the Lie derivative of S vanishes i.e.,

$$L_{\xi_i} S = 0. \tag{2.31}$$

for any $i = 1, \ldots, r$.

3. Weakly symmetric almost r-para contact manifold of LP-Sasakian type

In this section we suppose that the considered weakly symmetric manifold is almost r-para contact manifold of LP-Sasakian type. Then we obtain

Theorem 3.1 : Any weakly symmetric almost r-para contact manifold of LP-Sasakian type M, satisfies $\alpha + \beta + \delta = 0$.

Proof : Since the manifold is weakly symmetric, by putting $X = \xi_i$ in (2.3), we have

$$\begin{aligned}(\nabla_{\xi_i} S)(Z, U) = \alpha(\xi_i)S(Z, U) + \beta(Z)S(\xi_i, U) + \delta(U)S(Z, \xi_i)\\ +\beta(R(\xi_i, Z)U) + \delta(R(\xi_i, U)Z)\end{aligned} \tag{3.1}$$

By virtue of (2.21) and (2.31) we obtain

$$(\nabla_{\xi_i} S)(Z, U) = 0 \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned}\alpha(\xi_i)S(Z, U) + \beta(Z)S(\xi_i, U) + \delta(U)S(Z, \xi_i)\\ +\beta(R(\xi_i, Z)U) + \delta(R(\xi_i, U)Z) = 0\end{aligned} \tag{3.3}$$

Putting $Z = U = \xi_i$ in (3.3) and using (2.24), we get

$$[\alpha(\xi_i) + \beta(\xi_i) + \delta(\xi_i)]S(\xi_i, \xi_i) = 0 \tag{3.4}$$

which gives

$$\alpha(\xi_i) + \beta(\xi_i) + \delta(\xi_i) = 0. \tag{3.5}$$

This shows that $\alpha + \beta + \delta = 0$ over the vector field ξ_i on M.

Now we will show that $\alpha + \beta + \delta = 0$ holds for all vector fields on M.

Taking $X = Z = \xi_i$ in (2.3), we obtain

$$\begin{aligned}(\nabla_{\xi_i} S)(\xi_i, U) = \alpha(\xi_i)S(\xi_i, U) + \beta(\xi_i)S(\xi_i, U) + \delta(U)S(\xi_i, \xi_i)\\ +\beta(R(\xi_i, \xi_i)U) + \delta(R(\xi_i, U)\xi_i)\end{aligned} \tag{3.6}$$

Replacing U by X in (3.6), we get

$$\begin{aligned}\alpha(\xi_i)S(\xi_i, X) + \beta(\xi_i)S(\xi_i, X) + \delta(X)S(\xi_i, \xi_i)\\ +\beta(R(\xi_i, \xi_i)X) + \delta(R(\xi_i, X)\xi_i) = 0\end{aligned} \tag{3.7}$$

In (2.3), taking $X = U = \xi_i$, we have

$$\begin{aligned}(\nabla_{\xi_i} S)(Z, \xi_i) = \alpha(\xi_i)S(Z, \xi_i) + \beta(Z)S(\xi_i, \xi_i) + \delta(\xi_i)S(Z, \xi_i)\\ +\beta(R(\xi_i, Z)\xi_i) + \delta(R(\xi_i, \xi_i)Z)\end{aligned} \tag{3.8}$$

Using (3.2) in (3.8) and replacing Z by X, we obtain

$$\begin{aligned}\alpha(\xi_i)S(X, \xi_i) + \beta(X)S(\xi_i, \xi_i) + \delta(\xi_i)S(X, \xi_i)\\ +\beta(R(\xi_i, X)\xi_i) + \delta(R(\xi_i, \xi_i)X) = 0\end{aligned} \tag{3.9}$$

In (2.3), taking $Z = U = \xi_i$, we have

$$\begin{aligned}(\nabla_X S)(\xi_i, \xi_i) = \alpha(X)S(\xi_i, \xi_i) + \beta(\xi_i)S(X, \xi_i) + \delta(\xi_i)S(\xi_i, X)\\ +\beta(R(X, \xi_i)\xi_i) + \delta(R(X, \xi_i)\xi_i)\end{aligned} \tag{3.10}$$

Here also we have

$$(\nabla_X S)(\xi_i, \xi_i) = 0 \tag{3.11}$$

Using (3.11) in (3.10), we obtain

$$\begin{aligned}\alpha(X)S(\xi_i, \xi_i) + \beta(\xi_i)S(X, \xi_i) + \delta(\xi_i)S(\xi_i, X)\\ +\beta(R(X, \xi_i)\xi_i) + \delta(R(X, \xi_i)\xi_i) = 0\end{aligned} \tag{3.12}$$

adding (3.7), (3.9) and (3.12) and then using (3.5), we get

$$[\alpha(X) + \beta(X) + \delta(X)]S(\xi_i, \xi_i) = 0 \tag{3.13}$$

Hence from (3.13), we obtain

$$\alpha(X) + \beta(X) + \delta(X) = 0, \qquad \forall X.$$

Thus

$$\alpha + \beta + \delta = 0.$$

Hence the theorem is proved.

4. Weakly Ricci-symmetric almost r-para contact manifold of LP-Sasakian type

In this section we suppose that the weakly Ricci-symmetric manifold is almost r-para contact manifold of LP-Sasakian type. Then we have

Theorem 4.1 : Any weakly Ricci-symmetric almost r-para contact manifold of LP-Sasakian type M satisfies $\rho + \mu + \nu = 0$.

Proof. Since M is weakly Ricci-symmetric almost r-para contact manifold of LP-Sasakian type, then

by putting $X = \xi_i$ in (2.2) we get

$$(\nabla_{\xi_i} S)(Y, Z) = \rho(\xi_i)S(Y, Z) + \mu(Y)S(\xi_i, Z) + \nu(Z)S(\xi_i, Y) \tag{4.1}$$

Using (3.2) in (4.1), we have

$$\rho(\xi_i)S(Y, Z) + \mu(Y)S(\xi_i, Z) + \nu(Z)S(\xi_i, Y) = 0 \tag{4.2}$$

Replacing Y and Z by ξ_i in (4.2), we obtain

$$[\rho(\xi_i) + \mu(\xi_i) + \nu(\xi_i)]S(\xi_i, \xi_i) = 0 \tag{4.3}$$

which gives

$$\rho(\xi_i) + \mu(\xi_i) + \nu(\xi_i) = 0 \tag{4.4}$$

Taking $X = Y = \xi_i$ in (2.2) and using (3.2), then putting $Z = X$, we get

$$\rho(\xi_i)S(\xi_i, X) + \mu(\xi_i)S(\xi_i, X) + \nu(X)S(\xi_i, \xi_i) = 0. \tag{4.5}$$

In (2.2), taking $X = Z = \xi_i$ and using (3.2), then replacing Y by X, we obtain

$$\rho(\xi_i)S(X, \xi_i) + \mu(X)S(\xi_i, \xi_i) + \nu(\xi_i)S(\xi_i, X) = 0 \tag{4.6}$$

Putting $Y = Z = \xi_i$ in (2.2) and using (3.11), we obtain

$$\rho(X)S(\xi_i, \xi_i) + \mu(\xi_i)S(X, \xi_i) + \nu(\xi_i)S(X, \xi_i) = 0 \tag{4.7}$$

Adding (4.5), (4.6) and (4.7) and then using (4.4), we have

$$[\rho(X) + \mu(X) + \nu(X)]S(\xi_i, \xi_i) = 0 \tag{4.8}$$

Now from (4.8), we have

$$\rho(X) + \mu(X) + \nu(X) = 0, \qquad \forall X.$$

Thus

$$\rho + \mu + \nu = 0.$$

Hence the theorem is proved.

5. Weakly symmetric almost r-para contact Riemannian manifold of Kenmotsu type

Here we assume that the weakly symmetric manifold is almost r-para contact Riemannian manifold of Kenmotsu type. Then we have

Theorem 5.1 : Any weakly symmetric almost r-para contact Riemannian manifold of Kenmotsu type M satisfies $\alpha + \beta + \delta = 0$.

Proof . Since M is weakly symmetric, by taking $X = \xi_i$ in (2.3), we have

$$(\nabla_{\xi_i} S)(Z, U) = \alpha(\xi_i)S(Z, U) + \beta(Z)S(\xi_i, U) + \delta(U)S(Z, \xi_i) + \beta(R(\xi_i, Z)U) + \delta(R(\xi_i, U)Z) \tag{5.1}$$

By virtue of (2.26) and (2.31), we obtain

$$(\nabla_{\xi_i} S)(Z, U) = 0 \tag{5.2}$$

From (5.1) and (5.2), we have

$$\alpha(\xi_i)S(Z, U) + \beta(Z)S(\xi_i, U) + \delta(U)S(Z, \xi_i) + \beta(R(\xi_i, Z)U) + \delta(R(\xi_i, U)Z) = 0 \tag{5.3}$$

Putting $Z = U = \xi_i$ in (5.3) and using (2.29), we get

$$[\alpha(\xi_i) + \beta(\xi_i) + \delta(\xi_i)]S(\xi_i, \xi_i) = 0 \tag{5.4}$$

which gives

$$\alpha(\xi_i) + \beta(\xi_i) + \delta(\xi_i) = 0. \tag{5.5}$$

This shows that $\alpha + \beta + \delta$ vanishes over the vector field ξ_i on M.

Now we will show that $\alpha + \beta + \delta = 0$ holds for all vector fields on M. In (2.3), taking $X = Z = \xi_i$, we obtain

$$(\nabla_{\xi_i} S)(\xi_i, U) = \alpha(\xi_i)S(\xi_i, U) + \beta(\xi_i)S(\xi_i, U) + \delta(U)S(\xi_i, \xi_i) + \beta(R(\xi_i, \xi_i)U) + \delta(R(\xi_i, U)\xi_i) \tag{5.6}$$

By putting $U = X$ in (5.6), we get

$$\alpha(\xi_i)S(\xi_i, X) + \beta(\xi_i)S(\xi_i, X) + \delta(X)S(\xi_i, \xi_i) + \beta(R(\xi_i, \xi_i)X) + \delta(R(\xi_i, X)\xi_i) = 0 \tag{5.7}$$

In (2.3), taking $X = U = \xi_i$, we get

$$(\nabla_{\xi_i} S)(Z, \xi_i) = \alpha(\xi_i)S(Z, \xi_i) + \beta(Z)S(\xi_i, \xi_i) + \delta(\xi_i)S(Z, \xi_i) + \beta(R(\xi_i, Z)\xi_i) + \delta(R(\xi_i, \xi_i)Z) \tag{5.8}$$

Using (5.2) in (5.8) and then replacing Z by X, we have

$$\alpha(\xi_i)S(X, \xi_i) + \beta(X)S(\xi_i, \xi_i) + \delta(\xi_i)S(X, \xi_i) + \beta(R(\xi_i, X)\xi_i) + \delta(R(\xi_i, \xi_i)X) = 0 \tag{5.9}$$

Again in (2.3), taking $Z = U = \xi_i$, we get

$$(\nabla_X S)(\xi_i, \xi_i) = \alpha(X)S(\xi_i, \xi_i) + \beta(\xi_i)S(X, \xi_i) + \delta(\xi_i)S(\xi_i, X) + \beta(R(X, \xi_i)\xi_i) + \delta(R(X, \xi_i)\xi_i) \tag{5.10}$$

Here also we have

$$(\nabla_X S)(\xi_i, \xi_i) = 0 \tag{5.11}$$

Using (5.11) in (5.10), we obtain

$$\alpha(X)S(\xi_i, \xi_i) + \beta(\xi_i)S(X, \xi_i) + \delta(\xi_i)S(\xi_i, X) + \beta(R(X, \xi_i)\xi_i) + \delta(R(X, \xi_i)\xi_i) = 0 \tag{5.12}$$

adding (5.7), (5.9) and (5.12) and then using (5.5), we get

$$[\alpha(X) + \beta(X) + \delta(X)]S(\xi_i, \xi_i) = 0 \tag{5.13}$$

Hence from (5.13), we obtain

$$\alpha(X) + \beta(X) + \delta(X) = 0, \qquad \forall X.$$

Thus

$$\alpha + \beta + \delta = 0.$$

Hence the theorem is proved.

6. Weakly Ricci-symmetric almost r-para contact Riemannian manifold of Kenmotsu type

We suppose that the weakly Ricci-symmetric manifold is almost r-para contact Riemannian manifold of Kenmotsu type. Then we have

Theorem 6.1 : Any weakly Ricci-symmetric almost r-para contact Riemannian manifold of Kenmotsu type M satisfies $\rho + \mu + \nu = 0$.

Proof . Since M is weakly Ricci-symmetric almost r-para contact Riemannian manifold of Kenmotsu type,

Putting $X = \xi_i$ in (2.2) we get

$$(\nabla_{\xi_i} S)(Y,Z) = \rho(\xi_i)S(Y,Z) + \mu(Y)S(\xi_i,Z) + \nu(Z)S(\xi_i,Y) \tag{6.1}$$

Using (5.2) in (6.1), we have

$$\rho(\xi_i)S(Y,Z) + \mu(Y)S(\xi_i,Z) + \nu(Z)S(\xi_i,Y) = 0 \tag{6.2}$$

Replacing Y and Z by ξ_i in (6.2), we obtain

$$[\rho(\xi_i) + \mu(\xi_i) + \nu(\xi_i)]S(\xi_i,\xi_i) = 0 \tag{6.3}$$

which gives

$$\rho(\xi_i) + \mu(\xi_i) + \nu(\xi_i) = 0 \tag{6.4}$$

Taking $X = Y = \xi_i$ in (2.2) and using (5.2), then replacing Z by X, we obtain

$$\rho(\xi_i)S(\xi_i,X) + \mu(\xi_i)S(\xi_i,X) + \nu(X)S(\xi_i,\xi_i) = 0 \tag{6.5}$$

In (2.2), taking $X = Z = \xi_i$ and using (5.2), we get

$$\rho(\xi_i)S(Y,\xi_i) + \mu(Y)S(\xi_i,\xi_i) + \nu(\xi_i)S(\xi_i,Y) = 0 \tag{6.6}$$

Replacing Y by X in (6.6), we have

$$\rho(\xi_i)S(X,\xi_i) + \mu(X)S(\xi_i,\xi_i) + \nu(\xi_i)S(\xi_i,X) = 0 \tag{6.7}$$

Putting $Y = Z = \xi_i$ in (2.2) and using (5.11), we obtain

$$\rho(X)S(\xi_i,\xi_i) + \mu(\xi_i)S(X,\xi_i) + \nu(\xi_i)S(X,\xi_i) = 0 \tag{6.8}$$

Adding (6.5), (6.7) and (6.8) and then using (6.4), we have

$$[\rho(X) + \mu(X) + \nu(X)]S(\xi_i,\xi_i) = 0 \tag{6.9}$$

Now from (6.9), we have

$$\rho(X) + \mu(X) + \nu(X) = 0, \qquad \forall X.$$

Thus

$$\rho + \mu + \nu = 0.$$

Hence the theorem is proved.

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On Four Dimensional Finsler Space Satisfying T-Conditions

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Abstract

The purpose of the present paper is to consider the four dimensional Finsler spaces with $T_{hijk} = 0$ and generalize the idea of Landsberg angle to four dimensional Finsler spaces. The properties of a Finsler space satisfying $T-$condition has been studied in a three dimensional Finsler space by various authors ([2], [3], [4], [8], [10]). But from the relativistic point of view the importance of four dimensional Finsler space is not negligible. In relativity the fourth coordinate is taken as time, from this point of view we discuss the properties of four dimensional Finsler space satisfying $T-$condition. The results which are reducible to the three dimensional case also.

1. Introduction

H. Kawaguchi and M. Matsumoto have introduced the T-tensor in a Finsler space independently ([6], [5]). It is indicatrised tensor and studied by several authors ([1], [2], [3], [4], [8]). The vanishing of $T-$tensor is called $T-$condition. Hashiguchi [1] noticed the importance of $T-$tensor from the stand point of Landsberg spaces. It has been proved by him that a necessary and sufficient condition for a Landsberg space to be conformally invariant is that it satisfy $T-$condition.

The Landsberg angle θ was introduced by Landsberg in 1908. The coordinate system (L, θ) in a tangent plane M_x is regarded as a generalization of the polar coordinate system (r, θ) of a Euclidean plane. M. Matsumoto [9] gave the idea of Landsberg angle in two and three dimensional Finsler space.

In this paper we have considered four dimensional Finsler space with $T_{hijk} = 0$, and generalized the idea of Landsberg angle to four dimensional Finsler spaces.

Let M^4 be four dimensional Finsler space endowed with a fundamental function $L = L(x, y)$, where $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of M^4. The metric tensor g_{ij} and (h) hv-torsion tensor C_{ijk} of M^4 is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}. \tag{1.1}$$

If $[g^{ij}]$ denote the inverse matrix of $[g_{ij}]$ then, we have $g_{ij}g^{jk} = \delta_i^k$. The $T-$tensor T_{ijkl} is defined as

$$T_{hijk} = LC_{hij}\,|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}\ , \tag{1.2}$$

where $l_i = L^{-1}\, g_{ir} y^r$ and '$|$' denotes the $v-$covariant derivative with respect to Cartan connection $C\Gamma$ of M^4. For instance the $v-$covariant derivative of a tensor field $T^i_j(x, y)$ is defined by

$$T^i_j \Big|_k = \dot{\partial}_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk}\ , \tag{1.3}$$

where $\dot{\partial}_k = \frac{\partial}{\partial y^i}$, $\partial_k = \frac{\partial}{\partial x^k}$.

2. Scalar components in Miron frame

Let M^4 be a four dimensional Finsler space with the fundamental function $L(x, y)$. The frame $\{e^i_{\alpha)}\}$, $\alpha = 1, 2, 3, 4$ is called the Miron's frame of M^4, where $e^i_{1)} = l^i = y^i/L$ is the normalized supporting element, $e^i_{2)} = m^i = C^i/C$ is the normalized torsion vector, $e^i_{3)} = n^i$, $e^i_{4)} = p^i$ are constructed by $g_{ij}e^i_{\alpha)}e^j_{\beta)} = \delta_{\alpha\beta}$. Here C is the length of torsion vector $C_i = C_{ijk}g^{jk}$. The Greek letters $\alpha, \beta, \gamma, \delta$ varies from 1 to 4. Summation convention is applied for both the Greek and Latin indices.

In Miron's frame an arbitrary tensor field can be expressed by scalar components along the unit vectors $e^i_{\alpha)}$, $\alpha = 1, 2, 3, 4$. For instance, let T^i_j be a tensor field of type (1, 1), then the scalar components $T_{\alpha\beta}$ of T^i_j are defined by $T_{\alpha\beta} = T^i_j e_{\alpha)i} e^j_{\beta)}$ and the components T^i_j are expressed as $T^i_j = T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j}$. From the equation $g_{ij}e^i_{\alpha)}\, e^j_{\beta)} = \delta_{\alpha\beta}$, we have

$$g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j. \tag{2.1}$$

The C-tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^i}$ satisfies $C_{ijk}l^k = 0$ and is symmetric in i, j, k therefore if $C_{\alpha\beta\gamma}$ be the scalar components of LC_{ijk}, i.e. if

$$LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha)i} e_{\beta)j} e_{\gamma)k}, \tag{2.2}$$

then, we have [10]

$$\begin{aligned} LC_{ijk} &= C_{222} m_i m_j m_k + C_{333} n_i n_j n_k + C_{444} p_i p_j p_k + C_{233}\pi_{(ijk)}(m_i n_j n_k) \\ &+ C_{244}\pi_{(ijk)}(m_i p_j p_k) + C_{344}\pi_{(ijk)}(n_i p_j p_k) + C_{322}\pi_{(ijk)}(m_i m_j n_k) \\ &+ C_{433}\pi_{(ijk)}(n_i n_j p_k) + C_{422}\pi_{(ijk)}(m_i m_j p_k) + C_{234}\pi_{(ijk)}\{m_i(n_j p_k + n_k p_j)\}, \end{aligned} \tag{2.3}$$

where $\pi_{(ijk)}$ denote the cyclic permutation of indices i, j, k and summation. For instance

$$\pi_{(ijk)}(A_i B_j C_k) = A_i B_j C_k + B_i C_j A_k + C_i A_j B_k.$$

Contracting (2.2) with g^{jk}, we get $LCm_i = C_{\alpha\beta\beta} e_{\alpha)i}$. Thus if we put

$$\begin{aligned} &C_{222} = H, \quad C_{233} = I, \quad C_{244} = K, \quad C_{333} = J, \\ &C_{344} = J', \quad C_{444} = H', \quad C_{433} = I', \quad C_{234} = K', \end{aligned} \tag{2.4}$$

then we have

$$H + I + K = LC, \qquad C_{322} = -(J + J'), \qquad C_{422} = -(H' + I'). \tag{2.5}$$

The eight scalars $H, I, J, K, H', I', J', K'$ are called the main scalars of a four dimensional Finsler space.

The $v-$covariant derivative of the frame field $e_{\alpha)i}$ is given by

$$Le_{\alpha)i}|_j = V_{\alpha)\beta\gamma} e_{\beta)i} e_{\gamma)j}, \tag{2.6}$$

where $V_{\alpha)\beta\gamma}$, γ being fixed are given by

$$V_{\alpha)\beta\gamma} = \begin{bmatrix} 0 & \delta_{2\gamma} & \delta_{3\gamma} & \delta_{4\gamma} \\ \delta_{2\gamma} & 0 & u_\gamma & v_\gamma \\ \delta_{3\gamma} & -u_\gamma & 0 & w_\gamma \\ \delta_{4\gamma} & -v_\gamma & -w_\gamma & 0 \end{bmatrix} \quad \text{and} \quad \begin{aligned} V_{2)3\gamma} &= -V_{3)2\gamma} = u_\gamma \\ V_{2)4\gamma} &= -V_{4)2\gamma} = v_\gamma \\ V_{3)4\gamma} &= -V_{4)3\gamma} = w_\gamma \end{aligned} \tag{2.7}$$

Thus, in a four dimensional Finsler space there exists three $v-$connection vectors u_i, v_i, w_i whose scalar components with respect to the frame $\{e^i_{\alpha)}\}$ are u, v, w, i.e.

$$u_i = ue_{\gamma)i}, \qquad v_i = ve_{\gamma)i} \qquad w_i = we_{\gamma)i}. \tag{2.8}$$

In view of equations (2.8), the equation (2.6) may be explicitly written as

$$(2.9)\qquad \begin{aligned} Ll_i|_j &= m_i m_j + n_i n_j + p_i p_j & Lm_i|_j &= -\,l_i m_j + n_i u_j + p_i v_j,\\ Ln_i|_j &= -l_i n_j - m_i u_j + p_i w_j, & Lp_i|_j &= -\,l_i p_j - m_i v_j - n_i w_j. \end{aligned}$$

Since m_i, n_i, p_i are homogeneous functions of degree zero in y_i, we have

$$Lm_i|_j l^j = Ln_i|_j l^j = Lp_i|_j l^j = 0,$$

which in view of equations (2.8) and (2.9) gives $u_1 = 0$, $v_1 = 0$, $w_1 = 0$. Therefore

Lemma (2.1). The first scalar components u_1, v_1, w_1 of the $v-$connection vectors u_i, v_i, w_i vanishes identically, that is u_i, v_i, w_i are orthogonal to l^i.

3. Four-dimensional Finsler space satisfying the T-condition

The scalar derivative of the adopted components $T_{\alpha\beta}$ of T^i_j is defined as [9]

$$(3.1)\qquad T_{\alpha\beta;\gamma} = L(\partial_k T_{\alpha\beta})e_{\gamma)}{}^k + T_{\mu\beta}V_{\mu)\alpha\gamma} + T_{\alpha\mu}V_{\mu)\beta\gamma},$$

Thus $T_{\alpha\beta;\gamma}$ are adopted components of $LT^i_j\,|_k$, i.e.

$$(3.2)\qquad LT^i_j\,|_k\ = T; e^i_{\alpha)}e_{\beta)j}e_{\gamma)k}.$$

If the tensor field T^i_j is positively homogeneous of degree zero in y^i, $T_{\alpha\beta}$ is also positively homogeneous of degree zero in y^i, so equation (3.1) gives

$$T_{\alpha\beta;1} = T_{\mu\beta}V_{\mu)\alpha 1} + T_{\alpha\mu}V_{\mu)\beta 1},$$

which in view of (2.7) and lemma (2.1) gives $T_{\alpha\beta;1} = 0$. Therefore we have the following:

Proposition (3.1). If the tensor field T^i_j is positively homogeneous of degree zero in y^i, then $T_{\alpha\beta;1} = 0$.

Now, let T^i_j be positively homogenous of degree r in y^i and $T_\alpha\beta$ be the scalar components of $L^{-r}T^i_j$, then $L(L^{-r}T^i_j)\,|_k\ = T_{\alpha\beta;\gamma}e^i_{\alpha)}e_{\beta)j}e_{\gamma)k} = L^{-r+1}T^i_j\,|_k - rL^{-r}$ $T^i_j e_{1)k}$, which implies

$$(3.3)\qquad L^{-r+1}T^i_j\,|_k\ = (T_{\alpha\beta;\gamma} + rT_{\alpha\beta}\delta_{1\gamma})e^i_{\alpha)}e_{\beta)j}e_{\gamma)k}.$$

Hence we have

Proposition (3.2). If the tensor field T^i_j is positively homogeneous of degree r in y^i and $T_{\alpha\beta}$ be the scalar components of $L^{-r}T^i_j$, then the scalar components of $L^{-r+1}T^i_j\,|_k$ are given by $T_{\alpha\beta;\gamma} + rT_{\alpha\beta}\delta_{1\gamma}$.

Definition (3.1). The Finsler space M^4 is said to satisfy the T-condition if the $T-$tensor T_{hijk} of M^4 vanishes identically.

The C-tensor C_{ijk} is positively homogeneous of degree -1 in y^i, therefore from proposition (3.2) the scalar components of $L^2C_{ijk}\,|_h$ are given by

$$L^2C_{ijk}\,|_h = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta})e_{\alpha)i}e_{\beta)j}e_{\gamma)k}e_{\delta)h}, \tag{3.4}$$

And the scalar components $T\alpha\beta\gamma\delta$ of LT_{hijk} are given by

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma} + C_{\alpha\beta\gamma}\delta_{1\delta}. \tag{3.5}$$

We know that the $T-$tensor is indicatrized tensor and is symmetric in all indices, therefore $T_{hijk}\,l^k = 0$ i.e. $T_{\alpha\beta\gamma 1} = 0$. Therefore, the surviving scalar components of LT_{hijk} are given by

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} \qquad \alpha,\beta,\gamma,\delta = 2,3,4. \tag{3.6}$$

Since $C_{hij}\,|_k = C_{hik}\,|_j$, from (3.4) we have

$$C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta} = C_{\alpha\beta\delta;\gamma} - C_{\alpha\beta\delta}\delta_{1\gamma}. \tag{3.7}$$

In case of $(\gamma,\delta) = (1,2)$, $(1,3)$ and $(1,4)$ the above relation is trivial and when $(\gamma,\delta) =$ (2, 3), (2, 4), (3, 4), we get

$$C_{\alpha\beta 3;2} = C_{\alpha\beta 2;3}, \qquad C_{\alpha\beta 4;2} = C_{\alpha\beta 2;4}, \qquad C_{\alpha\beta 4;3} = C_{\alpha\beta 3;4}. \tag{3.8}$$

These equations are trivial for $\alpha,\beta = 1$. Consequently, we put $(\alpha,\beta) =$ (2, 2), (2, 3), (2, 4), (3, 3), (3.4), (4, 4) in equation (3.8). For instance $C_{223;2} = C_{222;3}$ etc. In view of (2.4) and (2.7), this equation is explicitly written as

$$(\dot\partial_i C_{223})e^i_{2)} + 2C_{\mu 23}V_{\mu)22} + C_{\mu 22}V_{\mu)32} = (\dot\partial_i C_{222})e^i_{3)} + 3C_{\mu 22}V_{\mu)23},$$

Or

$$\begin{aligned} &-(J+J')_{;2} + (H-2I)\,u_2 - 2K'v_2 + (H'+I')\,w_2 \\ &\quad = H_{;3} + 3(J+J')u_3 + 3(H'+I')\,v_3. \end{aligned} \tag{3.9)(a}$$

Similarly, from (2.4), (2.7) and (3.8), we get

$$I_{;2} - (3J+2J')u_2 - I'v_2 - 2K'w_2 \tag{3.9)(b}$$

$$= -(J+J')_{;3} + (H-2I)\,u_3 - 2K'v_3 + (H'+I')\,w_3,$$

(c)
$$K'_{;2} - (H'+2I')\,u_2 - (J+2J')\,v_2 + (I-K)\,w_2$$
$$= (H'+I')_{;3} - 2K'u_3 + (H-2K)\,v_3 - (J+J')\,w_3,$$
$$= (J+J')_{;4} + (H-2I)\,u_4 - 2K'\,v_4 + (H'+I')\,w_4,$$

(d)
$$J_{;2} + 3Iu_2 - 3I'\,w_2 = I_{;3} - (3J+2J')\,u_3 - I'v_3 - 2K'w_3,$$

(e)
$$I'_{;2} + 2K'u_2 + Iv_2 + (J-2J')w_2$$
$$= K'_{;3} - (H'+2I')u_3 - (J+2J')v_3 + (I-K)w_3$$
$$= I_{;4} - (3J+2J')u_4 - I'v_4 - 2K'w_4,$$

(f)
$$J'_{;2} + Ku_2 + 2K'v_2 + (2I'-H')w_2 = K_{;3} - J'u_3 - (3H'+2I')v_3 + 2K'w_3$$
$$= K'_{;4} - (H'+2I')u_4 - (J+2J')v_4 + (I-K)w_4,$$

(g)
$$-(H'+I')_{;2} - 2K'u_2 + (H-2K)v_2 - (J+J')w_2$$
$$= H_{;4} + 3(J+J')u_4 + 3(H'+I')v_4,$$

(h)
$$K_{;2} - J'u_2 - (3H'+2I')v_2 + 2K'\,w_2$$
$$= -(H'+I')_{;4} - 2K'u_4 + (H-2K)v_4 - (J+J')w_4,$$

(i)
$$H'_{;2} + 3Kv_2 + 3J'w_2 = K_{;4} - J'u_4 - (3H'+2I')v_4 + 2K'w_4,$$

(j)
$$I'_{;3} + 2K'u_3 + Iv_3 + (J-2J')w_3 = J_{;4} + 3Iu_4 - 3I'w_4,$$

(k)
$$J'_{;3} + Ku_3 + 2K'v_3 + (2I'-H')w_3 = I'_{;4} + 2K'u_4 + Iv_4 + (J-2J')w_4,$$

(l)
$$H'_{;3} + 3Kv_3 + 3J'w_3 = J'_{;4} + Ku_4 + 2K'v_4 + (2I'-H')w_4.$$

Since T_{hijk} is symmetric in all indices and $T_{1\beta\gamma\delta} = 0$, $\beta, \gamma, \delta = 2, 3, 4$, therefore, the surviving independent components are fifteen and they are

$$T_{2222},\quad T_{2223},\quad T_{2224},\quad T_{2234},\quad T_{2244},\quad T_{2233},$$
$$T_{2333},\quad T_{2334},\quad T_{2344},\quad T_{2444},\quad T_{3333},\quad T_{3334},$$
$$T_{3344},\quad T_{3444},\quad T_{4444}.$$

In view of (2.4), (2.7), (3.6) and (3.9) these scalar components are explicitly written as

$$T_{2222} = H_{;2} + 3(J+J')u_2 + 3(H'+I')v_2,$$
$$T_{2223} = H_{;3} + 3(J+J')u_3 + 3(H'+I')v_3$$

$$= -(J+J')_{;2} + (H-2I)u_2 - 2K'v_2 + (H'+I')w_2,$$

$$T_{2224} = H_{;4} + 3(J+J')u_4 + 3(H'+I')v_4$$
$$= -(H'+I')_{;2} - 2K'u_2 + (H-2K)v_2 - (J+J')w_2,$$

$$T_{2234} = -(J+J')_{;4} + (H-2I)u_4 - 2K'v_4 + (H'+I')w_4$$
$$= -(H'+I')_{;3} - 2K'u_3 + (H-2K)v_3 - (J+J')w_3$$
$$= K'_{;2} - (H'+2I')u_2 - (J+2J')v_2 + (I-K)w_2,$$

$$T_{2244} = -(H'+I')_{;4} - 2K'u_4 + (H-2K)v_4 - (J+J')w_4$$
$$= K_{;2} - J'u_2 - (3H'+2I')v_2 + 2K'w_2,$$

$$T_{2233} = -(J+J')_{;3} + (H-2I)u_3 - 2K'v_3 + (H'+I')w_3$$
$$= I_{;2} - (3J+2J')u_2 - I'v_2 - 2K'w_2,$$

$$T_{2333} = I_{;3} - (3J+2J')u_3 - I'v_3 - 2K'w_3 = J_{;2} + 3Iu_2 - 3I'w_2,$$

$$T_{2334} = I_{;4} - (3J+2J')u_4 - I'v_4 - 2K'w_4$$
$$= K'_{;3} - (H'+2I')u_3 - (J+2J')v_3 + (I-K)w_3$$
$$= I'_{;2} + 2K'u_2 + Iv_2 + (J-2J')w_2,$$

$$T_{2344} = K'_{;4} - (H'+2I')u_4 - (J+2J')v_4 + (I-K)w_4$$
$$= K_{;3} - J'u_3 - (3H'+2I')v_3 + 2K'w_3$$
$$= J'_{;2} + Ku_2 + 2K'v_2 + (2I'-H')w_2,$$

$$T_{2444} = K_{;4} - J'u_4 - (3H'+2I')v_4 + 2K'w_4 = H'_{;2} + 3Kv_2 + 3J'w_2,$$

$$T_{3333} = J_{;3} + 3Iu_3 - 3I'w_3,$$

$$T_{3334} = J_{;4} + 3Iu_4 - 3I'w_4 = I'_{;3} + 2K'u_3 + Iv_3 + (J-2J')w_3,$$

$$T_{3344} = I'_{;4} + 2K'u_4 + Iv_4 + (J-2J')w_4 = J'_{;3} + Ku_3 + 2K'v_3 + (2I'-H')w_3,$$

$$T_{3444} = J'_{;4} + Ku_4 + 2K'v_4 + (2I'-H')w_4 = H'_{;3} + 3Kv_3 + 3J'w_3,$$

$$T_{4444} = H'_{;4} + 3Kv_4 + 3J'w_4.$$

Now, we consider four dimensional Finsler space with vanishing $T-$tensor, then all the scalar components $T_{\alpha\beta\gamma\delta} = 0,\ \alpha,\beta,\gamma,\delta = 1,2,3,4$. Thus $T_{2234} = T_{3334} = T_{3444} = 0$ gives

$$-(J+J')_{;4} + (H-2I)u_4 - 2K'v_4 + (H'+I')w_4 = 0, \tag{3.10}$$

(3.11) $$J;_4 + 3Iu_4 - 3I'w_4 = 0,$$

(3.12) $$J';_4 + Ku_4 + 2K'v_4 + (2I' - H')w_4 = 0.$$

Adding (3.11), (3.12) and (3.10), we get

(3.13) $$(H + I + K)u_4 = 0.$$

Using (2.5) in (3.13) we get $LCu_4 = 0$. Since $LC \neq 0$, we have $u_4 = 0$.

Similarly, from $T_{2223} = T_{2233} = T_{2333} = T_{2344} = T_{3333} = T_{3344} = 0$, we get $u_3 = u_4 = 0$. Thus $u_\alpha = 0$ for $\alpha = 1, 2, 3, 4$ which implies $u_i = 0$.

Again $T_{2244} = T_{3344} = T_{4444} = 0$ gives

(3.14) $$-(H' + I');_4 - 2K'u_4 + (H - 2K)v_4 - (J + J')w_4 = 0,$$

(3.15) $$I';_4 + 2K'u_4 + Iv_4 + (J - 2J')w_4 = 0,$$

(3.16) $$H';_4 + 3Kv_4 + 3J'w_4 = 0.$$

Adding (3.14), (3.15) and (3.16) we get $(H+I+K)v_4 = 0$ which implies $v_4 = 0$.

Similarly, $T_{2224} = T_{2234} = T_{2334} = T_{2444} = T_{3334} = T_{3444} = 0$ give $v_2 = 0 = v_3$. Therefore $v_\alpha = 0$ for $\alpha = 1, 2, 3, 4$ which implies $v_i = 0$. Putting $u_2 = 0, u_3 = 0, v_2 = 0$, $v_3 = 0, u_4 = 0, v_4 = 0$ in $T_{2222} = 0, T_{2223} = 0$ and $T_{2224} = 0$ we get, $H;_2 = 0$, $H;_3 = 0$ and $H;_4 = 0$. Thus $H;_\alpha = 0$, for $\alpha = 2, 3, 4$. Putting $u_2 = 0$, $v_2 = 0$ in $T_{2234} = 0$, $u_3 = 0$, $v_3 = 0$ in $T_{2344} = 0$ and $u_4 = 0$, $v_4 = 0$ in $T_{2444} = 0$, we get

(3.17) $$K';_2 + (I - K)w_2 = 0, \qquad K';_3 + (I - K)w_3, \qquad K;_4 + 2K'w_4 = 0.$$

We consider two cases.

Case 1. If $I \neq K$ and $K';_\alpha = 0$ for $\alpha = 2, 3, 4$, then from (3..17) we get $w_\alpha = 0$ for $\alpha = 2$, 3, 4 i.e. $w_i = 0$. Hence $T_{\alpha\beta\gamma\delta} = 0$ gives $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 2$, 3, 4. Since the main scalars H, I, J, K, H', I', J' are positively homogeneous of degree one in y^i, we have $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 1$. Hence the main scalars H, I, J, K, H', I', J' does not depend on y^i. Therefore we have the following:

Theorem (3.1). If main scalar K' is independent of directional arguments y^i, and $I \neq K$, the $T-$condition for a non-Riemannian Finsler space of four dimension is equivalent to the fact that the $v-$connection vectors u_i, v_i, and w_i

vanishes identically and the remaining seven main scalars $H, I, J, K,\ H', I', J'$ are also functions of position alone.

Case 2. If $I = K$ then equation (3.17) gives $K';_\alpha = 0$ for $\alpha = 2, 3, 4$. Also $u_i = 0$, $v_i = 0$ gives $H;_\alpha = 0$. Putting these values in $T_{2233} = 0$, $T_{2244} = 0$, $T_{2333} = 0$, $T_{2344} = 0$, $T_{2334} = 0$, and $T_{2444} = 0$, we get

$$(3.17) \qquad \begin{aligned} I;_2 - 2K'w_2 &= 0, & K;_2 + 2K'w_2 &= 0 \\ I;_3 - 2K'w_3 &= 0, & K;_3 + 2K'w_3 &= 0, \\ I;_4 - 2K'w_4 &= 0, & K;_4 + 2K'w_4 &= 0. \end{aligned}$$

These equations gives $I;_\alpha + K;_\alpha = 0$ for $\alpha = 2, 3, 4$. Since $I = K$, we have $I;_\alpha = K;_\alpha = 0$ for $\alpha = 2, 3, 4$. Putting these values in (3.17) we get $w_2 = w_3 = w_4 = 0$, provided $K' \neq 0$. This implies that $w_i = 0$. Hence $T_{\alpha\beta\gamma\delta} = 0$ gives $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 2, 3, 4$. Since the main scalars $H, I, J, K,\ H', I', J'$ are positively homogeneous of degree one in y^i, we have $H;_\alpha = I;_\alpha = J;_\alpha = K;_\alpha = H';_\alpha = I';_\alpha = J';_\alpha = 0$ for $\alpha = 1$. Hence all the eight main scalars $H, I, J, K,\ H', I', J', K'$ are functions of position alone. Therefore we have the following:

Theorem (3.2). If main scalars I and K are equal, and $K' \neq 0$, the $T-$condition for a non-Riemannian Finsler space of four dimensions is equivalent to the fact that the $v-$connection vectors u_i, v_i, and w_i vanishes identically and all the main scalars $H, I, J, K,\ H', I', J', K'$ are functions of position alone.

Remark (3.1). It should be remarked here that the conditions $I \neq K$ and $K';_\alpha = 0$ in theorem (3.1) and $I = K$ and $K' \neq 0$ in theorem (3.2) is only necessary for a Finsler space satisfying $T-$condition to vanish $v-$connection vectors and all the main scalars to be functions of position alone. On the other hand if all the $v-$connection vectors vanish and all the main scalars are functions of position alone, then a four dimensional Finsler space satisfies $T-$condition.

Theorem (3.3)[1]. The tensor T_{hijk} vanishes if and only if the tensor P^i_{jkl} be invariant under any conformal transformation.

In view of theorems (3.2) and (3.3) we have the following:

Theorem (3.4). If $v-$connection vectors u_i, v_i, and w_i of a four dimensional Finsler space M^4 vanishes, and all the main scalars are functions of position alone, then (v) $hv-$curvature tensor P^i_{jkl} of M^4 is conformally invariant under any conformal transformation.

Theorem (3.5)[1]. A Landsberg space remains to be a Landsberg space by any conformal transformation if and only if $T_{hijk} = 0$.

In view of theorems (3.5) and (3.2) we have the following:

Theorem (3.6). If $v-$connection vectors u_i, v_i, and w_i of a four dimensional Finsler space M^4 vanishes, and all the main scalars are functions of position alone, then a Landsberg space remains to be a Landsberg space under any conformal transformation.

4. Landsberg angle in four dimensional Finsler space

In this section we consider Landsberg angle in four dimensional Finsler space M^4. The Landsberg angle θ, ϕ of three dimensional Finsler space with $v-$connection vector $v_i = 0$ is given by [9]

$$\dot{\partial}_i\theta = L^{-1}m_i, \qquad \dot{\partial}_i\phi = L^{-1}n_i. \tag{4.1}$$

The class of four dimensional Finsler spaces with v-connection vectors $u_i = v_i = w_i = 0$ is interested from the view point that we can generalize the Landsberg angle θ, ϕ of three dimensional Finsler space to four dimensions as follows:

We consider the differential equations

$$\dot{\partial}_i\theta = L^{-1}m_i, \qquad \dot{\partial}_i\phi = L^{-1}n_i, \qquad \dot{\partial}_i\psi = L^{-1}p_i, \tag{4.2}$$

Proposition (4.1). If the $v-$connection vectors u_i, v_i and w_i of a four dimensional Finsler space M^4 vanish identically, there exist three scalar fields θ, ϕ and ψ satisfying the differential equation (4.2).

These scalars θ, ϕ, ψ are defined up to additional functions of position only and may be called the Landsberg angles of such a special four dimensional Finsler space.

On account of (2.9) with $u_i = v_i = w_i = 0$ it is easy to show that these equations are completely integrable. The L, θ, ϕ and ψ are regarded as polar coordinates of a kind of the tangent space and

$$\frac{\partial y^i}{\partial L} = l^i, \qquad \frac{\partial y^i}{\partial \theta} = Lm^i, \qquad \frac{\partial y^i}{\partial \phi} = Ln^i, \qquad \frac{\partial y^i}{\partial \psi} = Lp^i, \tag{4.3}$$

are immediately derived.

Let g be the determinant of the fundamental tensor g_{ij} then from $\dot{\partial}_i = 2g\,C_i = 2g\,Cm_i$, it follows that

$$\frac{\partial g}{\partial L} = 0, \qquad \frac{\partial g}{\partial \theta} = 2(LC)g, \qquad \frac{\partial g}{\partial \phi} = 0, \text{ and } \frac{\partial g}{\partial \psi} = 0.$$

Proposition (4.2). The determinant g of the fundamental tensor g_{ij} of a four dimensional non-Riemannian Finsler space with the vanishing $v-$connection vectors u_i, v_i, w_i is of the form $g = te^{2\theta(LC)}$ where t and LC are the functions of position alone. LC is the unified main scalar and θ is the first Landsberg angle.

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On Generalized W_2-recurrent $(LCS)_n$-manifolds

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Abstract

The object of the present paper is to study generalized recurrent and generalized W_2-recurrent $(LCS)_n$-manifolds.

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1. Introduction

In 2003 A. A. Shaikh [8] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example. An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp. non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp. $\leq 0, = 0, > 0$) [1, 4].

Recurrent spaces have been of great interest and were studied by a large number of authors such as Ruse [7], Patterson [5], U. C. De and N. Guha [2], Y. B. Maralabhavi and M. Rathnamma [3] etc. In this paper, I have studied a special type of Lorentzian manifolds called $(LCS)_n$-manifolds with generalized recurrent and generalized W_2-recurrent $(LCS)_n$-manifolds. The paper is organized as follows: Section 2 is concerened about basic identities of $(LCS)_n$-manifolds. In section 3, we study generalized recurrent $(LCS)_n$-manifolds. Here it is proved that such a manifold is Einstein if and only if $\beta = 2\alpha\rho$. The last section deals with generalized W_2-recurrent $(LCS)_n$-manifold and proved that if such a manifold is Einstein with $r = n(n-1)(\alpha^2 - \rho)$, then it reduces to a W_2-recurrent manifold. Finally, sufficient condition for a generalized W_2-recurrent manifold to be a generalized recurrent manifold is given.

2. $(LCS)_n$-manifolds

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X) \tag{2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{3}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{4}$$

where ρ being a certain scalar function. By virtue of (2), (3) and (4), it follws that

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{5}$$

where $\beta = -(\xi\rho)$ is a scalar function. Next if we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{6}$$

then from (3) and (6) we have

$$\phi X = X + \eta(X)\xi, \tag{7}$$

from which it follws that ϕ is symmetric $(1,1)$ tensor and is called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefy $(LCS)_n$-manifold) [8, 9]. In a $(LCS)_n$-manifold, the following relations hold

$$(a) \qquad \eta(\xi) = -1, \qquad (b)\phi\xi = 0, \qquad (c)\eta(\phi X) = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{9}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi], \tag{10}$$

$$\eta(R(X, Y)Z) = (\rho - \alpha^2)[g(Y, Z)X - g(X, Z)Y], \tag{11}$$

$$S(X, \xi) = (n - 1)(\rho - \alpha^2)\eta(X), \tag{12}$$

$$R(X, Y)\xi = (\rho - \alpha^2)[\eta(Y)X - \eta(X)Y], \tag{13}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)(\rho - \alpha^2)\eta(X)\eta(Y), \tag{14}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

3. Generalized recurrent $(LCS)_n$-manifolds

Definition 3.1 : A $(LCS)_n$-manifold M^n is called generalized recurrrent if its curvature tensor R satisfies the condition([2])

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U + B(X)[g(Z, U)Y - g(Y, U)Z] \tag{15}$$

where, A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, \rho_1), \;\; B(X) = g(X, \rho_2), \tag{16}$$

ρ_1 and ρ_2 are vector fields associated with 1-froms A and B, respectively. If the 1-form B vanishes, then the manifold reduces to recurrent manifold.

This section deals with generalized recurrent $(LCS)_n$-manifolds.

Theorem 3.1 : A generalized recurrent $(LCS)_n$-manifold is Einstein if and only if $\beta = 2\alpha\rho$.

Let us consider a generalized recurrent $(LCS)_n$-manifold. From (15) it follows that

$$\begin{aligned} g((\nabla_X R)(Y, Z)U, V) &= A(X)g(R(Y, Z)U, V) \\ &+ B(X)[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)]. \end{aligned} \tag{17}$$

Let $\{e_i\}$, $i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y = V = e_i$, $1 \leq i \leq n$, we get

$$(\nabla_X S)(Z, U) = A(X)S(Z, U) + (n - 1)B(X)g(Z, U). \tag{18}$$

Replacing U by ξ in (18) and using (12) we have

$$(\nabla_X S)(Z, \xi) = (n - 1)[(\rho - \alpha^2)A(X) + B(X)]\eta(Z). \tag{19}$$

Now we have

$$(\nabla_X S)(Z,\xi) = \nabla_X S(Z,\xi) - S(\nabla_X Z,\xi) - S(Z,\nabla_X \xi). \tag{20}$$

which yields by virtue of (3), (4) and (12) that

$$(\nabla_X S)(Z,\xi) = (n-1)[(2\alpha\rho-\beta)\eta(X)\eta(Z)+\alpha(\alpha^2-\rho)g(X,Z)]-\alpha S(X,Z). \tag{21}$$

From (19) and (21), it follows that

$$\begin{aligned}\alpha S(X,Z) &= (n-1)[(2\alpha\rho-\beta)\eta(X)\eta(Z)+\alpha(\alpha^2-\rho)g(X,Z)]\\ &\quad -(n-1)[(\alpha^2-\rho)A(X)+B(X)]\eta(Z).\end{aligned} \tag{22}$$

Hence setting $Z = \phi Z$ in (22) and then using (8c) we have

$$S(X,Z) = (n-1)(\alpha^2-\rho)g(X,Z). \tag{23}$$

If the manifold under consideration is Einstein, then (23) implies $(\alpha^2-\rho) =$ constant and hence $2\alpha\rho-\beta = 0$. Conversely, if $2\alpha\rho-\beta = 0$, then $\nabla_X(\alpha^2-\rho) = 0$. Consequently $(\alpha^2-\rho) =$ constant. This result was proved by A.A. Shaikh [10] for generalized Ricci-recurrent $(LCS)_n$-manifolds.

Next, the nature of scalar curvature r in terms of contact forms $\eta(\rho_1)$ and $\eta(\rho_2)$ is discussed.

Theorem 3.2 : The scalar curvature r of a generalized recurrent $(LCS)_n$-manifold is related in terms of contact forms $\eta(\rho_1)$ and $\eta(\rho_2)$ as given by

$$r = [(n-1)/\eta(\rho_1)][2(\alpha^2-\rho)\eta(\rho_1)-(n-2)\eta(\rho_2)]. \tag{24}$$

Let us consider a generalized recurrent $(LCS)_n$-manifold. In (15) changing X, Y, Z; cyclically in and then adding the results, we obtain

$$\begin{aligned}&(\nabla_X R)(Y,Z)U + (\nabla_Y R)(Z,X)U + (\nabla_Z R)(X,Y)U\\ &\quad = A(X)R(Y,Z)U + B(X)[g(Z,U)Y - g(Y,U)Z]\\ &\quad +A(Y)R(Z,X)U + B(Y)[g(X,U)Z - g(Z,U)X]\\ &\quad +A(Z)R(X,Y)U + B(Z)[g(Y,U)X - g(X,U)Y].\end{aligned} \tag{25}$$

By virtue of second Bianchi identity, we have

$$\begin{aligned}&A(X)g(R(Y,Z)U,V) + B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)]\\ &+A(Y)g(R(Z,X)U,V) + B(Y)[g(X,U)g(Z,V) - g(Z,U)g(X,V)]\\ &+A(Z)g(R(X,Y)U,V) + B(Z)[g(Y,U)g(X,V) - g(X,U)g(Y,V)] = 0.\end{aligned} \tag{26}$$

Contraction (26) with respect to Z and U, we get

$$\begin{aligned}&A(X)S(Y,V)+(n-1)B(X)g(Y,V)\\&-A(Y)S(X,V)-(n-1)B(Y)g(X,V)\\&-A(R(X,Y)V)+B(Y)g(X,V)-B(X)g(Y,V)=0.\end{aligned} \tag{27}$$

Again, by contraction (26) with respect to Y and V, we get

$$A(X)r+(n-1)(n-2)B(X)-2S(X,\rho_1)=0. \tag{28}$$

Taking $X=\xi$ and then using (12) and (18), we have the required result.

4. Generalized W_2-recurrent $(LCS)_n$-manifolds

In 1970 G. P. Pokhariyal and R. S. Mishra [6] introduced the notion of a new curvature tensor, denoted by W_2 and studied its relativistic significance. The W_2-curvature tensor of type $(0,4)$ is defined by

$$W_2(Y,Z,U,V)=R(Y,Z,U,V)+\frac{1}{n-1}[g(Y,U)S(Z,V)-g(Z,U)S(Y,V)] \tag{29}$$

where S is the Ricci tensor of type $(0,2)$.

Definition 4.1 : A $(LCS)_n$-manifold M^n is called generalized W_2-recurrrent if its curvature tensor W_2 satisfies the condition

$$(\nabla_X W_2)(Y,Z)U=A(X)W_2(Y,Z)U+B(X)[g(Z,U)Y-g(Y,U)Z] \tag{30}$$

where A and B are as defined as in (16).

Theorem 4.1 : A generalized W_2-recurrent $(LCS)_n$-manifold is Einstein if and only if $\beta=2\alpha\rho$.

Let us consider a generalized W_2-recurrent $(LCS)_n$-manifolds. From (30) it follows that

$$\begin{aligned}g((\nabla_X W_2)(Y,Z)U,V)&=A(X)g(W_2(Y,Z)U,V)\\&+B(X)[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)].\end{aligned} \tag{31}$$

Let $\{e_i\}$, $i=1,2,...,n$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y=V=e_i$, $1\le i\le n$, we get

$$\begin{aligned}&(\nabla_X S)(Z,U)\\&=A(X)S(Z,U)+\tfrac{1}{n-1}[S(Z,U)-rg(Z,U)]A(X)+(n-1)B(X)g(Z,U).\end{aligned} \tag{32}$$

Replacing U by ξ in (32) and using (12) we have

$$(\nabla_X S)(Z,\xi) = \left[\left\{n(\alpha^2-\rho) - \frac{r}{n-1}\right\} A(X) + (n-1)B(X)\right]\eta(Z). \tag{33}$$

From (33) and (21), it follows that

$$\begin{aligned}\alpha S(X,Z) &= (n-1)[(\alpha^2-\rho)\alpha g(X,Z) + (2\alpha\rho-\beta)\eta(X)\eta(Z)] \\ &+ \left[\left\{\frac{r}{n-1} - n(\alpha^2-\rho)\right\} A(X) + (n-1)B(X)\right]\eta(Z).\end{aligned} \tag{34}$$

Hence setting $Z = \phi Z$ in (22) and then using (8c) we have

$$S(X,\phi Z) = (n-1)(\alpha^2-\rho)g(X,\phi Z). \tag{35}$$

If the manifold under consideration is Einstein, then (35) implies $(\alpha^2-\rho)=$ constant and hence $2\alpha\rho-\beta=0$. Conversely, if $2\alpha\rho-\beta=0$, then $\nabla_X(\alpha^2-\rho)=0$. Consequently $(\alpha^2-\rho)=$ constant.

Theorem 4.2 : An Einstein generalized W_2-recurrent $(LCS)_n$-manifold with $r = n(n-1)(\alpha^2-\rho)$ is a W_2-recurrent $(LCS)_n-$ manifold.

If generalized W_2-recurrent $(LCS)_n$-manifold is Einstein, then $\alpha^2-\rho$ is constant and hence $2\alpha\rho-\beta=0$. Consequently, from (34) we have

$$\begin{aligned}\alpha S(X,Z) &= (n-1)(\alpha^2-\rho)\alpha g(X,Z) \\ &+ \left[\left\{\frac{r}{n-1} - n(\alpha^2-\rho)\right\} A(X) + (n-1)B(X)\right]\eta(Z).\end{aligned} \tag{36}$$

By putting $Z=\xi$ in (36), we obtain

$$B(X) = -\frac{1}{n-1}\left[\frac{r}{n-1} - n(\alpha^2-\rho)\right]A(X). \tag{37}$$

If $r = n(n-1)(\alpha^2-\rho)$, then from (37) we get $B(X)=0$. Hence, generalized W_2-recurrent $(LCS)_n-$manifold reduces to W_2-recurrent $(LCS)_n-$ manifold.

Sufficient condition for a generalized W_2-recurrent manifold to be a generalized recurrent manifold

Theorem 4.3 : An Einstein generalized W_2-recurrent manifold with vanishing scalar curvature is a generalized recurrent manifold.

If a generalized W_2-recurrent manifold is Einstein. So we have

$$S(X,Y) = \frac{r}{n}g(X,Y) \tag{38}$$

From which it follows that

$$dr(X) = 0 \quad \text{and} \quad (\nabla_Z S)(X,Y) = 0 \quad \text{for all} \quad \text{X, Y, Z}. \tag{39}$$

Using (38) and (39) in (29), we have

$$(\nabla_X W_2)(Y,Z,U,V) = (\nabla_X R)(Y,Z,U,V). \tag{40}$$

In view of (30), the relation (40) takes the form

$$\begin{aligned}&(\nabla_X R)(Y,Z,U,V)\\ &\quad = A(X)\left\{R(Y,Z,U,V) + \tfrac{1}{n-1}[g(Y,U)S(Z,V) - g(Z,U)S(Y,V)]\right\}\\ &\quad +B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)].\end{aligned} \tag{41}$$

Again, in an Einstein generalized W_2-recurrent $(LCS)_n-$manifold if $r = 0$, then we have $S(X,Y) = 0$ for all X, Y and hence (41) yields (15). This shows that the manifold if generalized recurrent.

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On Cartan Spaces with Generalized $(\alpha, \beta)-$metric

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Abstract

In 1933 E.Cartan [1] introduced a new space known as Cartan space. It is considered as dual of Finsler space. H.Rund [4] , F.Brickell [2] and others studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron ([6] , [7]). T.Igrashi ([10] , [11]) introduced the notion of $(\alpha, \beta$)-metric in Cartan spaces and obtained the metric tensor and the invariants ρ and σ which characterize the special classes of Cartan spaces with $(\alpha, \beta$)-metric. Later on H.G.Nagaraja [3] studied Cartan spaces with $(\alpha, \beta$)-metric in 2007. This paper deals with a study of Cartan spaces with Generalized $(\alpha, \beta$)-metric admitting h-metrical d-connection. The conditions for these spaces to be locally Minkowaski and conformally flat have been obtained.

Keywords and Phrases : Cartan spaces, Generalized(α, β)-metric, h-metrical d-connection, locally Minkowaski and conformally flat spaces.
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1. Introduction

In 1978, M.Matsumoto and H.Shimada [5] introduced the concept of 1-form metric $L(\beta_\lambda)$, where $L(\beta_\lambda)$ is positively homogeneous function of degree one in n-arguments $\beta_\lambda(x, y)$, where $\beta_\lambda(x, y) = b_{(\lambda)i}(x)y^i$, $1 \leq \lambda \leq n$, are n-linearly independent 1-forms. In this paper we consider a Cartan metric

$$K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}), \quad 1 \leq \lambda \leq n, \tag{1.1}$$

where (1.1) is a p-homogeneous function with respect to $\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}$ and $\alpha(x, p) = \left(a^{ij}p_ip_j\right)^{\frac{1}{2}}$ together with $\beta^{(r)}(x, p) = b^{(r)i}(x)p_i, r = 1, ..., \lambda$, which are $\lambda-$ linearly independent 1-forms. For $\lambda = 1$ this metric is nothing but (α, β)-metric.

Let M be a real smooth manifold and (T M, π, M) its cotangent bundle. Let $C^n = (M, K(x,p))$, where $K : T^*M \longrightarrow R$ is a scalar function which is differentiable on T*M=TM-{0} and is homogeneous on fibres of T*M. The hessian of K^2 i.e., $g^{ij}(x,p) = \frac{1}{2}\partial^{\cdot i}\partial^{\cdot j}K^2$, where $\partial^{\cdot i} = \frac{\partial}{\partial p^i}$, is positively homogeneous on T^*M. Here C^n is called the Cartan space and the functions $K(x,p)$ and $g^{ij}(x,p)$ are called, respectively, the fundamental function and the metric tensor of the Cartan space C^n. The reciprocal $g_{ij}(x,p)$ of $g^{ij}(x,p)$ is given by $g_{ij}(x,p)g^{ik}(x,p) = \delta^k_j$, where $g_{ij}(x,p)$ and $g^{ij}(x,p)$ are both symmetric and homogeneous of order 0 in p_j.

A Cartan space $C^n = (M, K(x,p))$ is said to be with generalized $(\alpha, \beta$)-metric if $K(x,p)$ is a function of the variables $\alpha(x,p) = \left(a^{ij}p_ip_j\right)^{\frac{1}{2}}, \beta^{(r)}(x,p) = b^{(r)i}(x)p_i, r = 1, ..., \lambda$, where $a^{ij}(x)$ is a Riemannian metric and $b^{(r)i}(x)$ is a vector field depending only on x. Clearly, K must satisfy the conditions imposed to the fundamental functions of a Caratn space.

2. Generalized (α, β)-metric

Definition (2.1). A Cartan metric $K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$ is called Generalized (α, β)-metric.

In this paper we consider the Cartan spaces with generalized (α, β)-metric admitting h-metrical d-connection and their conformal change. To find the angular metric tensor g^{ij} of $C^n = (M, K(x,p))$ we use the following results:

$$\partial^{\cdot i}\alpha = \frac{p^i}{\alpha}, \partial^{\cdot i}\beta^{(r)} = b^{(r)i}, \partial^{\cdot i}K = l^i, \partial^{\cdot j}l^i = \frac{1}{K}h^{ij}, \tag{2.1}$$

where

$$\partial^{\cdot i} = \frac{\partial}{\partial p^i}, \quad h^{ij} = g^{ij} - l^i l^j = K\,\frac{\partial^2 K}{\partial p_i \partial p_j} \text{ and } p^i = a^{ij}p_j.$$

The successive differentiation of (1.1) with respect to p_i and p_j gives

$$l^i = K_\alpha \frac{p^i}{\alpha} + \sum_{r=1}^{\lambda} K_{\beta^{(r)}} b^{(r)i} \tag{2.2}$$

$$h^{ij} = \frac{KK_{\alpha\alpha}p^ip^j}{\alpha^2} + \sum_{r=1}^{\lambda} \frac{KK_{\alpha\beta^{(r)}}}{\alpha}\left(b^{(r)i}p^j + b^{(r)j}p^i\right) + \frac{KK_\alpha}{\alpha}a^{ij} \tag{2.3}$$

$$-\frac{KK_\alpha}{\alpha^3}p^ip^j+\sum_{r=1}^{\lambda}\sum_{s=1}^{\lambda}KK_{\beta^{(r)}\beta^{(s)}}b^{(r)i}b^{(s)j},$$

where

$$K_\alpha=\frac{\partial K}{\partial \alpha},\qquad K_{\beta^{(r)}}=\frac{\partial K}{\partial \beta^{(r)}},\qquad K_{\alpha\alpha}=\frac{\partial^2 K}{\partial \alpha^2},\qquad K_{\alpha\beta^{(r)}}=\frac{\partial^2 K}{\partial\alpha\partial \beta^{(r)}},$$

$$K_{\beta^{(r)}\beta^{(s)}}=\frac{\partial^2 K}{\partial \beta^{(r)}\ \partial\beta^{(s)}}.$$

From (2.2) and (2.3) , we get the metric tensor of C^n, given by

$$g^{ij}=\rho a^{ij}+\sum_{r=1}^{\lambda}\sum_{s=1}^{\lambda}\rho^{rs}b^{(r)i}b^{(s)j}+\sum_{r=1}^{\lambda}\rho^r\left(b^{(r)i}p^j+b^{(r)j}p^i\right)+\sigma p^ip^j, \tag{2.4}$$

where ρ^{rs}, ρ^r and σ are functions of α and $\beta^{(r)}$, given by

$$\rho=\frac{KK_\alpha}{\alpha},\quad \rho^{rs}=KK_{\beta^{(r)}\beta^{(s)}}+K_{\beta^{(r)}}K_{\beta^{(s)}},\rho^r=\frac{KK_{\alpha\beta^{(r)}}+K_\alpha K_{\beta^{(r)}}}{\alpha} \tag{2.5}$$

and

$$\sigma=\frac{KK_{\alpha\alpha}-\alpha^{-1}KK_\alpha+K_\alpha^2}{\alpha^2}.$$

The homogeneity of K in α and $\beta^{(1)},\beta^{(2)},...,\beta^{(\lambda)}$ gives the identity

$$\sum_{r=1}^{\lambda}\rho^{rs}\beta^{(r)}+\rho^s\alpha^2=KK_{\beta^{(s)}}. \tag{2.6}$$

Let γ^i_{jk} denote the christoffel symbol constructed from a_{ij} and $F_\gamma=\{\gamma^i_{jk},\gamma^i_{0j} = p^k\gamma^i_{kj},0\}$ be the linear Finsler connection of the space C^n, induced from the Riemannian connection $\gamma=(\gamma^i_{jk}(x))$ of the associated Riemannian space $(M^n,\alpha$). We denote ':' the covariant differentiation with respect to F_γ.Then $a_{ij:k}=0$, $a^{ij}_{:k}=0, p^i_{:k}=0$. Since $p_i= a_{ij}p^j$, it follows that $p_{i:k}=0$. Also, $\alpha^2=a^{ij}(x)p_ip_j$ gives $\alpha_{:k}=0$. Now, if we assume that $b^{(r)i}_{:k}=0$ for $r=1,...,\lambda$, then $\beta^{(r)}=b^{(r)i}p_i$ gives $\beta^{(r)}_{:k}=0$ for $r=1$,...,λ. Consequently, (1.1) gives

$$K_{:k}=K_\alpha\alpha_{:k}+\sum_{r=1}^{\lambda}L_{\beta^{(r)}}\beta^{(r)}_{:k}=0.$$

Since K_α is a function of $\alpha,\beta^{(1)},\beta^{(2)},...,\beta^{(\lambda)}$, we have $(K_\alpha)_{:k}=0$. Similarly, $K_{\alpha\beta^{(r)}:k}=0, K_{\beta^{(r)}\beta^{(s)}:k}=0$, which in view of (2.5) give $\rho_{:k}=0$, $\rho^{rs}_{:k}=0,\rho^r_{:k}=0$ and $\sigma_{:k}=0$. Therefore, from (2.4) it follows that $g^{ij}_{:k}=0$.

Further, F_γ has vanishing (h) h-torsion tensor T, deflection tensor D and (h) hv-torsion tensor C. Therefore, by the definition of Rund connection, we have

Proposition (2.1). If $b^{(r)i}_{:k} = 0, r = 1, ..., \lambda,$ is satisfied in a Cartan space C^n with generalized $(\alpha, \beta)-$ metric then the linear Cartan connection F_γ is nothing but the Rund's connection $R\Gamma$ of C^n.

It is remarked that the h-covariant derivative with respect to $R\Gamma$ coincides with that with respect to the Cartan connection $C\Gamma$.

Using the Christoffel symbols $\Gamma^i_{jk}(p) = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ constructed from $g_{ij}(x, p)$, we can define canonical N-connection

$$N_{ij} = \Gamma^k_{ij}\rho_k - \frac{1}{2}\Gamma^k_{hr}\rho_k\rho^r\partial^{\cdot h} g_{ij}. \tag{2.7}$$

We consider the canonical d-connection

$$D\Gamma = \left(N_{jk}, H^i_{jk}, C^{jk}_i\right), \tag{2.8}$$

where

$$H^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}). \tag{2.9}$$

The d-tensor field C^{jk}_i of type (2, 1) is given by

$$C^{jk}_i = -\frac{1}{2} g_{ir}\partial^{\cdot r} g^{jk} = g_{ir}C^{rjk}. \tag{2.10}$$

Let $_{|k}$ denote the h-covariant derivative with respect to $D\Gamma$, then we have

Definition (2.2). A d-connection $D\Gamma$ of a Cartan space C^n with generalized $(\alpha, \beta)-$metric is called h-metrical d-connection if it satisfies the following conditions:

(i) h-deflection tensor $D_{ij} = (p_{i|j}) = 0$,

(ii) $a^{ij}_{|k} = 0$,

(iii) $g^{ij}_{|k} = 0$.

3. Cartan Spaces with generalized (α, β)- metric admitting h-metrical d-connection

From the condition (i) of definition (2.2), we get $D_{ij} = p_{i|j} = 0$, therefore, the equation $K^2 = g^{ij}p_i p_j$ and condition (iii) of definition (2.2) give $K_{|k} = 0$.

Again, by the condition (i) and (ii), on the basis of the equation $p^i = a^{ij}(x)p_j$ and $\alpha^2 = a^{ij}(x)\ p_ip_j$, we get

$$\alpha_{\shortmid k} = 0, \qquad p^i_{\shortmid k} = 0. \tag{3.1}$$

Since $K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$, $1 \leq \lambda \leq n$, on the basis of (3.1), we get

$$K_{\shortmid k} = \sum_{r=1}^{\lambda} K_{\beta^{(r)}}\beta^{(r)}_{\shortmid k} = 0.$$

Therefore, $\beta^{(r)}_{\shortmid k} = 0$ for $r = 1, ..., \lambda$ and $K_{\beta^{(r)}}$ are linearly independent. Since, $\beta^{(r)}\ (x, p) = b^{(r)i}\ (x)p_i, r = 1, ..., \lambda$, on the basis of condition (i) of definition(2.3), we get

$$\beta^{(r)}_{\shortmid k} = b^{(r)i}_{\shortmid k}\ (x)p_i = 0,\ r = 1, ..., \lambda. \tag{3.2}$$

Since, $K_{\shortmid k} = 0$, $\alpha_{\shortmid k} = 0$, $\beta^{(r)i}_{\shortmid k} = 0$ for $r = 1, ..., \lambda$ and $K_\alpha, K_{\alpha\alpha}, K_{\alpha\beta^{(r)}}, K_{\beta^{(r)}\beta^{(s)}}$ are functions of $\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}$, therefore, $\rho_{\shortmid k} = 0$, $\rho^r_{\shortmid k} = 0$, $\rho^{rs}_{\shortmid k} = 0, \sigma_{\shortmid k} = 0$. Hence, h-covariant derivative of (2.4), on the basis of the conditions(ii) and (iii) of definition (2.2) gives

$$g^{ij}_{\shortmid k} = 0 = \sum_{r=1}^{\lambda} b^{(r)i}_{\shortmid k}\left(\sum_{s=1}^{\lambda}\rho^{rs}b^{(r)j} + \rho^s p^j\right) + \sum_{s=1}^{\lambda} b^{(s)i}_{\shortmid k}\left(\sum_{r=1}^{\lambda}\rho^{rs}b^{(r)i} + \rho^s p^i\right).$$

Contracting by p_j and using the identity (2.6) and equation (3.2), we get

$$\sum_{r=1}^{\lambda} K_{\beta^{(r)}} b^{(r)i}_{\shortmid k} = 0.$$

Since $K_{\beta^{(r)}}$ are linearly independent, we have

$$b^{(r)i}_{\shortmid k} = 0, r = 1, ..., \lambda. \tag{3.3}$$

Now from $a^{ij}_{\shortmid k} = 0$, we get $H^i_{jk} = \gamma^i_{jk}$. Hence, we have

$$b^{(r)i}_{:k} = 0, \qquad r = 1, ..., \lambda. \tag{3.4}$$

Also, the curvature tensor D^i_{hjk} of $D\Gamma$ coincides with the curvature tensor R^i_{hjk} of Riemannian connection $R\Gamma = \left(\gamma^i_{jk}, \gamma^i_{jk}p_i, 0\right)$. If $R^i_{hjk} = 0$, then $D^i_{hjk} = 0$. Thus, we have the following proposition:

Proposition (3.1). A Cartan space C^n with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection is locally flat if and only if the associated Riemannian space is locally flat.

If the connection $D\Gamma$ is h-metrical, then $g^{ij}_{|h} = 0$, $\alpha_{|h} = 0, a^{ij}_{|h} = 0, b^{(r)k}_{|h} = 0, p^k_{|h} = 0$. From (2.1), (2.4) and (2.5) it follows that $C^{ijk} = -\frac{1}{2}\partial^{\cdot k} g^{ij}$ can be determined in terms of $a^{ij}, p^i, b^{(r)i}, K$ and its partial derivatives of first, second and third orders with respect to α and $\beta^{(r)}, (r = 1, ..., \lambda)$. Since the h-covariant derivative of all these quantities vanishes, we have $C^{ijk}_{|h} = 0$. Hence, in view of (2.10) and condition (iii) of definition (2.2), it follows that

$$C^{ij}_{k|h} = 0. \tag{3.5}$$

Definition (3.1). A Cartan space C^n is a Berwald space if and only if $C^{ij}_{k|h} = 0$.

Hence, from (3.5), we have the following proposition:

Proposition (3.2). A Cartan space C^n with generalized $(\alpha, \beta)-$metric admitting h-metrical d-connection is a Berwald space.

As we know [9] a locally Minkowaski space is a Berwald space in which the curvature tensor vanishes. Hence, from the propositions (3.1) and (3.2), we have the following theorem:

Theorem (3.1). A Cartan space C^n with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection is locally Minkowaski if and only if the associated Riemannian space is locally flat.

4. Conformal change of Cartan space

Let $C^n = (M, K(x, p))$ be an n-dimensional Cartan space with generalized $(\alpha, \beta)-$metric $K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$, $1 \leq \lambda \leq n$, by a conformal change $\eta : K \longrightarrow \overline{K}$ such that $\overline{K}\left(\overline{\alpha}, \overline{\beta}^{(1)}, \overline{\beta}^{(2)}, ..., \overline{\beta}^{(\lambda)}\right) = e^{\eta} K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$, $1 \leq \lambda \leq n$, we have another Cartan space $\overline{C}^n = \left(M, \overline{K}\left(\overline{\alpha}, \overline{\beta}^{(1)}, ..., \overline{\beta}^{(\lambda)}\right)\right)$, where $\overline{\alpha} = e^{\eta}\alpha$ and $\left(\overline{\beta}^{(1)}, \overline{\beta}^{(2)}, ..., \overline{\beta}^{(\lambda)}\right) = e^{\eta}\left(\beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}\right)$. Putting $\alpha(x, p) = \left(a^{ij} p_i p_j\right)^{\frac{1}{2}}$ and $\beta^{(r)}(x, p) = b^{(r)i}(x) p_i$, $r = 1, ..., \lambda$, we get $\overline{a}^{ij} = e^{2\eta} a^{ij}$ and $\overline{b}^{(r)i} = e^{\eta} b^{(r)i}$. Then the Christoffel symbol $\overline{\gamma}^i_{jk}$ constructed from $\overline{a}^{ij}$ is written as

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + B^i_{jk}, \tag{4.1}$$

where

$$B^i_{jk} = \delta^i_j \eta_k + \delta^i_k \eta_j - \eta^i a_{jk}, \qquad \eta^i = \eta_j a^{ij}.$$

Taking covariant derivative of $\overline{b}^{(r)i}$ with respect to $\overline{\gamma}^i_{jk}$, we get

$$\overline{b}^{(r)i}_{:k} = e^\eta \sum_{r=1}^{\lambda} \left(b^{(r)i}_{:k} + 2\eta_k b^{(r)i} + b^{(r)j} \eta_j \delta^i_k - \eta^i a_{jk} b^{(r)j} \right).$$

Transvecting by $\overline{b}^{(r)k}$ and putting

$$M^i = \frac{1}{B^2} \sum_{r=1}^{\lambda} \left(b^{(r)k} b^{(r)i}_{:k} - \frac{1}{n+4} b^{(r)j}_{:j} b^{(r)i} \right), \tag{4.2}$$

we have

$$\eta^i = \overline{M}^i - M^i, \text{ from which we get, } \sigma_i = \overline{M}_i - M_i.$$

Substituting this in (4.1) and putting $D^i_{hj} = \gamma^i_{hj} + \delta^i_h M_j - M^i a_{hj}$, we have

$$\overline{D}^i_{hj} = D^i_{hj} \tag{4.3}$$

D^i_{hj} is a symmetric conformally invariant linear connection on M. Thus we have the following proposition:

Proposition (4.1). In a Cartan space with generalized $(\alpha, \beta)-$ metric there exists a conformally invariant symmetric linear connection D^i_{hj}.

If we denote by D^i_{hjk}, the curvature tensor of D^i_{hj}, we have from (4.3)

$$\overline{D}^i_{hjk} = D^i_{hjk} \tag{4.4}$$

Since $b^{(r)i}_{:k} = 0$, we have $M^i = 0$. Hence, $D^i_{jk} = \gamma^i_{jk}$ and $D^i_{hjk} = R^i_{hjk}$. Thus, we have the next proposition:

Proposition (4.2). In a Cartan space with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection $M^i = 0$ and there exists a conformally invariant symmetric linear connection D^i_{jk} such that $D^i_{jk} = \gamma^i_{jk}$ and $D^i_{hjk} = R^i_{hjk}$.

If the associated Riemannian space (M, α) is locally flat $\left(R^i_{hjk} = 0\right)$, then from (4.4) and proposition (4.2), we have $\overline{D}^i_{hjk} = 0$, i.e., the space C^n is conformally flat. Thus we conclude that

Theorem (4.1). A Cartan space C^n with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection is conformally flat if and only if the associated Riemannian space (M, α) is locally flat $\left(R^i_{hjk} = 0\right)$.

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On Special Curvature Tensor in a Generalized 2-recurrent Smooth Riemannian Manifold

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Abstract

In this paper, we have discussed on a special type of curvature tensor in a smooth Riemannian manifold and have studied its cyclic and differentiable properties. We have also studied the 2-recurrence properties of the tensor S, T, F and H in Riemannian manifold as well as in an Einstein manifold.

1. Introduction

Special curvature tensor has been introdued and studied by Singh and Khan [6]. Let M_n be an $n-$dimensional smooth Riemannian manifold and X, Y, Z and W be differentiable vector field on M_n. A special curvature tensor $H(X, Y, Z)$ of type (1, 3) has been defined as [6] :

$$H(X,Y,Z) = R(X,Y,Z) + R(X,Z,Y), \tag{1.1}$$

$$< H(X,Y,Z), W > = < R(X,Y,Z), W > + < R(X,Z,Y), W >, \tag{1.2}$$

or

$$'H(X,Y,Z,W) = {}'R(X,Y,Z,W) + {}'R(X,Z,Y,W). \tag{1.3}$$

It is obvious that

$$H(X,Y,Z) = H(X,Z,Y),$$

which shows that it is symmetric in last two slots. Sinha [5] has defined and studied certain tensors of type (1, 3) in a smooth Riemannian manifold. They are

$$S(X,Y,Z) = Ric(Y,Z)\,X + Ric(Z,X)\,Y + Ric(X,Y)\,Z, \tag{1.4}$$

$$T(X,Y,Z) = < Y,Z > \; X + < Z,X > \; Y + < X,Y > \; Z, \tag{1.5}$$

$$F(X,Y,Z) = < Y,Z > \; K(X) + < Z,X > \; K(Y) + < X,Y > \; K(Z), \tag{1.6}$$

which are symmetric in X, Y, Z.

A special curvature tensor $H(X, Y, Z)$ has cyclic property [4].

$$H(X, Y, Z) + H(Y, Z, X) + H(Z, X, Y) = 0. \tag{1.7}$$

In 1972 A. K. Roy generalized the notion of 2-recurrent manifold. A Riemannian manifold (M^n, g) is called generalized 2-recurrent, if the Riemannian curvature tensor satisfies the condition

$$(D_V D_U R)(X, Y)Z = A(V)(D_U R)(X, Y, Z) + B(U, V)R(X, Y, Z) \tag{1.8}$$

where A is non-zero 1-form and B is non-zero 2-form tensor. D denote the covariant differentiation with respect to metric tensor.

In a recent paper, De and Bandyopadhyay [2] introduced and studied generalized Ricci 2-recurrent Riemannian manifold which are defined as : A non-flat Riemannian manifold is called generalized Ricci 2-recurrent Riemannian manifold if the Ricci tensor is non-zero and satisfies the condition

$$(D_V D_U Ric)(X, Y)Z = A(V)(D_U Ric)(X, Y) + B(U, V)Ric(X, Y) \tag{1.9}$$

where A and B are stated earlier. If the 2-form $B(U, V)$ becomes zero, then the space reduces to Ricci recurrent space.

An $n-$dimensional smooth Riemannian manifold M_n is called an Einstein manifold, if for all $X, Y \in \chi(M_n)$

$$Ric(X, Y) = k < X, Y >, \tag{1.10}$$

where $k : M_n \rightarrow R$ real is a valued function.

In this paper, we have studied some theorems about special curvature tensor $H(X, Y, Z)$. In section two of this paper, we have studied the 2-recurrent properties of the tensors S, T, F and H in a smooth Riemannian manifold as well as in an Einstein manifold. In section third of this paper, we have studied its cyclic and bi-covariant differentiation properties in generalized 2-recurrent smooth Riemannian manifold.

2. Recurrence Properties of (1, 3) type Tensors in a Generalized 2-recurrent Smooth Riemannian Manifold

Let M_n be an $n-$dimensional smooth Riemannian manifold. Then, M_n is called generalized 2-recurrent smooth Riemannian manifold with respect to the tensor $H(X, Y, Z)$, if

$$(D_V D_U H)(X, Y, Z) = A(V)(D_U H)(X, Y, Z) + B(U, V)H(X, Y, Z), \tag{2.1}$$

where A is $1-$form and B is $2-$form known as recurrence parameter. A smooth Riemannian manifold M_n is called $2-$recurrent with respect to tensor $S(X,Y,Z)$, $T(X,Y,Z)$ and $F(X,Y,Z)$ defined by equations (1.4), (1.5) and (1.6) respectively, if

$$(D_V D_U Q)(X,Y,Z) = A(V)(D_U Q)(X,Y,Z) + B(U,V)Q(X,Y,Z), \qquad (2.2)$$

where A and B are stated earlier and Q stands for S, T and F, respectively. We now prove the following :

Theorem (2.1). An $n-$dimensional smooth Riemannian manifold M_n is generalized $2-recurrent$ with respect to the tensor $H(X,Y,Z)$, if it is a generalized 2-recurrent smooth Riemannian manifold with the same recurrence parameter.

Proof. Taking bi-covariant derivative of equation (1.1) with respect to 'U' and 'V', we get

$$(D_V D_U H)(X,Y,Z) = (D_V D_U R)(X,Y,Z) + (D_V D_U R)(X,Z,Y). \qquad (2.3)$$

On using equation (1.8) in equation (2.3), we get

$$(D_V D_U H)(X,Y,Z) = A(V)(D_U R)(X,Y,Z) + B(U,V)R(X,Y,Z)$$
$$+A(V)(D_U R)(X,Z,Y) + B(U,V)R(X,Z,Y). \qquad (2.4)$$

On using equation (1.1) in equation (2.4), we get

$$(D_V D_U H)(X,Y,Z) = A(V)(D_U H)(X,Y,Z) + B(U,V)H(X,Y,Z). \qquad (2.5)$$

That is, M_n is generalized $2-$recurrent with respect to tensor $H(X,Y,Z)$.

Theorem (2.2). If a smooth Riemannian manifold M_n is generalized $2-$recurrent with respect to the special tensor $H(X,Y,Z)$, then

$$A(V)(D_U H)(X,Y,Z) + B(U,V)H(X,Y,Z) = (D_U D_V R)(X,Y,Z)$$
$$+(D_U D_V R)(X,Z,Y). \qquad (2.6)$$

Proof. Let M_n be 2-recurrent Riemannian manifold with respect to the tensor $H(X,Y,Z)$, then from equation (2.1), we have

$$A(V)\{(D_U R)(X,Y,Z) + (D_U R)(X,Z,Y) + B(U,V)\{R(X,Y,Z) + R(X,Z,Y)\}$$
$$= (D_U D_V R)(X,Y,Z) + (D_U D_V R)(X,Z,Y), \qquad (2.7)$$

$$(D_U D_V R)(X,Y,Z) - A(V)(D_U R)(X,Y,Z) - B(U,V)R(X,Y,Z)$$
$$+(D_U D_V R)(X,Z,Y) = -A(V)(D_U R)(X,Z,Y) - B(U,V)R(X,Z,Y) = 0. \qquad (2.8)$$

Since M_n is generalized 2-recurrent with respect to tensor $H(X, Y, Z)$. Therefore, on using equations (1.8) and (1.1) in equation (2.8), we get the required result.

Theorem (2.3). An Einstein manifold M_n is generalized 2−recurrent with respect to the tensor $T(X, Y, Z)$, if it is generalized Ricci 2-recurrent for the same recurrence 2-form.

Proof. On using equation (1.5) in equation (2.8), we get

$$T(X,Y,Z) = \frac{1}{k}\,[Ric\,(Y,Z)X + Ric(Z,X)Y + Ric(X,Y)Z]. \tag{2.9}$$

Taking bi-covariant derivative of equation (2.9), with respect to 'U' and 'V', we get

$$(D_U D_V T)(X,Y,Z) = \frac{1}{k}\,[(D_U D_V Ric)\,(Y,Z)X + (D_U D_V Ric)(Z,X)Y + (D_U D_V Ric)(X,Y)Z]. \tag{2.10}$$

Now, let M_n be a generalized Ricci 2-recurrent Riemannian manifold, then using equation (1.9) in equation (2.10), we get

$$\begin{aligned}(D_U D_V T)(X,Y,Z) = \frac{1}{k}\,[&A(V)(D_U Ric)\,(Y,Z)X + B(U,V)Ric(Y,Z)X \\ &+A(V)(D_U Ric)\,(Z,X)Y + B(U,V)Ric(Z,X)Y \\ &+A(V)(D_U Ric)\,(X,Y)Z + B(U,V)Ric(X,Y)Z].\end{aligned} \tag{2.11}$$

On using equation (2.9) in equation (2.11), we get

$$(D_U D_V T)(X,Y,Z) = [A(V)(D_U T)\,(X,Y,Z) + B(U,V)T(X,Y,Z)]. \tag{2.12}$$

That is, M_n is 2-recurrent with respect to tensor $T(X, Y, Z)$.

Theorem (2.4). If an Einstein manifold M_n is generalized 2−recurrent with respect to the tensor $T(X, Y, Z)$, then

$$\begin{aligned}&\{(D_U D_V Ric)(Y,Z) - A(V)(D_U Ric)(Y,Z) - B(U,V)Ric(Y,Z)\}X \\ &+ \{(D_U D_V Ric)(Z,X) - A(V)(D_U Ric)(Z,X) - B(U,V)Ric(Z,X)\}Y \\ &+ \{(D_U D_V Ric)(X,Y) - A(V)(D_U Ric)(X,Y) - B(U,V)Ric(X,Y)\}Z = 0.\end{aligned} \tag{2.13}$$

Proof. Let M_n be generalized 2-recurrent with respect to the tensor $T(X, Y, Z)$, then from equations (2.2) and (2.9), we have

$$A(V)(D_U T)(X,Y,Z) + B(U,V)T(X,Y,Z) = \frac{1}{k}[(D_U D_V Ric)(Y,Z)X +$$

$$(D_U D_V Ric)(Z,X)Y + (D_U D_V Ric)(X,Y)Z]. \tag{2.14}$$

On using equation (1.9) in equation (2.14), we get the required result.

Theorem (2.5). An Einstein smooth Riemannian manifold M_n is generalized $2-$recurrent with respect to the tensor $T(X,Y,Z)$, if and only if M_n is recurrent with respect to the tensor $S(X,Y,Z)$ for the same recurrence parameter.

Proof. From equations (2.8) and (1.4), we have

$$S(X,Y,Z) = kT(X,Y,Z). \tag{2.15}$$

Taking bi-covariant derivative of equation (2.15) with respect to 'U' and 'V', we get

$$(D_U D_V S)(X,Y,Z) = k(D_U D_V T)(X,Y,Z). \tag{2.16}$$

From equation (2.16), it is evident that, if M_n is 2-recurrent with respect to the tensor $S(X,Y,Z)$, then M_n is also 2-recurrent with respect to the tensor $T(X,Y,Z)$ and vice-versa.

We now prove the following :

Theorem (2.6). An $n-$dimensional smooth Riemannian manifold is 2-recurrent with respect to tensor $S(X,Y,Z)$, if it is Ricci 2-recurrent with the same recurrence parameter.

Proof. Taking bi-covariant derivative of equation (1.4) with respect to 'U' and 'V', we get

$$(D_U D_V S)(X,Y,Z) = (D_U D_V Ric)(Y,Z)X + (D_U D_V Ric)(Z,X)Y$$
$$+(D_U D_V Ric)(X,Y)Z. \tag{2.17}$$

Now, let M_n be Ricci 2-recurrent Riemannian manifold, then using equations (1.9) and (1.4) in equation (2.17), we get M_n as 2-recurrent Riemannian manifold with respect to the tensor $S(X,Y,Z)$.

Theorem (2.7). If a smooth Riemannian manifold M_n is 2-recurrent with respect to tensor $S(X,Y,Z)$, then

$$\{(D_U D_V Ric)(Y,Z) - A(V)(D_U Ric)(Y,Z) - B(U,V)Ric(Y,Z)\}X$$
$$+\{(D_U D_V Ric)(Z,X) - A(V)(D_U Ric)(Z,X) - B(U,V)Ric(Z,X)\}Y$$
$$+\{(D_U D_V Ric)(X,Y) - A(V)(D_U Ric)(X,Y) - B(U,V)Ric(X,Y)\}Z = 0.$$

Proof. Let M_n be 2-recurrent with respect to the tensor $S(X,Y,Z)$, then using equation (2.2) in equation (2.17), we have

$$A(V)(D_U S)(X,Y,Z) + B(U,V)S(X,Y,Z) = (D_U D_V Ric)(Y,Z)X$$

$$+(D_U D_V Ric)(Z,X)Y + (D_U D_V Ric)(X,Y)Z. \quad (2.18)$$

On using equation (1.4) in equation (2.18), we get the required results.

Corollary (2.1). An Einstein manifold M_n is 2-recurrent with respect to the tensor $T(X,Y,Z)$ if and only if M_n is 2-recurrent with respect to the tensor $F(X,Y,Z)$ for the same recurrence parameter.

3. Some Properties of Special Curvature Tensor $H(X,Y,Z)$ in Generalized 2-recurrent Smooth Riemannian Manifold

Theorem (3.1). In an $n-$dimensional smooth Riemannian manifold M_n, the special curvature tensor $H(X,Y,Z)$ has the following properties :

(i) If special curvature tensor $H(X,Y,Z)$ has cyclic property defined by equation (1.7), then it also has

$$\{(D_U H)(X,Y,Z) + (D_U H)(Y,Z,X) + (D_U H)(Z,Y,X)\} = 0,$$

and

(ii) $$(D_U D_X H)(Y,Z,W) + (D_U D_Y H)(Z,X,W) + (D_U D_Z H)(X,Y,W)$$

$$= (D_U D_X R)(Y,Z,W) + (D_U D_Y R)(Z,W,X) + (D_U D_Z R)(X,W,Y).$$

Proof.(i). Taking bi-covariant derivative of equation (1.7) with respect to 'U' and 'V', we get

$$(D_U D_V H)(X,Y,Z) + (D_U D_V H)(Y,Z,X) + (D_U D_V H)(Z,X,Y) = 0 \quad (3.1)$$

On using equations (2.1) and (1.7) in equation (3.1), we get the required result.

(ii) We have

$$H(Y,Z,W) = R(Y,Z,W) + R(Y,W,Z).$$

Taking bi-covariant derivative of the above equation with respect to 'X' and 'U', we get

$$(D_U D_X H)(Y,Z,W) = (D_U D_X R)(Y,Z,W) + (D_U D_X R)(Y,W,Z). \quad (3.2)$$

Taking cyclic permutation of equation (3.2) in X, Y, Z; adding the three equations and then using Bianchi's second identity, we get the required result.

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Pseudo-Slant Submanifolds of a Generalised Almost Contact Metric Structure Manifold

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Abstract

In this paper we have studied pseudo-slant submanifolds of a Generalised almost contact metric structure manifold and established integrability conditions of distributions and some interesting results on this submanifold.

Keywords and Phrases : Generalised Almost Contact Metric Structure Manifold, Slant Submanifold Pseudo-Slant Submanifold.

1. Introduction

The geometry of slant submanifolds was initiated by B. Y. Chen. He defined slant immersions in the complex geometry as a natural generalization of both holomorphic and totally real immersions [4]. A. Lotta introduced the notion of slant immersions of a Riemannian manifold into an almost contact metric manifold [5]. In [2], J. L. Cabererizo et. all studied and characterised slant submanifolds of K-contact and Sasakian manifolds with several examples. Recently Khan and Khan studied Pseudo-slant submanifolds of a Sasakian manifold [5].

The purpose of this paper is to study pseudo-slant submanifolds of Generalised almost contact metric structure manifold. In section 3 we defined slant immersions and slant distributions on Generalised almost contact metric structure manifold and Hyperbolic Hermite manifold and proved some characterisation theorem. In section 4 we defined pseudo-slant submanifolds of these manifolds and established a relation between them. We also worked out integrability conditions of distributions on pseudo-slant submanifolds of Generalised almost contact metric structure manifold.

2. Preliminaries

First we define a Generalised almost contact metric structure manifold.

Definition (2.1) [8]. An odd dimensional Riemannian manifold $(\overline{M}, g)$ is said to be a Generalised almost contact metric structure manifold if, there exits a tensor ϕ of the type (1, 1) and a global vector field ξ and a 1-form η satisfying the following equations:

$$\phi^2 X = a^2 X + \eta(X)\xi \tag{1}$$

$$\eta(\phi X) = 0 \tag{2}$$

$$\eta(\xi) = -a^2 \tag{3}$$

$$\phi(\xi) = 0 \tag{4}$$

$$\eta(X) = g(X, \xi) \tag{5}$$

$$g(\phi X, \phi Y) = -a^2 g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

where $X, Y \in T\,\overline{M}$, a be a complex number and g be the metric of $\overline{M}$.

From above definition it is clear that almost contact metric manifold is a particular case of a Generalised almost contact metric structure manifold for $a^2 = -1$.

If $'\Phi$ is a 2-form defined on $\overline{M}$ as

$$'\Phi(X, Y) = g(\phi X, Y),$$

then $'\Phi$ is alternating i.e.

$$'\Phi(Y, X) = -'\Phi(X, Y)$$

or

$$g(\phi X, Y) = -g(\phi Y, X). \tag{7}$$

Now let M be a submanifold immersed in $\overline{M}$ and we denote by the same symbol g the induced metric on M. let TM be the Lie algebra of the vector fields in M and $T^{\perp}M$ denote the set of all vector fields normal to M. Then, the Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{8}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}{}_X V, \tag{9}$$

for all $X, Y \in TM$, $V \in T^{\perp}M$.

Where $\overline{\nabla}$, ∇ are respectively the Levi-Civita connexions on $\overline{M}$ and M and $\nabla^{\perp}$ is induced connexion in normal bundle of M i.e. $T^{\perp}M$, h is symmetric bilinear vector valued function called second fundamental form and A_V is the shape operator associated with V. The second fundamental form h and the shape operator A are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{10}$$

For any$X \in TM$, we write,

$$\phi X = TX + NX, \tag{11}$$

where TX is the tangential component of ϕX and NX is the normal component of ϕX. Similarly for any V in $T^{\perp}M$, we write

$$\phi V = tV + nV, \tag{12}$$

where tV (resp. nV) denotes the tangential (resp. normal) component of ϕV.

The submanifold M is said to be an invariant submanifold if N is identically zero i.e. $\phi X = TX$ for any $X \in TM$. On the other hand the submanifold M is called anti-invariant submanifold in T is identically zero i.e. $\phi X = NX$.

The covariant derivatives of T and N are defined as

$$(\overline{\nabla}_X T)Y = \nabla_X(TY) - T(\nabla_X Y) \tag{13}$$

and

$$(\overline{\nabla}_X N)Y = \nabla^{\perp}{}_X(NY) - N(\nabla_X Y). \tag{14}$$

The distribution spanned by the structure vector ? is denoted by $< \xi >$.

3. Slant distributions and slant immersions

Let M be a Riemannian manifold, isometrically immersed in a Generalised almost contact metric structure manifold $(\overline{M}, \phi, g, a, \eta, \xi)$. Suppose that the structure vector ξ is tangent to M. if we denote by D the orthogonal distribution to ξ in TM. Then

$$TM = D \oplus < \xi > .$$

For each nonzero vector X tangent to M at x, such that X is not proportional to ξ_x, we denote by $\theta(X)$ the angle between ϕX and $T_x M$. Since $\phi(\xi) = 0$, thus $\theta(X)$ is the angle between ϕX and D_x.

Definition (3.1) : M is said to be slant if the angle $\theta(X)$ is constant, i.e. which is independent of the choice of $x \in M$ and $X \in TM - <\xi_x>$. The angle θ of a slant immersion is called the slant angle of the immersion.

From this definition, it is evident that invariant and anti-invariant immersions slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A slant immersion, which is neither invariant nor anti-invariant, is called proper slant immersion.

A useful characterization of slant submanifolds in Generalised almost contact metric structure manifold is given by the following theorem.

Theorem (3.1) : Let M be a submanifold isometrically immersed in a Generalised almost contact metric structure manifold $(\overline{M}, \phi, g, a, \eta, \xi)$ such that $\xi \in TM$, then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2 = a^2\lambda I + \lambda\eta \otimes \xi. \tag{15}$$

Furthermore, in this case, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Proof : Let $X, Y \in TM$, then for any slant submanifold, we have

$$\begin{aligned}
&g(TX, TY) = \cos^2\theta . g(\phi X, \phi Y) \\
&\Leftrightarrow g(TX, TY) = \cos^2\theta . [-a^2 g(X, Y) - \eta(X)\eta(Y)] \text{ from (6)} \\
&\Leftrightarrow -g(T^2X, Y) = -\cos^2\theta . [a^2 g(X, Y) + \eta(X)\eta(Y)] \ \Theta \ g(TX, Y) = -g(X, TY) \\
&\Leftrightarrow g(T^2X, Y) = \cos^2\theta . [a^2 g(X, Y) + \eta(X)\eta(Y)] \ \forall \ Y \in TM \\
&\Leftrightarrow T^2X = \cos^2\theta . [a^2 X + \eta(X)\xi] \ \forall \ X \in TM \\
&\Leftrightarrow T^2 = \cos^2\theta . [a^2 I + \eta \otimes \xi] \\
&\Leftrightarrow T^2 = a^2\lambda I + \lambda\eta \otimes \xi
\end{aligned}$$

where $\lambda = \cos^2\theta$, θ is the slant angle.

Hence the theorem.

Now we define slant distributions.

Definition (3.2) : A differentiable distribution ν on M is said to be a slant distribution if for each $x \in M$ and each nonzero vector $X \in \nu_x$, the angle $\theta\nu(X)$ between ϕX and the vector space ν_x is constant, i.e. which is independent of the choice of$x \in M$ and $X \in \nu_x$. In this case the constant angle $\theta\nu$ is called the slant angle of the distribution ν.

Thus we see that if a submanifold is slant, then there exists a slant distribution on M.

The following theorem provides a useful characterization for the existence of a slant distribution on a Generalised almost contact metric structure manifold.

Theorem (3.2) : Let ν be a distribution on M, orthogonal to ξ. Then ν is slant if and only if there exists a constant $\lambda \in [0,1]$ such that $(PT)^2X = a^2\lambda X$, for any $X \in \nu$.

Furthermore, in this case, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Proof : The proof is straightforward and may be obtained from theorem (3.1).

Now we define slant distributions on a submanifold of Hyperbolic Hermite manifold.

Definition (3.2) : Given a submanifold S, isometrically immersed in a Hyperbolic Hermite manifold $(\overline{S}, J, g_1)$, a differentiable distribution D on S is said to be a slant distribution if for any nonzero vector $X \in D_x$, $x \in S$, the angle between JX and the vector space Dx is constant, i.e. which is independent of the choice of $x \in S$ and $X \in D_x$. In this case the constant angle is called the slant angle of the distribution D (compare with the definition (3.2)).

4. Pseudo-slant submanifolds of Generalised almost contact metric structure manifold

We first define pseudo-slant submanifolds of Hyperbolic Hermite manifold.

Definition (4.1) : A submanifold S of a Hyperbolic Hermite manifold $(\overline{S}, J, g_1)$ is called a pseudo-slant submanifold, if there exists on S, two differentiable orthogonal distributions D_1 and D_2 such that $TM = D_1 \oplus D_2$, where D_1 is totally real distribution i.e. $JD_1 \subset T^{\perp}S$ and D_2 is slant distribution with slant angle $\theta \neq \pi/2$, in particular if dim $D_1 = 0$ and $\theta \in (0, \pi/2)$, then S is proper slant submanifold of $(\overline{S}, J, g_1)$.

In the following paragraph we show that there is a relationship between slant submanifold of Generalised almost contact metric structure manifold and pseudo-slant submanifolds of Hyperbolic Hermite manifold.

Let $(\overline{M}, \phi, g, a, \eta, \xi)$ be a Generalised almost contact metric structure manifold. Then we consider the manifold $\overline{M} \times R$. We denote by $\left(X, f\frac{d}{dt}\right)$ a vector field on $\overline{M} \times R$, where X is tangent to $\overline{M}$, t is the coordinate of R and f is a differentiable function on $\overline{M} \times R$.

If we define a tensor J of type (1, 1) on $\overline{M} \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \frac{1}{a}\left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right) \tag{16}$$

Then we have, $J^2\left(X, f\frac{d}{dt}\right) = \frac{1}{a}J\left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$ from (16)

$$\begin{aligned}
&= \tfrac{1}{a}\cdot\tfrac{1}{a}\left(\phi(\phi X - f\xi) - \eta(X)\xi, \eta(\phi X - f\xi)\tfrac{d}{dt}\right)\\
&= \tfrac{1}{a^2}\left(\phi^2 X - f\phi\xi - \eta(X)\xi, (\eta(\phi X) - f\eta(\xi))\tfrac{d}{dt}\right)\\
&= \tfrac{1}{a^2}\left(a^2 X, (a^2 f)\tfrac{d}{dt}\right), \text{ from (1), (2), (3) and (4)}\\
&= \left(X, f\tfrac{d}{dt}\right)
\end{aligned}$$

i.e.

$$J^2\left(X, f\frac{d}{dt}\right) = \left(X, f\frac{d}{dt}\right). \tag{17}$$

Now we define the metric g_1 on $\overline{M} \times R$ as

$$g_1\left[\left(X, f\frac{d}{dt}\right), \left(Y, h\frac{d}{dt}\right)\right] = g(X, Y) + fh. \tag{18}$$

Then we obtain

$$g_1\big[J\big(X, f\frac{d}{dt}\big), J\big(Y, h\frac{d}{dt}\big)\big] = g_1\big[\frac{1}{a}\big(\phi X - f\xi, \eta(X)\frac{d}{dt}\big), \frac{1}{a}\big(\phi Y - h\xi, \eta(Y)\frac{d}{dt}\big)\big],$$

by (16)

$$\begin{aligned}
&= \tfrac{1}{a^2}g_1\left[\left(\phi X - f\xi, \eta(X)\tfrac{d}{dt}\right), \left(\phi Y - h\xi, \eta(Y)\tfrac{d}{dt}\right)\right]\\
&= \tfrac{1}{a^2}\left[g\left(\phi X - f\xi, \phi Y - h\xi\right) + \eta(X)\eta(Y)\right] \text{ by (18)}\\
&= \tfrac{1}{a^2}\left[g(\phi X, \phi Y) - g(\phi X, h\xi) - g(f\xi, \phi Y) + g(f\xi, h\xi) + \eta(X)\eta(Y)\right]\\
&= \tfrac{1}{a^2}\left[-a^2 g(X, Y) - \eta(X)\eta(Y) - a^2 fh + \eta(X)\eta(Y)\right],\\
&\qquad \text{by (3), (4), (5), (6) and (7)}\\
&= -\left[g(X, Y) + fh\right]\\
&= -g_1\left[\left(X, f\tfrac{d}{dt}\right), \left(Y, h\tfrac{d}{dt}\right)\right], \text{ by (18)}
\end{aligned}$$

Therefore we have

$$g_1\left[J\left(X, f\frac{d}{dt}\right), J\left(Y, h\frac{d}{dt}\right)\right] = -g_1\left[\left(X, f\frac{d}{dt}\right), \left(Y, h\frac{d}{dt}\right)\right], \tag{19}$$

from (17) and (19), we see that $(\overline{M} \times R, J, g_1)$ is a Hyperbolic Hermite structure manifold.

Now we state the following theorem, which provides a method to obtain a pseudo- slant submanifold of $\overline{M} \times R$ from slant submanifold of $\overline{M}$.

Theorem (4.1) : Let M be a non anti-invariant slant submanifold of a Generalised almost contact metric structure manifold $\overline{M}$ with slant distribution D and ξ is orthogonal to M. then $M \times R$ is a pseudo-slant submanifold of the Hyperbolic Hermite manifold $\overline{M} \times R$ with totally real distribution $D_1 = \left\{(0.\frac{d}{dt})\right\}$ and slant distribution $D_2 = \{(X, 0)|X \in D\}$.

Proof : Since we have,

$$g_1 \left[(X, 0), \left(0, \frac{d}{dt}\right)\right] = g(X, 0) + 0 = 0.$$

and $\left(X, f\frac{d}{dt}\right) = (X, 0) + f\left(0, \frac{d}{dt}\right), \forall \left(X, f\frac{d}{dt}\right) \in T(M \times R)$,

therefore $T(M \times R) = D_1 \oplus D_2$ is an orthogonal direct decomposition.

Also $\quad J\left(0, \frac{d}{dt}\right) = \frac{1}{a}(-\xi, 0) \subset T^{\perp}(M \times R)$ from (16)

$\therefore D_1$ is totally real distribution. It is easy to see that D_2 is slant distribution with slant angle θ (which is slant angle of D) in the sense of Papaghuic [9].

To introduce pseudo-slant submanifold of a Generalised almost contact metric structure manifold; first we define bislant submanifolds of a Generalised almost contact metric structure manifold.

Definition (4.2) : M is said to be a bislant submanifold of a Generalised almost contact metric structure manifold $\overline{M}$ if there exists two orthogonal distributions D_1 and D_2 such that

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus <\xi>$

(ii) The distribution D_1 is slant with angle θ_1

(iii) The distribution D_2 is slant with angle θ_2.

Now we define pseudo-slant submanifold of a Generalised almost contact metric structure manifold as a particular case of bislant submanifold.

Definition (4.3) : M is said to be a pseudo-slant submanifold of a Generalised almost contact metric structure manifold $\overline{M}$ if there exists two orthogonal distributions D_1 and D_2, such that

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus <\xi>$

(ii) The distribution D_1 is anti-invariant i.e. $\phi D_1 \subset T^{\perp}M$

(iii) The distribution D2 is slant with angle $\theta \neq \pi/2$.

If we denote by d_i, the dimension of D_i, for $i = 1, 2$, then we find the following cases

(a) If $d_2 = 0$, then M is an anti-invariant submanifold.

(b) If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.

(c) If $d_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ.

(d) If $d_1 \neq 0$ and $\theta = 0$, then M is a semi invariant submanifold.

Let M be a pseudo-slant submanifold of a Generalised almost contact metric structure manifold $\overline{M}$. Then, for any $X \in TM$, we write

$$X = P_1X + P_2X + \eta(X)\xi \tag{20}$$

where P_i denotes the projection map on the distribution D_i, $i = 1, 2$.

Now operating on both sides of the equ. (20), we obtain

$$\phi X = NP_1X + TP_2X + NP_2X, \tag{21}$$

because

$$\phi P_1X = NP_1X, \qquad TP_1X = 0. \tag{22}$$

It is easy to see that

$$TX = TP_2X \, NX = NP_1X + NP_2X \tag{23}$$

and

$$TP_2X \in D_2. \tag{24}$$

Since D_2 is slant distribution, by theorem (3.2)

$$T^2X = a^2\cos^2\theta X, \qquad \forall X \in D_2. \tag{25}$$

Now we have the following theorem.

Theorem (4.2) : Let M be a submanifold of a Generalised almost contact metric structure manifold $\overline{M}$, such that $\xi \in TM$. Then M is a pseudo-slant submanifold is and only if there exists a constant $\lambda \in (0, 1]$, such that

(i) $D = \{X \in TM | T^2X = a^2\lambda X\}$ is a distribution on M.

(ii) For any $X \in TM$, orthogonal to D, $TX = 0$.

Furthermore, in this case, $\lambda = \cos^2\theta$ where θ denotes the slant angle of D.

Proof : Putting $\lambda = \cos^2\theta$, it is obvious that for any $X \in D$, $T^2X = a^2\cos^2\theta X$ therefore $D = D_2$ from equ. (25).

Thus D is a distribution on M.

Also for any $X \in TM$, orthogonal to D, we have

$$\phi X \in T^{\perp}M \text{ and } \phi\xi = 0, \text{ i.e. } TX = 0.$$

Hence the condition is necessary.

Conversely, consider the orthogonal direct decomposition $TM = D \oplus D^{\perp} \oplus <\xi>$, then by (i) and theorem (3.2), we find D is a slant distribution. From (ii) it is evident that $D^{\perp}$ is an anti-invariant distribution.

Therefore M is a pseudo-slant submanifold, hence the theorem.

In the following paragraph, we discuss on the integrability conditions of the distributions involved in a pseudo-slant submanifolds of $\overline{M}$.

If μ be the invariant subspace of $T^{\perp}M$, then in case of pseudo-slant submanifold, consider the direct decomposition of $T^{\perp}M$ as

$$T^{\perp}M = \mu \oplus ND_1 \oplus ND_2 \tag{26}$$

Since D_1 and D_2 are orthogonal, therefore $g(Z, X) = 0$. $\forall X \in D_1, Z \in D_2$

This implies that $g(NZ, NX) = g(\phi Z, \phi X) = 0$ Q $g(TZ, NX) = 0$.

Therefore (26) gives orthogonal direct decomposition of $T^{\perp}M$.

First, we prove some important lemmas.

Lemma (4.1) : $A_{\phi X}Y = A_{\phi Y}X$, if and only if

$$g((\overline{\nabla}_z\phi)X, Y) = 0, \qquad \forall X, Y \in D_1, Z \in TM.$$

Proof : Let $X, Y \in D_1$ and $Z \in TM$, then

$$g(A_{\phi Y}X, Z) = g(h(X, Z), \phi Y)$$
$$= g(h(Z, X), \phi Y) = g(\overline{\nabla}_Z X - \nabla_Z X, \phi Y) = g(\overline{\nabla}_Z X, \phi Y) = -g(\phi(\overline{\nabla}_Z X), Y)$$
$$= -g(\overline{\nabla}_Z(\phi X) - (\overline{\nabla}_Z \phi)X, Y) = -g(-A_{\phi X}Z + \nabla^{\perp}_Z \phi X, Y) + g((\overline{\nabla}_Z \phi)X, Y)$$
$$= g(A_{\phi X}Z, Y) + g((\overline{\nabla}_Z \phi)X, Y) = g(A_{\phi X}Y, Z) + g((\overline{\nabla}_Z \phi)X, Y) \tag{27}$$

By (27), we have the lemma.

Lemma (4.2) : $[X, \xi] \in D_1$ if and only if

$$g((\nabla_X \phi)\xi, Z) = g((\nabla_\xi \phi)X, Z, \quad \forall X \in D_1, Z \in D_2.$$

Proof : For any $X \in D_1$ and $Z \in D_2$, we have

$$g([X, \xi], TZ) = g(\overline{\nabla}_X \xi - \overline{\nabla}_\xi X, TZ)$$
$$= g(\nabla_X \xi - \nabla_\xi X, \phi Z) = -g(\phi(\nabla_X \xi - \nabla_\xi X), Z) \text{ using equ. (8)}$$
$$= g((\nabla_X \phi)\xi + \nabla_\xi(\phi X) - (\nabla_\xi \phi)X, Z) = g((\nabla_X \phi)\xi - (\nabla_\xi \phi)X, Z).$$

Hence the lemma is followed by last equation.

Lemma (4.3) : For any $X, Y \in D_1 \oplus D_2$, $[X, Y] \in D_1 \oplus D_2$, if and only if

$$g(\phi Y, (\overline{\nabla}_X \phi)\xi) = g(\phi X, (\overline{\nabla}_Y \phi)\xi).$$

Proof : We have for any $X, Y \in D_1 \oplus D_2$,

$$g([X, Y], \xi) = g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, \xi). \tag{28}$$

Now

$$g(Y, \xi) = 0 \Rightarrow g(\overline{\nabla}_X Y, \xi) = -g(Y, \overline{\nabla}_X \xi) \tag{29}$$

and $\quad g(\phi Y, \phi Z) = -a^2 g(Y, Z) \ \forall Z \in \overline{TM}.$

Replacing Z by $\overline{\nabla}_X \xi$ in the last equ., we obtain

$$g(Y, \overline{\nabla}_X \xi) = -\frac{1}{a^2} g(\phi Y, \phi(\overline{\nabla}_X \xi))$$
$$= \frac{1}{a^2} g(\phi Y, (\overline{\nabla}_X \phi)\xi), \tag{30}$$

making the use of (29) and (30) in (28), we obtain

$$g([X, Y], \xi) = \frac{1}{a^2}[g(\phi X, (\overline{\nabla}_Y \phi)\xi) - g(\phi Y, (\overline{\nabla}_X \phi)\xi)],$$

but $[X,Y] \in D_1 \oplus D_2$, if and only if $g([X,Y],\xi) = 0$.

Hence the lemma follows from last equation.

For any $X, Y \in D_1$ and $Z \in TM$, we have

$$g([X,Y],TP_2Z) = -g(\phi[X,Y],P_2Z) = -g(\phi(\overline{\nabla}_XY - \overline{\nabla}_YX),P_2Z)$$
$$= -g(\overline{\nabla}_X(\phi Y) - (\overline{\nabla}_X\phi)Y - \overline{\nabla}_Y(\phi X) + (\overline{\nabla}_Y\phi)X, P_2Z)$$
$$= -g(-A_{\phi Y}X + \nabla^{\perp}_X(\phi Y) + A_{\phi X}Y - \nabla^{\perp}_Y(\phi X) - (\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X, P_2Z),$$
using (27)
$$= g((\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X, P_2Z) + g(\overline{\nabla}_{P_2Z}\phi)X, Y), \tag{31}$$

Since, $[X,Y] \in D_1$ if and only if $g([X,Y],TP_2Z) = 0$.

Thus, the required integrability conditions are obtained from (31) and lemma (4.1).

Similarly, for the distribution $D_1\oplus < \xi >$, the integrability conditions are obtained from (31) and lemma (4.2).

Now, for any $X, Y \in D_2$ and $Z \in D_1$, we have

$$g(\phi[X,Y],\phi Z) = -\,a^2 g([X,Y],Z)$$
$$\Rightarrow a^2g([X,Y],Z) = -g(\phi[X,Y],NZ) = -g(\overline{\nabla}_X(\phi Y) - \overline{\nabla}_Y(\phi X) - (\overline{\nabla}_X\phi)Y$$
$$+(\overline{\nabla}_Y\phi)X, NZ)$$
$$= g(h(Y,TX) - h(X,TY) + \nabla^{\perp}_Y NX - \nabla^{\perp}_X NY + (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X, NZ). \tag{32}$$

Therefore, the integrability of the slant distribution D_2 is obtained from lemma (4.3), and the fact that ND_1 and ND_2 are orthogonal in the equ. (32).

In similar manner we easily find the integrability conditions for the distribution $D_2\oplus < \xi >$.

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Study on Kaehlerian Recurrent and Symmetric Spaces of Second Order

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Abstract

Tachibana (1967), Singh (1971) studied and defined the Bochner curvature tensor and Kaehlerian spaces with recurrent Bochner curvature tensor. Further , Negi and Rawat (1994), (1997) studied some bi-recurrence and bi-symmetric properties in a Kaehlerian space and Kaehlerian spaces with recurrent and symmetric Bochner curvature tensor.

In the present paper, we have studied Kaehlerian recurrent and symmetric spaces of second order by taking different curvature tensor and relations between them. Also several theorems have been established therein.

1. Introduction

Let X_{2n} be a $2n-$dimensional almost-complex space and its almost-complex structure, then by definition, we have

$$F_j^s F_s^i = \delta_j^i. \tag{1.1}$$

An almost-complex space with a positive definite Riemannian metric g_{ji} satisfying

$$g_{rs} F_j^r F_i^s = g_{ji} \tag{1.2}$$

is called an almost-Hermitian space. From (1.2) it follows that $F_{ji} = g_{ri} F_j^r$ is skew-symmetric.

If an almost-Hermitian space satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \tag{1.3}$$

where ∇_j denotes the operator of covariant derivative with respect to the symmetric Riemannian connection, then it is called an almost-Kaehlerian space and if it satisfies

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0 \tag{1.4}$$

Then it is called a $K-$space. In an almost-Hermitian space, if

$$\nabla_j F_{ih} = 0. \tag{1.5}$$

Then it is called a Kaehlerian space or briefly a K_n space.

The Riemannian curvature tensor which are denoted by R^h_{ijk} is given by (Weatherburn 1938)

$$R^h_{ijk} = \partial_i \left\{ \begin{array}{c} h \\ jk \end{array} \right\} - \partial_j \left\{ \begin{array}{c} h \\ ik \end{array} \right\} + \left\{ \begin{array}{c} h \\ ip \end{array} \right\} \left\{ \begin{array}{c} p \\ jk \end{array} \right\} - \left\{ \begin{array}{c} h \\ jp \end{array} \right\} \left\{ \begin{array}{c} p \\ ik \end{array} \right\} \tag{1.6}$$

The Ricci-tensor and scalar curvature are respectively given by

$$R_{ij} = R^a_{aij} \qquad \text{and} \qquad R = R_{ij} g^{ij}.$$

If we define a tensor S_{ij} by

$$S_{ij} = F^a_i R_{aj}, \tag{1.7}$$

Then, we have

$$S_{ij} = -\, S_{ji}, \tag{1.8}$$

and

$$F^a_i S_{aj} = -\, S_{ia}\, F^a_j. \tag{1.9}$$

The holomorphically projective curvature tensor and the H-Concircular curvature tensor are respectively given by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2F^h_k S_{ij}) \tag{1.10}$$

and

$$C^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{1.11}$$

The equation (1.10), in view of (1.11) may be expressed as

$$P^h_{ijk} = C^h_{ijk} + \frac{1}{n(n+2)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k) -$$
$$- \frac{R}{(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{1.12}$$

If we put

$$L_{ij} = R_{ij} - \frac{R}{n} g_{ij} \tag{1.13}$$

and

$$M_{ij} = F^a_i S_{aj} = S_{ij} - \frac{R}{n} F_{ij}, \tag{1.14}$$

Then (1.12) reduces to the form

$$P^h_{ijk} = C^h_{ijk} + \frac{R}{n(n+2)}(L_{ik}\delta^h_j - L_{jk}\delta^h_i + M_{ik}F^h_j - M_{jk}F^h_i + 2M_{ij}F^h_k). \quad (1.15)$$

Now, we have the following :

2. Kaehlerian Recurrent Space of Second Order

Definition (2.1) : A Kaehler space K_n satisfying the relation

$$\nabla_b \nabla_a R^h_{ijk} = \lambda_{ab} R^h_{ijk}, \quad (2.1)$$

For some non- zero tensor λ_{ab}, will be called a Kaehlerian recurrent space of second order and is called Ricci-recurrent (or, semi-recurrent) space of second order, if it satisfies

$$\nabla_b \nabla_a R_{ij} = \lambda_{ab} R_{ij}, \quad (2.2)$$

Multiplying the above equation by g^{ij}, we have

$$\nabla_b \nabla_a R = \lambda_{ab} R, \quad (2.3)$$

Remark (2.1) : From (2.1) and (2.2), it follows that every Kaehlerian recurrent space of second order is Ricci-recurrent space of second order but the converse is not necessarily true.

Definition (2.2) :A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a P^h_{ijk} = \lambda_{ab} P^h_{ijk}, \quad (2.4)$$

For some non-zero tensor λ_{ab}, will be called a Kaehlerian $H-$Projective recurrent space of second order or, briefly a $K_n - P$ space.

Definition (2.3) : A Kaehler space K_n satisfying the relation

$$\nabla_b \nabla_a C^h_{ijk} = \lambda_{ab} C^h_{ijk}, \quad (2.5)$$

For some non-zero tensor λ_{ab}, will be called a Kaehlerian $H-$Concircular recurrent space of second order or, briefly a $K_n - C$ space.

Theorem (2.1) : Every Kaehlerian recurrent space of second order is $K_n - C$ space.

Proof : Differentiating (1.11) covariantly with respect to x^a, again differentiate the result thus obtained covariantly with respect to x^b, we have

$$\nabla_b\nabla_a C^h_{ijk} = \nabla_b\nabla_a R^h_{ijk} + \frac{\nabla_b\nabla_a R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{2.6}$$

Multiplying (1.11) by λ_{ab}, then subtracting from (2.6), we obtain

$$\nabla_b\nabla_a C^h_{ijk} - \lambda_{ab}C^h_{ijk} = \nabla_b\nabla_a R^h_{ijk} - \lambda_{ab}R^h_{ijk} + \frac{(\nabla_b\nabla_a R - \lambda_{ab}R)}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i$$
$$+F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{2.7}$$

Now, let the space be Kaehlerian recurrent space of second order, then equation (2.7) with the help of equations (2.1) and (2.3) becomes

$$\nabla_b\nabla_a C^h_{ijk} - \lambda_{ab}C^h_{ijk} = 0,$$

Or,

$$\nabla_b\nabla_a C^h_{ijk} = \lambda_{ab}C^h_{ijk},$$

Which shows that the space is $K_n - C$ space.

Similarly, in view of equations (1.10),(2.1),(2.2) and (1.7), we have the following :

Theorem (2.2) : Every Kaehlerian recurrent space of second order is $K_n - P$ space.

Theorem (2.3) : The necessary and sufficient condition for a $K_n - C$ space to be a $K_n - P$ space is that

$$(\nabla_b\nabla_a L_{ik} - \lambda_{ab}L_{ik})\delta^h_j - (\nabla_b\nabla_a L_{jk} - \lambda_{ab}L_{jk})\delta^h_i + (\nabla_b\nabla_a M_{ik} - \lambda_{ab}M_{ik})F^h_j$$
$$-(\nabla_b\nabla_a M_{jk} - \lambda_{ab}M_{jk})F^h_i + 2(\nabla_b\nabla_a M_{ij} - \lambda_{ab}M_{ij})F^h_k = 0. \tag{2.8}$$

Proof : Suppose $K_n - C$ space is a $K_n - P$ space.

Differentiating (1.15) covariantly w.r.t. x^a, again differentiate the result thus obtained covariantly w.r.t. x^b, we have

$$\nabla_b\nabla_a P^h_{ijk} = \nabla_b\nabla_a C^h_{ijk} + \frac{1}{(n+2)}(\nabla_b\nabla_a L_{ik}\delta^h_j - \nabla_b\nabla_a L_{jk}\delta^h_i + \nabla_b\nabla_a M_{ik}F^h_j$$
$$-\nabla_b\nabla_a M_{jk}F^h_i + 2\nabla_b\nabla_a M_{ij}F^h_k) \tag{2.9}$$

Transvecting (1.15) by λ_{ab} and subtracting from the above equation (2.9), we have

$$\nabla_b\nabla_a P^h_{ijk} - \lambda_{ab}P^h_{ijk} = \nabla_b\nabla_a C^h_{ijk} - \lambda_{ab}C^h_{ijk} + \frac{1}{(n+2)}[(\nabla_b\nabla_a L_{ik} - \lambda_{ab}L_{ik})\delta^h_j$$
$$-(\nabla_b\nabla_a L_{jk} - \lambda_{ab}L_{jk})\delta^h_i + (\nabla_b\nabla_a M_{ik} - \lambda_{ab}M_{ik})F^h_j$$
$$-(\nabla_b\nabla_a M_{jk} - \lambda_{ab}M_{jk})F^h_i + 2(\nabla_b\nabla_a M_{ij} - \lambda_{ab}M_{ij})F^h_k] \quad (2.10)$$

Since a $K_n - C$ space is a $K_n - P$ space, then equation (2.10), in view of (2.4) and (2.5) reduces to (2.8).

Conversely, if $K_n - C$ space satisfies the condition (2.8), then (2.10) in view of (2.5) reduces to

$$\nabla_b\nabla_a P^h_{ijk} - \lambda_{ab}P^h_{ijk} = 0,$$

which shows that the space is $K_n - P$ space.

This completes the proof.

Theorem (2.4) : If in a Kaehler space satisfying any two of the following properties :

(i) the space is Kaehlerian Ricci- recurrent space of second order,

(ii) the space is Kaehlerian Projective recurrent space of second order,

(iii) the space is H-Concircular recurrent space of second order , then it must also satisfies third.

Proof : Differentiating (1.12) covariantly w.r.t. x^a, again differentiate the result thus obtained covariantly w.r.t. x^b, we have

$$\nabla_b\nabla_a P^h_{ijk} = \nabla_b\nabla_a C^h_{ijk} + \frac{1}{(n+2)}(\nabla_b\nabla_a R_{ik}\delta^h_j - \nabla_b\nabla_a R_{jk}\delta^h_i + \nabla_b\nabla_a S_{ik}F^h_j$$
$$-\nabla_b\nabla_a S_{jk}F^h_i + 2\nabla_b\nabla_a S_{ij}F^h_k - \frac{\nabla_b\nabla_a R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i$$
$$+F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k), \quad (2.11)$$

Multiplying (1.12) by λ_{ab} and subtracting the result from (2.11), we have

Kaehlerian Ricci-recurrent space of second order, Kaehlerian Projective recurrent space of second order and Kaehlerian $H-$Concircular recurrent space of second order are respectively characterized by the equations (2.2), (2.4) and (2.5).

The statement of the above theorem follows in view of equations (2.2), (2.4), (2.5) and (2.12).

3. Kaehlerian Symmetric Space of Second Order

Definition (3.1) : A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a R^h_{ijk} = 0, \qquad \text{or equivalently} \qquad \nabla_b \nabla_a R_{ijkl} = 0, \tag{3.1}$$

Will be called Kaehlerian symmetric space of second order and is called Kaehlerian Ricci-symmetric or (semi-symmetric) space of second order, if it satisfies

$$\nabla_b \nabla_a R_{ij} = 0, \tag{3.2}$$

Multiplying the above equation by g^{ij}, we have

$$\nabla_b \nabla_a R = 0, \tag{3.3}$$

Remark (3.1) : From (3.1) and (3.2), it follows that every Kaehlerian symmetric space of second order is Kaehlerian Ricci-symmetric space of second order , but the converse is not necessarily true.

Definition (3.2) : A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a P^h_{ijk} = 0, \qquad \text{or equivalently} \qquad \nabla_b \nabla_a P_{ijkl} = 0, \tag{3.4}$$

will be called a Kaehlerian $H-$Projective symmetric space of second order or, briefly a $^*K_n - P$ space.

Definition (3.3) : A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a C^h_{ijk} = 0, \qquad \text{or equivalently} \qquad \nabla_b \nabla_a C_{ijkl} = 0, \tag{3.5}$$

will be called a Kaehlerian $H-$Concircular symmetric space of second order or, briefly $^*K_n - C$ space.

Theorem (3.1) : The necessary and sufficient condition for a $^*K_n - C$ space to be a $^*K_n - P$ space is that

$$\nabla_b \nabla_a L_{ik}\delta^h_j - \nabla_b \nabla_a L_{jk}\delta^h_i + \nabla_b \nabla_a M_{ik}F^h_j - \nabla_b \nabla_a M_{jk}F^h_i + 2\nabla_b \nabla_a M_{ij}F^h_k = 0. \tag{3.6}$$

Proof : From equations (1.5), (2.9) and (3.5), we have

$$\nabla_b \nabla_a P^h_{ijk} = \frac{1}{(n+2)}(\nabla_b \nabla_a L_{ik}\delta^h_j - \nabla_b \nabla_a L_{jk}\delta^h_i + \nabla_b \nabla_a M_{ik}F^h_j - \nabla_b \nabla_a M_{jk}F^h_i$$
$$+2\nabla_b \nabla_a M_{ij}F^h_k = 0. \tag{3.7}$$

Since ${}^{*}K_n - C$ space is a ${}^{*}K_n - P$ space, hence equation (3.7) reduces to the form

$$\nabla_b\nabla_a L_{ik}\delta_j^h - \nabla_b\nabla_a L_{jk}\delta_i^h + \nabla_b\nabla_a M_{ik}F_j^h - \nabla_b\nabla_a M_{jk}F_i^h + 2\nabla_b\nabla_a M_{ij}F_k^h = 0. \tag{3.8}$$

Conversely, if a ${}^{*}K_n - C$ space satisfies equation (3.6), then (3.7) reduces to the form

$$\nabla_b\nabla_a P_{ijk}^h = 0$$

which shows that the space is ${}^{*}K_n - P$ space.

Theorem (3.2) :A necessary and sufficient condition for a $H-$Concircular symmetric space of second order to be Kaehlerian-Ricci symmetric space of second order is that

$$\nabla_b\nabla_a R_{ijk}^h + \lambda_{ab}[C_{ijk}^h - R_{ijk}^h - \frac{R}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)] = 0 \tag{3.9}$$

Proof : If the space is a $H-$Concircular symmetric space of second order, then equation (2.7) in view of (3.5) reduces to the form

$$\nabla_b\nabla_a R_{ijk}^h - \lambda_{ab}R_{ijk}^h + \lambda_{ab}C_{ijk}^h + \frac{(\nabla_b\nabla_a R - \lambda_{ab}R)}{n(n+2)}[g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)] = 0 \tag{3.10}$$

Now, if the space is Kaehlerian- Ricci symmetric space of second order then (3.2) is satisfied and equation (3.10), in view of (3.2) reduces to (3.9).

Conversely, if $H-$Concircular symmetric space of second order satisfies the condition (3.9), then equation (2.7) gives

$$\frac{\nabla_b\nabla_a R}{n(n+2)}[g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)] = 0$$

which gives $\nabla_b\nabla_a R = 0$

$\nabla_b\nabla_a g^{ij}R_{ij} = 0$ since $R = R_{ij}g^{ij}$

Or $\nabla_b\nabla_a R_{ij} = 0$ since $g^{ij} \neq 0$

which shows that the space is Kaehlerian Ricci-symmetric space of second order.

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Implications of One-Loop Quantum Correction in the Background Geometry of 5-Dimensional Kaluza-Klein Cosmology

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Abstract

Through dimensional reduction and one-loop quantum correction of scalar and spinor fields, time-dependent cosmological constant Λ_{eff}, effective gravitational constant G_{eff} and fine structure constant are derived in 5-dimensional Kaluza-Klein model for cosmology. If the internal manifold contracts with time and stabilizes itself at some later time, one possibility gets fine-structure constant equal to

$$\frac{1}{137}, \ \ G_{eff} \simeq G_N \ \ and \ \ \Lambda_{eff} \simeq 0 \ .$$

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1. Introduction

In the context of unification of gravity with other fundamental forces, Kaluza-Klein theory is important. Basically, in this theory 5-dimensional manifold is considered as $M^4 \times S^1$ where M^4 is the 4-dimensional manifold and S^1 is a circle. Our observable universe is 4-dimensional, so it is expected that radius of S^1 is extremely small (undetectable). Hence, it is very natural to think that if extra manifold was a reality at very high energy scale and is undetectable now because of nonavailability of energy of required order, it should manifest itself in some way or the other. Employing the method of heat - Kernel method, Toms [3] calculated one-loop effective action in 5-dimensional background geometry and obtained induced cosmological constant, gravity and Maxwell's term as manifestation of fifth dimension of the space. But the cosmological constant obtained by him is very large. The model, considered by him (Toms) contains static component of metric tensor corresponding to extra space which completely ignores its dynamical contribution.

This note offers calculation of time-dependent cosmological constant, effective gravitational constant(time dependent) as well as Maxwell's terms using the heat-Kernal method (adapted by Toms) to evaluate one-loop effective action for scalar fields as well as Dirac spinors. The 5-dimensional cosmological model proposed here is given by the line-element

$$ds^2 = dt^2 - a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - b^2(t)(dy - kA_\mu(x)dx^\mu)^2 \qquad (1)$$

where t is the cosmic time, $a(t)$ is the expanding scale factor for spatially flat subspace of M^4, $b(t)$ is the contracting scale factor for S^1, $A_\mu(\mu = 0, 1, 2, 3)$ is the four-dimensional electromagnetic field and k is a constant of $(mass)^{-1}$ dimension to make $kA_\mu(x)$ dimensionless.

Using horizontal lift basis [4,5] the action in the background geometry given by (1) is written as

$$S = -\frac{1}{16\pi G_5}\int d^4x dy\sqrt{-g_5}R_5 + \int d^4x dy\sqrt{-g_5}\frac{1}{2}[g^{m'n'}(D_{m'}\Phi)^* (D_{n'}\Phi) - \xi R_5\Phi^*\Phi - M_0^2\Phi^*\Phi] + \frac{1}{2}\int d^4x dy\sqrt{-g_5}\bar{\Psi}(ir^{m'}D_{m'} - M_{\frac{1}{2}})\Psi \qquad (2)$$

where $G_5 = G_N L$ (G_N is the Newtonian gravitational constant equal to M_p^{-2} where M_p is Planck mass, $0 \le y \le L$). 5-dim. Ricci scalar $R_5 = R_4 - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu}$ (R_4 is 4-dim. Ricci scalar, $F_{\mu\nu} = D_\nu A_\mu - D_\mu A_\nu$, $D_\mu = \bigtriangledown_\mu + kA_\mu$, $D_5 = \bigtriangledown_5$ ($\bigtriangledown_\mu$ and $\bigtriangledown_5$ are convariant derivatives in curved space). 5-dim. Dirac matrices $\gamma^{m'}(m' = 0, 1, 2, 3, 5)$ in curved space are given as $\gamma^{m'} = h_a^{m'}\tilde{\gamma}^a(\tilde{\gamma}^0, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)$ are Dirac matrices in 4-dimensional flat space and $\tilde{\gamma}^5 = \tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3, h_a^{m'}$ are defined as $h_a^{m'}h_b'\eta^{ab} = g^{m'n'}$ with $\eta^{ab} = diag(1, -1, -1, -1, -1)$, ξ is a coupling constant, Φ is a scalar field with mass M_o, Ψ is the Dirac spinor with mass $M_{\frac{1}{2}}$ and g_5 is the determinant of the metric tensor $g_{m'n'}$ given as

$$g_{m'n'} = diag(1, -a^2, -a^2, -a^2, -b^2)$$

in horizontal lift basis. $\hbar = c = 1$ is used as fundamental unit where $\hbar$ and c have their usual meaning.

2. Gravity

5-dimensional action for gravity given by (2) can be reduced to 4-dimensional action employing the method of Pollock[6]. In this method, $g_{m'n'}$ can be conformally transformed to $g'_{m'n'}$ as

$$g_{m'n'} = b^2(t)g'_{m'n'} = b^2(t)\begin{pmatrix}\tilde{\tilde{g}}_{\mu\nu} & 0 \\ 0 & -1\end{pmatrix} \quad (3)$$

where $\tilde{\tilde{g}}$ is the resulting metric tensor on M^4. So, on ignoring term of total divergence,

$$S_g = -\frac{1}{16\pi G_5}\int d^4x dy \sqrt{-\tilde{\tilde{g}}_4}\, b^3 \left[\tilde{\tilde{R}}_4 - 12b^{-2}(\tilde{\tilde{\nabla}} b)^2 - \frac{1}{4}b^{-2}k^2\tilde{\tilde{F}}_{\mu\nu}\tilde{\tilde{F}}^{\mu\nu}\right] \quad (4)$$

where $\tilde{\tilde{\nabla}}$ is the covariant derivative, $\tilde{\tilde{R}}_4$ is Ricci scalar and $\tilde{\tilde{F}}_{\mu\nu}$ is electromagnetic field strength corresponding to $\tilde{\tilde{g}}_{\mu\nu}$.

Further conformal transformation is done over $\tilde{\tilde{g}}_{\mu\nu}$ only as

$$\tilde{\tilde{g}}_{\mu\nu} = e^{2v}g_{\mu\nu} \quad (5)$$

where v is function of $b(t)$. Now using this conformal transformation and integrating over y,

$$\begin{aligned} S_g^{(4)} = &-\frac{1}{16\pi G_N}\int d^4x\sqrt{-g_4(x)}\,b^3e^{2v} \\ &\left[R_4 - \frac{1}{4}b^{-2}k^2e^{-2v}F^{\mu\nu}F_{\mu\nu} - 12(\dot{v})^2 - 12(\frac{\dot{b}}{b})^2 - 18\dot{v}(\frac{\dot{b}}{b})\right] \end{aligned} \quad (6)$$

where dot (.) denotes derivative with respect to t(time). Choosing $v = -\frac{3}{2}ln\ b(t)$, one gets 4-dimensional action for gravity as

$$S_g^{(4)} = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g_4(x)}\left[R_4 - \frac{1}{4}k^2F_{\mu\nu}F^{\mu\nu} - 12(\frac{\dot{b}}{b})^2\right] \quad (7)$$

constant k was introduced with intention to keep the theory dimensionally correct. So, without any harm to physics, k may be identified with $(16\pi G_N)^{\frac{1}{2}}$.

3. Scalar fields

The extra manifold is a circle which is not simply-connected, hence any field on it can be either untwisted(periodic in y) or twisted(anti-periodic in y)[7]. Hence, in either case, one may write

$$\Phi(x^\mu, y) = [Lb(t)]^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \Phi_n(x^\mu) exp[i(n+\alpha)My] \tag{8}$$

where $M = 2\pi L^{-1}$ (L is circumference of S^1) and $\alpha = 0 \left(\frac{1}{2}\right)$ for untwisted (twisted) field.

Substituting $\Phi(x^\mu, y)$ given by (8) in the action for scalar field given by (2) and integrating over Y

$$S_\Phi^{(4)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d^4x \sqrt{-g_4}(x)[g^{\mu\nu}(D_\mu^{(n)}\Phi_n)^*(D_\nu^{(n)}\Phi_n) - M_n^2 \Phi_n^* \Phi_n - \xi(R_4 - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu})\Phi_n^*\Phi_n] \tag{9}$$

where

$$D_\mu^{(n)}\Phi_n = \bigtriangledown_\mu \Phi_n + iq_n A_n \Phi_n, \tag{10a}$$

$$M_n^2 = M_0^2 + \frac{(n+\alpha)^2}{b^2}M^2 - \frac{3}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{1}{4}\left(\frac{\dot{b}}{b}\right)^2 - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{b}}{b}\right) \tag{10b}$$

and

$$q_n = (n+\alpha)e = (n+\alpha)kM \tag{10c}$$

Here q_n is the charge of the scalar particle in nth mode which is integral (half-integral) multiple of e $(= kM)$ for untwisted (twisted) field.

Now one loop effective action for Φ_n is calculated for nth mode and summed up over all modes to get [3]

$$\Gamma_\Phi^{(1)} = \frac{i}{2} \sum_{n=-\infty}^{\infty} ln\, det \Delta_n \tag{11}$$

where Δ_n is the operator defined as

$$\Delta_n = g^{\mu\nu} D_\mu^{(n)} D_\nu^{(n)} + M_n^2 + \xi \left(R_4 - \frac{1}{4}k^2 F_{\mu\nu}F^{\mu\nu}\right) \tag{12}$$

Using the kernal $k_n(s, x, x)$ for Δ_n, (11) can be re-written as

$$\Gamma_\Phi^{(1)} = \frac{i}{2} \sum_{n=-\infty}^{\infty} \int d^4x \sqrt{-g_4} \int_0^\infty \frac{ds}{s} tr\; k_n(s, x, x) \tag{13}$$

where

$$k_n(s,x,x) = i\mu^{4-N}(4\pi is)^{-\frac{N}{2}} exp(-iM_n^2 s)\sum_{k=0}^{\infty}(is)^k a_k(x)$$

(N is the space-time dimension used as dimensional regulator with $N \to 4$ and μ is a constant of mass dimension to get dimensionless action). For Δ_n given by (12) [8,9]

$$a_0(x) = 1 \qquad (14a)$$

$$a_1(x) = (\frac{1}{6} - \xi)R_4 + \frac{1}{4}\xi k^2 F_{\mu\nu}F^{\mu\nu} \qquad (14b)$$

$$a_2(x) = -\frac{1}{12}k^2 M^2 (n+\alpha)^2 + \dots \qquad (14c)$$

Only relevant terms are mentioned here.

Integrating over s in (13) and using (14)

$$\Gamma_\Phi^{(1)} = -\frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g_4}\,[\lim_{N\to 4}\Gamma(-\frac{N}{2})\sum_{n=-\infty}^{\infty}\Big\{\frac{(n+\alpha)^2M^2}{b^2} + \bar{M}^2(t)\Big\}^{\frac{N}{2}} +$$
$$\lim_{N\to 4}\sqrt{(1-\frac{N}{2})}\sum_{n=-\infty}^{\infty}\Big\{\frac{(n+\alpha)^2M^2}{b^2} + \bar{M}^2(t)\Big\}^{\frac{N}{2}-1}(\frac{1}{6}-\xi)R_4 +$$
$$\lim_{N\to 4}\{\frac{1}{4}\xi k^2\sqrt{(1-\frac{N}{2})}\sum_{n=-\infty}^{\infty}[\frac{(n+\alpha)^2M^2}{b^2} + \bar{M}^2(t)]^{\frac{N}{2}-1} -$$
$$\frac{1}{12}\sqrt{2-\frac{N}{2}}\sum_{n=-\infty}^{\infty}k^2M^2(n+\alpha)^2[\frac{(n+\alpha)^2M^2}{b^2} + \bar{M}^2(t)]^{\frac{N}{2}-2}\} + \dots] \qquad (15)$$

where

$$M^{-2}(t) = M_0^2 - \frac{3}{2}\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{1}{4}\left(\frac{\dot{b}}{b}\right)^2 - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{b}}{b}\right)$$

Using the formulae (B6) of ref.[10],

$$\sum_{n=-\infty}^{\infty}[(n+c)^2 + d^2]^{-\lambda} = \pi^{\frac{1}{2}}d^{1-2\lambda}\frac{\sqrt{(\lambda-\frac{1}{2})}}{\sqrt{\lambda}} + 4\sin\pi\lambda f_\lambda(c,d)$$

(where Re $\lambda > \frac{1}{2}$ and c and d are real), series in (15), for $\bar{M}^2(t) > 0$ is summed to yield when $\alpha = 0$,

$$\Gamma^{(1)}_{\Phi} = -\frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g_4}[-\frac{8\pi}{15}\{\bar{M}(t)\}^5\frac{b}{M} + \frac{4\pi b}{3M}\{\bar{M}(t)\}^3\times \\ (\frac{1}{6}-\xi)R_4 + \frac{k^2}{4}\left(\frac{4\pi\xi b}{3M}\{\bar{M}(t)\}^3 + \frac{M^2\varsigma(3)}{24\pi^2}\right)F_{\mu\nu}F^{\mu\nu} + \ldots] \tag{16}$$

where $\varsigma(p)$ is the Riemann-zeta function.

When $\alpha = \frac{1}{2}$

$$\Gamma^{(1)}_{\Phi} = -\frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g_4}[-\frac{8\pi}{15}\{\bar{M}(t)\}^5 + \frac{4\pi b}{3M}\{\bar{M}(t)\}^3\times \\ (\frac{1}{6}-\xi)R_4 + \frac{k^2}{4}\left(\frac{4\pi\xi b}{3M}\{\bar{M}(t)\}^3 - \frac{M^2\varsigma(3)}{4\pi^2}\right)F_{\mu\nu}F^{\mu\nu} + \ldots] \tag{17}$$

If $N_0^+(N_0^-)$ is the number of untwisted (twisted) scalar fields in the theory,

$$\Gamma^{(1)}_{\Phi} = -\frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g_4}[-\frac{8\pi}{15}\frac{b}{M}\{\bar{M}(t)\}^5(N_0^+ + N_0^-) + \\ \frac{4\pi b}{3M}(N_0^+ + N_0^-)\{\bar{M}(t)\}^3(\frac{1}{6}-\xi)R_4 + \\ \frac{k^2}{4}\left(\frac{4\pi\xi b}{3M}\{\bar{M}(t)\}^3(N_0^+ + N_0^-) + \frac{M^2\varsigma(3)}{24\pi^2}(N_0^+ - \frac{3}{2}N_0^-)\right)F_{\mu\nu}F^{\mu\nu} + \ldots] \tag{18}$$

4. Dirac spinors

Like scalar fields, $\Psi(x^\mu, y)$ may also be written as

$$\Psi(x^\mu, y) = [Lb(t)]^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty}\Psi_n(x^\mu)exp[i(n+\alpha)My] \tag{19}$$

Using this anstaz for $\Psi(x^\mu, y)$ in the action for $\Psi(x^\mu, y)$ given by (2) and integrating over y,

$$S^{(4)}_{\Psi} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int d^4x\sqrt{-g_4}\bar{\Psi}_n\left[i\gamma^\mu D^{(n)}_\mu - \frac{\tilde{\gamma}^5(n+\alpha)M}{b} - M_{\frac{1}{2}}\right]\Psi_n \tag{20}$$

Under chiral rotations [11,12], the mass term for Ψ_n gets the canonical form

$$\bar{\Psi}_n \left[\frac{(n+\alpha)^2}{b^2} + M^2_{\frac{1}{2}} \right] \Psi_n \tag{21}$$

Now one-loop correction terms for Ψ_n can be calculated by repeating the procedure adopted for scalar fields with

$$t_r a_0(x) = p \tag{22a}$$

$$t_r a_1(x) = -\frac{1}{12} p R_4 + \frac{pk^2}{16} F_{\mu\nu} F^{\mu\nu} \tag{22b}$$

$$t_r a_2(x) = -\frac{p}{12} k^2 M^2 + (n+\alpha)^2 F_{\mu\nu} F^{\mu\nu} + \dots \tag{22c}$$

Here also only relevant terms are mentioned, p in (22) is the number of spinor components which is 4 for Ψ_n. If number of untwisted (twisted) spinors are $N^+_{\frac{1}{2}} (N^-_{\frac{1}{2}})$

$$\begin{aligned} \Gamma^{(1)}_{\Psi} = &- \frac{1}{2(4\pi)^2} \int d^4x \sqrt{-g_4}\, [-\frac{32\pi b}{15M} M^5_{\frac{1}{2}} (N^+_{\frac{1}{2}} + N^-_{\frac{1}{2}}) + \\ & \frac{4\pi b}{9M} M^3_{\frac{1}{2}} (N^+_{\frac{1}{2}} + N^-_{\frac{1}{2}}) R_4 + \frac{k^2}{4} \{ \frac{-4\pi b}{3M} M^3_{\frac{1}{2}} (N^+_{\frac{1}{2}} + N^-_{\frac{1}{2}}) - \\ & \frac{2M^2}{3\pi^2} \varsigma(3) (N^+_{\frac{1}{2}} - \frac{3}{2} N^-_{\frac{1}{2}}) \} F_{\mu\nu} F^{\mu\nu} + \dots \end{aligned} \tag{23}$$

5. Effective action for gravity

From (7), (18) and (23), effective action for 4-dimensional gravity is written as

$$\begin{aligned} S^{(4)eff}_g = \int d^4x \sqrt{-g_4}\, & [-\frac{1}{16\pi G_n} + \frac{b}{24\pi M} \{\bar{M}(t)\}^3 (N^+_0 + N^-_0)(\frac{1}{6} - \xi) + \\ & \frac{b}{72\pi M} M^3_{\frac{1}{2}} (N^+_{\frac{1}{2}} + N^-_{\frac{1}{2}}) R_4 + \frac{3}{4\pi G_N} \left(\frac{\dot{b}}{b} \right)^2 + \frac{1}{60\pi} \frac{b}{M} \{\bar{M}(t)\}^5 \times \\ & (N^+_0 + N^-_0) - \frac{b}{15\pi M} M^5_{\frac{1}{2}} (N^+_{\frac{1}{2}} + N^-_{\frac{1}{2}})] \end{aligned} \tag{24}$$

which yields the effective 4-dimensional gravitational constant as

$$\frac{1}{16\pi G_{eff}} = \frac{1}{16\pi G_N} + \frac{b}{72\pi M} [3\{\bar{M}(t)\}^3 (N^+_0 + N^-_0)(\frac{1}{6} - \xi) + M^3_{\frac{1}{2}} (N^+_{\frac{1}{2}} + N^-_{\frac{1}{2}})] \tag{25a}$$

and effective cosmological constant $\wedge_{eff}$ as

$$\frac{\wedge_{eff}}{8\pi G_{eff}} = \frac{3}{4\pi G_N}(\frac{\dot{b}}{b})^2 + \frac{b}{60\pi M}[\{\bar{M}(t)\}^5(N_0^+ + N_0^-) - 4M_{\frac{1}{2}}^5(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-)] \quad (25b)$$

Thus, one finds that G_{eff} and $\wedge_{eff}$ are time dependent. Also it is interesting to see that if $\xi > \frac{1}{6}$ and at a particular time t'

$$\frac{1}{16\pi G_N} < \frac{b(t')}{72\pi M}[3\{\bar{M}(t)\}^3(N_0^+ + N_0^-)(\xi - \frac{1}{6}) - M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-)] \quad (25c)$$

$G_{eff} < 0$. It means that under above circumstances, gravity becomes repulsive contrary to its usually believed nature. Possibility of anti-gravity has also been discussed by Yoshimura[13] in the context of his finite temperature theory of higher-dimensional Kaluza-Klein type cosmology. But if $\xi \leq \frac{1}{6}$, $G_{eff} > 0$. Even if $\xi > \frac{1}{6}$, $G_{eff} > 0$ is possible provided that a particular time t''

$$M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-) > 3\{\bar{M}(t'')\}^3(N_0^+ + N_0^-)(\xi - \frac{1}{6})$$

6. Induced Maxwell's terms

From (7),(18) and (23), induced Maxwell's term in the action is given as

$$S_{F^2}^{(4)} = \frac{1}{4}\int d^4x\sqrt{-g_4}\frac{e^2}{M^2}[\frac{b}{16\pi G_N} + \frac{4\pi\xi b}{3M}\{\bar{M}(t)\}^3(N_0^+ + N_0^-) + \frac{M^2\varsigma(3)}{6\pi^2}\times$$
$$(N_0^+ - \frac{3}{2}N_0^-) - \frac{4\pi b}{M}M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-) - \frac{2M^2}{3\pi^2}\varsigma(3)(N_{\frac{1}{2}}^+ - \frac{3}{2}N_{\frac{1}{2}}^-)F_{\mu\nu}F^{\mu\nu}] \quad (26)$$

The normalization condition for A_μ yields [14,15,16]

$$b(t)\left[\frac{M_p^2}{16\pi} + \frac{4\pi\xi}{3M}\{\bar{M}(t)\}^3(N_0^+ + N_0^-) - \frac{4\pi}{M}M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-)\right] +$$
$$\frac{M^2\varsigma(3)}{6\pi^2}(N_0^+ - \frac{3}{2}N_0^-) - \frac{2M^2}{3\pi^2}\varsigma(3)(N_{\frac{1}{2}}^+ - \frac{3}{2}N_{\frac{1}{2}}^-) = \frac{M^2}{e^2} \quad (27)$$

If $N_0^+ = 4N_{\frac{1}{2}}^+$ and $N_0^- = 4N_{\frac{1}{2}}^-$, (27) gets a more convenient from as

$$b(t)\left[\frac{M_p^2}{16\pi} + \frac{16\pi\xi}{3M}\{\bar{M}(t)\}^3 - \frac{4\pi}{M}M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-)\right] = \frac{M^2}{e^2} \quad (28)$$

It is interesting to see from (27) and (28) that e (gauge coupling constant for electromagnetic field) is time-dependent. As a result fine structure constant (for $N_0^+ = 4N_{\frac{1}{2}}^+$ and $N_0^- = 4N_{\frac{1}{2}}^-$) is given as

$$\frac{e^2}{4\pi} = \frac{M^2}{4\pi}[b(t)]^{-1}\left[\frac{M_p^2}{16\pi} + \frac{16\pi\xi}{3M}\{\bar{M}(t)\}^3 - \frac{4\pi}{M}M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-)\right]^{-1} \tag{29}$$

is time-dependent which shows that when $b \to \infty$, $\frac{e^2}{4\pi} \to 0$ and as $b \to 0$, $\frac{e^2}{4\pi} \to \infty$. But we know that at low mass scale (large t), $\frac{e^2}{4\pi} \simeq \frac{1}{137}$. This well-known result puts a constraint on $b(t)$ that $b(t)$ should stabilize itself at some time t_1, during the course of evolution of the universe around the value $b_1 = b(t_1)$ given by

$$\frac{1}{137} = \frac{M^2}{4\pi}b_1^{-1}\left[\frac{M_p^2}{16\pi} + \frac{16\pi\xi M_0^3}{3M} - \frac{4\pi}{M}M_{\frac{1}{2}}^3(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-)\right]^{-1} \tag{30}$$

In (30), if M_0 and $M_{\frac{1}{2}}$ are sufficiently small,

$$b_1 \simeq \frac{548M^2}{M_p^2} \tag{31}$$

The effective radius of the extra manifold (circle) is Lb(t). If extra manifold is hidden, at the compactification time t_c

$$Lb(t_c) \lesssim L_p \tag{32}$$

Constraint obtained above and the fact that $b(t)$ is a contracting scale factor, imply that

$$b(t_c) \geq b_1 \tag{33}$$

Thus, one gets

$$Lb_1 \leq Lb(t_c) \lesssim L_p \tag{34}$$

Now (31) and (34) imply compactification mass $M \lesssim \frac{M_p}{548}$ and $b_1 \lesssim 1.8 \times \bar{1}0^3$.

From (25a) and (34), one gets at $t = t_1$

$$\frac{1}{16\pi G_{eff}} \lesssim \frac{1}{16\pi G_N} + \frac{M_p^{-1}}{72\pi}\left\{12M_0^3(\frac{1}{6} - \xi) + M_{\frac{1}{2}}^3\right\}(N_{\frac{1}{2}}^+ + N_{\frac{1}{2}}^-) \approx \frac{1}{16\pi G_N} \tag{35}$$

(25b) and (34) imply that at $t = t_1$

$$\frac{\wedge_{eff}}{8\pi G_{eff}} \lesssim \frac{M_p^{-1}}{15\pi}(N_{\frac{1}{2}}^{+} + N_{\frac{1}{2}}^{-})(M_0^5 - N_{\frac{1}{2}}^5) \tag{36}$$

which shows that if $M_0 \simeq M_{\frac{1}{2}}, \wedge_{eff} = 0$, otherwise also $\wedge_{eff} \approx 0$.

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Einstein-Kaehlerian Recurrent Space of Second Order

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Abstract

Walker (1950) and Roter (1964) studied and defined Ruse's spaces of recurrent curvature and second order recurrent spaces respectively.

In the present paper, we have studied and defined Einstein-Kaehlerian recurrent space of second order and several theorems have been established therein.

1. Introduction

An $n(=2m)$ dimensional Kaehlerian space K^n is an even dimensional Riemannian space, with a mixed tensor field F_i^h and with Riemannian metric g_{ij} satisfying the following conditions

$$F_i^h F_j^i = -\delta_j^h, \tag{1.1}$$

$$F_{ij} = -F_{ji}, \ (F_{ij} = F_i^a g_{aj}) \tag{1.2}$$

and

$$F_{i,j}^h = 0, \tag{1.3}$$

where the $(,)$ followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, which we denote by R_{ijk}^h is given by

$$R_{ijk}^h = \partial_i \left\{ \begin{array}{c} h \\ jk \end{array} \right\} - \partial_j \left\{ \begin{array}{c} h \\ ik \end{array} \right\} + \left\{ \begin{array}{c} h \\ ii \end{array} \right\} \left\{ \begin{array}{c} l \\ jk \end{array} \right\} - \left\{ \begin{array}{c} h \\ jl \end{array} \right\} \left\{ \begin{array}{c} l \\ ik \end{array} \right\} \tag{1.4}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denote real local coordinates.

The Ricci-tensor and the scalar curvature are respectively given by

$$R_{ij} = R_{aij}^a \qquad \text{and} \qquad R = R_{ij} g^{ij}.$$

It is well known that these tensors satisfies the following identities

$$R^{a}_{ijk,a} = R_{jk,i} - R_{ik,j}, \tag{1.5}$$

$$R_{,i} = 2R^{a}_{i,a} \tag{1.6}$$

$$F^{a}_{i} R_{aj} = -\, R_{ia} F^{a}_{j}, \tag{1.7}$$

and

$$F^{a}_{i} R^{j}_{a} = R^{a}_{i} F^{j}_{a} \tag{1.8}$$

Let R_{hijk} be the components of the Riemannian curvature tensor.

We define a bi-recurrent space as a non-flat Riemannian V_n, the Riemannian Curvature tensor of which satisfies a relation of the form

$$R_{hijk,ab} = \lambda_{ab} R_{hijk} \tag{1.9}$$

where λ_{ab} is a non-zero tensor of the second order called the tensor of recurrence or recurrence tensor.

A Kaehlerian space K^n is said to be Kaehlerian recurrent space of second order if the curvature tensor field satisfy the condition

$$R_{hijk,ab} - \lambda_{ab} R_{hijk} = 0 \tag{1.10}$$

for some non-zero recurrence tensor λ_{ab}.

THe space is said to be Kaehlerian Ricci recurrent space of second order, if it satisfies the condition

$$R_{ij,ab} - \lambda_{ab} R_{ij} = 0 \tag{1.11}$$

Multiplying the above equation by g^{ij}, we get

$$R_{,ab} - \lambda_{ab} R = 0 \tag{1.12}$$

An immediate consequence of (1.9) and Bianchi identity

$$R_{hijk,a} + R_{hika,j} + R_{hiaj,k} = 0$$

gives for a bi-recurrent space

$$\lambda_{ab} R_{hijk} + \lambda_{jb} R_{hika} + \lambda_{kb} R_{hiaj} = 0 \tag{1.13}$$

In the case

$$R_{hijk,ab} = 0$$

(1.9) and (1.13) are satisfied for $\lambda_{ij} = 0$ and the space may or may not satisfy (1.13) for some non-zero tensor λ_{ij}

Let us suppose that a Kaehlerian space is an Einstein one, then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij}, \tag{1.14}$$

at every point of the space.

Theorem 1. If a recurrent space of second order (or bi-recurrent space) be Einstein, then the Ricci-curvature tensor vanishes.

Proof. Considering (1.13), transvecting by $g^{hk}g^{ij}$, we get

$$\lambda_{ab}R - \lambda_{jb}g^{ij}R_{ia} - \lambda_{kb}g^{hk}R_{ha} = 0$$

i.e.

$$\lambda_{ab}R - 2\lambda_{jb}g^{ij}R_{ia} = 0$$

Let a bi-recurrent space be Einstein one. Then making use of (1.14), in (1.15), we obtain

$$\lambda_{ab}R - 2\lambda_{jb}g^{ij}\frac{R}{n} g_{ia} = 0$$

whence

$$(n-2)\lambda_{ab}R = 0.$$

Since $\lambda_{ab} \neq 0$ and $n > 2$, $R = 0$ which is equivalent in an Einstein space to saying that $R_{ij} = 0$. This completes the proof.

Theorem 2. In an Einstein recurrent space of second order, the scalar $g^{rs}\lambda_{rs}$ vanishes.

Proof. Transvecting (1.13) by g^{hk} and with the aid of $R_{ij} = 0$, we get

$$\lambda_{kb}R^k_{iaj} = 0 \tag{1.16}$$

Transvecting (1.13) again by g^{ab} yields

$$\phi R_{hijk} - \lambda_{jb}g^{ab}R_{akhi} + \lambda_{kb}g^{ab}R_{ajhi} = 0 \tag{1.17}$$

where we have put the scalar $g^{ab}\lambda_{ab} = \phi$. Simplifying (1.17), we get

$$\phi R_{hijk} = \lambda_{jb}R^b_{khi} - \lambda_{kb}R^b_{jhi}.$$

This, by virtue of (1.16), gives

$$\phi R_{hijk} = 0.$$

Hence, either $\phi = 0$ or $R_{hijk} = 0$. But $R_{hijk} \neq 0$, because the case of flatness contradicts the definition of a recurrent space of second order (or, bi-recurrent space).

Therefore $\phi = 0$, i.e., $g^{ab}\lambda_{ab} = 0$ or, $g^{rs}\lambda_{rs} = 0$.

Which completes the proof of the theorem.

2. Condition for recurrent space of second order to be recurrent

We know the definition of a recurrent space. Evidently, a recurrent space is bi-recurrent or recurrent space of second order, but the converse is not true. It will however be shown in the form of a theorem that under certain conditions a recurrent space of second order (or, bi-recurrent space) becomes recurrent.

Theorem 3. A recurrent space of second order (or, bi-recurrent space) with $\lambda^{rs}\lambda_{rs} = 0$, $g^{rs}\lambda_{rs} \neq 0$ is recurrent when and only when the space is Ricci-recurrent.

Proof. If a recurrent space of second order is recurrent, then the space is Ricci-recurrent. Conversely, if $\lambda^{rs}\lambda_{rs} = 0$ and $g^{rs}\lambda_{rs} \neq 0$, then as shown by Roter [2], the curvature tensor of a recurrent space of second order (bi-recurrent space) has the following form

$$R_{hijk} = \frac{2}{R}(R_{hk}R_{ij} - R_{hj}R_{ik}), \tag{2.1}$$

we then consider those recurrent spaces of second order which are Ricci-recurrent having β_l as vector of recurrence.

Equation (2.1) thus yields

$$R_{hijk,a} = \frac{4}{R}\beta_l(R_{hk}R_{ij} - R_{hj}R_{ik}) - \frac{2}{R}\beta_l(R_{hk}R_{ij} - R_{hj}R_{ik} = \frac{2}{R}\beta_l(R_{hk}R_{ij} - R_{hj}R_{ik})$$
$$= \beta_l R_{hijk}.$$

Therefore, the space is recurrent.

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A Note on Affine Motion in a Birecurrent Finsler Space

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Abstract

Several authors discussed affine motion generated by contra, concurrent, special concircular, recurrent, concircular and torse forming vector fields in special spaces such as recurrent, birecurrent and symmetric Riemannian and Finsler spaces. The first author [20-22] for the first time obtained the necessary and sufficient conditions for the above vector fields to generate an affine motion in a general Finsler space. Recently Surendra Pratap Singh [26] discussed affine motion in a birecurrent Finsler space. The aim of this paper is to generalize the results of Surendra Pratap Singh.

Keywords and Phrases : Recurrent Finsler space, Birecurrent Finsler space, Contra vector field, Concurrent vector field, Affine motion.
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1. Introduction

Takano [27-28] studied certain types of affine motion generated by contra, concurrent, special concircular, torse forming and birecurrent vectors in non-Riemannian manifold of recurrent curvature. Following the techniques of Takano, the authors Sinha [25], Misra [5-7], Misra and Meher [8-10], Meher [4] and Kumar [1-3] studied the above mentioned types of affine motion in Finsler space of recurrent curvature and obtained various results. The first author obtained the necessary and sufficient conditions for above vector fields to generate an affine motion in a general Finsler space. Surendra Pratap Singh [26] discussed affine motion in birecurrent Finsler space. In the present paper we have generalized certain results of Surendra Pratap Singh and highlighted some results which are either trivial or meaningless in the aforesaid paper.

2. Preliminaries

Let $F_n\,(F, g, G)$ be an $n-$dimensional Finsler space of class at least C^7 equipped with metric function F, corresponding symmetric metric tensor g and Berwald's connection G. Connection coefficients of Berwald satisfy

(2.1) (a) $G^i_{j\,k} = G^i_{k\,j}$, (b) $G^i_{j\,k}\dot{x}^k = G^i_j$, (c) $\dot{\partial}_k G^i_j = G^i_{k\,j}$,

where $\dot{\partial}_k \equiv \frac{\partial}{\partial \dot{x}^k}$.

$G^i_{jk\,h} = \dot{\partial}_h G^i_{jk}$ constitute a tensor which are symmetric in its lower indices and satisfy

(2.2) $$G^i_{jkh}\dot{x}^h = G^i_{khj}\dot{x}^h = G^i_{hjk}\dot{x}^h = 0.$$

The covariant derivative $\mathcal{B}_k\,T^i_j$ of an arbitrary tensor T^i_j for the connection G is given by

(2.3) $$\mathcal{B}_k\,T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\,G^r_k + T^r_j\,G^i_{rk} - T^i_r\,G^r_{jk},$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$.

The operator $\mathcal{B}_k$ commutes with $\dot{\partial}_k$ and itself as follows

(2.4) $$(\dot{\partial}_j \mathcal{B}_k - \mathcal{B}_k \dot{\partial}_j)\,T^i_h = T^r_h\,G^i_{jkr} - T^i_r\,G^r_{jkh},$$

(2.5) $$(\mathcal{B}_j \mathcal{B}_k - \mathcal{B}_k \mathcal{B}_j)\,T^i_h = T^r_h\,H^i_{jkr} - T^i_r\,H^r_{jkh} - (\dot{\partial}_r T^i_h)\,H^r_{jk},$$

where H^i_{jkh} constitute Berwald's curvature tensor given by

(2.6) $$H^i_{jkh} = \partial_j G^i_{hk} - \partial_k G^i_{hj} + G^r_{hk}\,G^i_{jr} - G^r_{hj}\,G^i_{rk} + G^i_{rhj}\,G^r_k - G^i_{rhk}\,G^r_j.$$

This tensor is anti-symmetric in first two lower indices and is positively homogeneous of degree zero in $\dot{x}^i$. The tensor H^i_{jk} appearing in (2.5) is related with the curvature tensor as

(2.7)(a) $H^i_{jkh}\,\dot{x}^h = H^i_{jk}$, (b) $\dot{\partial}_h\,H^i_{jk} = H^i_{jkh}$,

and with deviation tensor H^i_j as

(2.8)(a) $H^i_{jk}\,\dot{x}^k = H^i_j$,

(b) $\frac{1}{3}(\dot{\partial}_k H^i_j - \dot{\partial}_j H^i_k) = H^i_{jk}$.

The associate vector y_i of $\dot{x}^i$ satisfies the relations [18]

(2.9)(a) $$y_i\dot{x}^i = F^2, \qquad (b) \qquad y_i H^i_{jk} = 0, \qquad (c) \qquad g_{ik}H^i_{mj} + y_i H^i_{mjk} = 0,$$

where g_{ij} are components of metric tensor g.

The curvature tensor fields satisfy the following Bianchi identities [23]

(2.10) $$\mathcal{B}_l H^i_{jkh} + \mathcal{B}_j H^i_{klh} + \mathcal{B}_k H^i_{ljh} + H^r_{jk} G^i_{rlh} + H^r_{kl} G^i_{rjh} + H^r_{lj} G^i_{rkh} = 0,$$

(2.11) $$\mathcal{B}_l H^i_{jk} + \mathcal{B}_j H^i_{kl} + \mathcal{B}_k H^i_{lj} = 0,$$

(2.12) $$\mathcal{B}_l H^i_k - \mathcal{B}_k H^i_l + (\mathcal{B}_r H^i_{kl})\,\dot{x}^r = 0.$$

Let us consider the infinitesimal transformation

(2.13) $$\bar{x}^i = x^i + \varepsilon\, v^i(x^j),$$

generated by a vector field $v^i(x^j)$, ε being an infinitesimal constant. The Lie derivatives of an arbitrary tensor T^i_j and the connection coefficients G^i_{jk} with respect to (2.13) are given by [29]

(2.14) $$\pounds\, T^i_j = v^r \mathcal{B}_r T^i_j - T^r_j \mathcal{B}_r v^i + T^i_r \mathcal{B}_j v^r + (\dot{\partial}_r T^i_j)\, \mathcal{B}_s v^r\, \dot{x}^s,$$

(2.15) $$\pounds\, G^i_{jk} = \mathcal{B}_j \mathcal{B}_k v^i + H^i_{mjk} v^m + G^i_{jkr} \mathcal{B}_s v^r\, \dot{x}^s.$$

The operator $\pounds$ commutes with the operators $\mathcal{B}_k$ and $\dot{\partial}_k$ according as

(2.16) $$(\pounds\, \mathcal{B}_k - \mathcal{B}_k\, \pounds)\, T^i_j = T^r_j\, \pounds\, G^i_{rk} - T^i_r\, \pounds\, G^r_{jk} - (\dot{\partial}_r T^i_j)\, \pounds\, G^r_k,$$

(2.17) $$(\dot{\partial}_k \pounds - \pounds \dot{\partial}_k)\Omega = 0,$$

where Ω is a vector, tensor or connection coefficients.

The infinitesimal transformation (2.13) defines an affine motion if it preserves parallelism of pair of vectors. The necessary and sufficient condition for the vector $v^i(x^j)$ to generate an affine motion is that [29]

(2.18) $$\pounds\, G^i_{jk} = 0.$$

Since the curvature tensor is Lie invariant with respect to an affine motion, in this case we have

(2.19) $$\pounds H^i_{jkh} = 0.$$

The vector field v^i is called contra and concurrent vector field according as it satisfies [27]

$$(2.20)(a) \qquad \mathcal{B}_k v^i = 0, \qquad (b) \qquad \mathcal{B}_k v^i = \lambda \delta^i_k,$$

λ being a constant.

The affine motion generated by the above vector fields is called a contra affine motion and a concurrent affine motion, respectively.

3. Special Finsler Spaces

A non-flat Finsler space F_n is called a recurrent Finsler space if the curvature tensor satisfies

$$(3.1) \qquad \mathcal{B}_l H^i_{jkh} = K_l H^i_{jkh},$$

where K_l is a non-zero vector field [2-4, 6-9, 16, 17, 25]. Pandey [17] proved that the recurrence vector K_l is independent of $\dot{x}^i$, in general.

Following identities are satisfied in a recurrent space [17]:

$$(3.2) \qquad K_l H^i_{jkh} + K_k H^i_{ljh} + K_j H^i_{klh} = 0,$$

$$(3.3) \qquad K_l H^i_{jk} + K_k H^i_{lj} + K_j H^i_{kl} = 0,$$

$$(3.4) \qquad H^r_{[jk} G^i_{l]m\,r} = 0,$$

where square bracket shows the skew-symmetric part with respect to the indices enclosed in it.

A non-flat Finsler space F_n is called a birecurrent Finsler space if the curvature tensor satisfies the relation

$$(3.5) \qquad \mathcal{B}_l \mathcal{B}_m H^i_{jkh} = A_{l\,m} H^i_{jkh},$$

where $A_{l\,m}$ is a non-zero tensor field, called birecurrence tensor field [1, 5, 12].

A birecurrent Finsler space satisfies the following:

$$(3.6) \qquad A_{lm} H^i_{jk} + A_{lk} H^i_{mj} + A_{lj} H^i_{km} = 0.$$

We may also define an r-recurrent Finsler space characterized by the condition

$$(3.7) \qquad \mathcal{B}_{l_1}\mathcal{B}_{l_2}\cdots\mathcal{B}_{l_r} H^i_{j\,k\,h} = A_{l_1\,l_2\cdots l_r} H^i_{jkh}.$$

In view of Bianchi identities, the tensor field H^i_{jk} satisfies

$$A_{l_1 l_2 \cdots l_{r-1} l_r} H^i_{jk} + A_{l_1 l_2 \cdots l_{r-1} k} H^i_{l_r j} + \cdots = 0. \tag{3.8}$$

4. Affine Motion in a Birecurrent Finsler Space F_n

Let us consider a Finsler space F_n admitting the affine motion (2.13). Then, we have (2.18) and (2.19). In view of the commutation formula exhibited by (2.16) and the equation (2.18), we find that the operators of covariant differentiation $\mathcal{B}_k$ and Lie-differentiation $\pounds$ are commutative for an arbitrary tensor $T^{\cdots}_{\cdots}$ of any order, i.e.

$$\pounds \mathcal{B}_m T^{\cdots}_{\cdots} = \mathcal{B}_m \pounds T^{\cdots}_{\cdots}. \tag{4.1}$$

In particular,

$$\begin{aligned} &\pounds \mathcal{B}_m H^i_{jkh} = \mathcal{B}_m \pounds H^i_{jkh}, \\ &\pounds \mathcal{B}_l \mathcal{B}_m H^i_{jkh} = \mathcal{B}_l \pounds \mathcal{B}_m H^i_{jkh} = \mathcal{B}_l \mathcal{B}_m \pounds H^i_{jkh}, \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ &\pounds \mathcal{B}_{m_1} \mathcal{B}_{m_2} \cdots \mathcal{B}_{m_r} H^i_{jkh} = \mathcal{B}_{m_1} \mathcal{B}_{m_2} \cdots \mathcal{B}_{m_r} \pounds H^i_{jkh} \end{aligned} \tag{4.2}$$

which, in view of (2.19), give

$$\begin{aligned} &\pounds \mathcal{B}_m H^i_{jkh} = 0, \\ &\pounds \mathcal{B}_l \mathcal{B}_m H^i_{jkh} = 0, \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ &\pounds \mathcal{B}_{m_1} \mathcal{B}_{m_2} \cdots \mathcal{B}_{m_r} H^i_{jkh} = 0. \end{aligned} \tag{4.3}$$

In view of (4.3), for a recurrent space, a birecurrent space and an r - recurrent space, we have

$$\pounds K_m = 0, \tag{4.4}$$

$$\pounds A_{lm} = 0 \tag{4.5}$$

and

$$\pounds A_{m_1 m_2 \dots m_r} = 0, \tag{4.6}$$

respectively.

Singh [26] considered a special birecurrent Finsler space (though he did not use the word "special") whose recurrence tensor A_{lm} is of the form

$$A_{lm} = \mathcal{B}_m K_l + K_m K_l. \tag{4.7}$$

He discussed affine motion in such space and obtained the following theorems :

Theorem 1. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the Lie-derivative of the recurrence tensor field A_{lm} satisfies the relation $\pounds A_{lm} = \pounds \mathcal{B}_m K_l$.

Theorem 2. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the recurrence tensor A_{lm} satisfies the identity $\pounds \mathcal{B}_n A_{[lm]} + \pounds \mathcal{B}_l A_{[mn]} + \pounds \mathcal{B}_m A_{[nl]} = 0$.

Theorem 3. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the recurrence tensor A_{lm} satisfies $\pounds (\dot{\partial}_r A_{[lm]}) = 0$.

Theorem 4. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the Bianchi identities satisfied by curvature tensor H^i_{jkh}, H^i_{jk} and H^i_k take the forms

$$(\pounds A_{ls}) \dot{x}^s H^i_{jkh} + (\pounds A_{ks}) \dot{x}^s H^i_{jhl} + (\pounds A_{hs}) \dot{x}^s H^i_{jlk} = 0,$$
$$(\pounds A_{ls}) H^i_{jk} + (\pounds A_{js}) H^i_{kl} + (\pounds A_{ks}) H^i_{lj} = 0$$

and

$$(\pounds A_{ls}) H^i_k - (\pounds A_{ks}) H^i_l + (\pounds A_{rs}) H^i_{kl} \dot{x}^r = 0,$$

respectively.

Theorem 5. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion in order that the vector field $v^i(x^j)$ spans a contra field, the relations $H^i_{sjk} v^s = 0$ and $H^i_{sjk} \pounds v^s = 0$ hold good.

Theorem 6. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion in order that the vector filed $v^i(x^j)$ determines concurrent field the relations $H^i_{sjk} v^s = 0$ and $H^i_{sjk} \pounds v^s = 0$ are necessarily true.

In view of (4.5), Theorem 1 is not correct while the next three theorems (Theorem 2, Theorem 3 and Theorem 4) reduce to $0 = 0$.

The Lie-derivative of a tensor field T^i_j with respect to the infinitesimal transformation (2.13) is given by (2.14).

In particular,

$$\pounds v^i = v^r \mathcal{B}_r v^i + (\dot{\partial}_r v^i) \mathcal{B}_s v^r \dot{x}^s - v^r \mathcal{B}_r v^i = 0. \tag{4.8}$$

The main finding in Theorem 5 and Theorem 6 of Singh [26] is that a contra or concurrent vector field $v^i(x^j)$ generating an affine motion in the so called birecurrent Finsler space satisfies

$$H^i_{sjk}\,\pounds\, v^s \,=\, 0. \tag{4.9}$$

In view of (4.8), it is trivial.

Pandey [20] proved that an infinitesimal transformation, generated by a contra vector field, is necessarily an affine motion in a general Finsler space. Therefore, it is an affine motion in a birecurrent Finsler space.

If a birecurrent Finsler space admits an infinitesimal transformation generated by a contra vector field $v^i(x^j)$, then the recurrence tensor A_{lm} satisfies (vide Pandey [20]):

(4.10)(a) $\quad A_{lm}\, v^m \,=\, 0, \qquad$ (b) $\quad A_{lm}\, v^l \,=\, 0.$

In case of recurrence tensor $A_{l\,m}$ considered by Singh [26] above conditions become

(4.11)(a) $\quad (\mathcal{B}_m K_l \,+\, K_m\, K_l)\, v^m \,=\, 0,$

(b) $\quad (\mathcal{B}_m K_l \,+\, K_m\, K_l)\, v^l \,=\, 0.$

In view of (4.11a) and (4.11b), we have

(4.12)(a) $\quad v^m\, \mathcal{B}_m\, K_l \,=\, -(K_m\, v^m) K_l,$

(b) $\quad \mathcal{B}_m\, (K_l\, v^l) \,=\, -(K_l\, v^l) K_m.$

If we put $K_l\, v^l \,=\, L$, then (4.12a) and (4.12b) reduce to

(4.13)(a) $\quad v^m\, \mathcal{B}_m\, K_l \,=\, -L\, K_l,$

(b) $\quad \mathcal{B}_m\, L \,=\, -L\, K_m.$

Using (2.14) for K_l and applying (2.20a), we have

$$\pounds\, K_l = v^m\, \mathcal{B}_m\, K_l. \tag{4.14}$$

From (4.13a) and (4.14), we obtain

$$\pounds\, K_l = \,-\, L K_l. \tag{4.15}$$

Thus, we have

Theorem 7. In a birecurrent Finsler space admitting an infinitesimal transformation generated by a contra vector field $v^i(x^j)$, if the birecurrence tensor A_{lm} is characterized by (4.7), then the vector K_l is Lie-recurrent.

Again, from (4.15), we observe that $\pounds\, K_l = 0$ if and only if $L = K_l\, v^l = 0$. Thus, we conclude that

Theorem 8. In a birecurrent Finsler space admitting an infinitesimal transformation generated by a contra vector field $v^i(x^j)$, if the birecurrence tensor A_{lm} is characterized by (4.7), then the necessary and sufficient condition for the vector K_l to be Lie-invariant is that K_l is orthogonal to the contra vector $v^i(x^j)$.

Pandey [20] proved that a birecurrent Finsler space does not admit any infinitesimal transformation generated by a concurrent vector field. Therefore, the study of a birecurrent Finsler space admitting a concurrent affine motion is wastage of precious time and is to indulge in unnecessary mechanical labour.

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On A Semi-Symmetric Non Metric Connection in Lorentzian Para-Cosympletic Manifold

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Abstract

In this paper we define and studied a semi-symmetric non metric connection on a Lorentzian Para-Cosympletic Manifold and prove its existence. We deduce the expression for curvature tensor and Ricci tensor of semi-symmetric non metric connection defined. A necessary and sufficient condition has been deduced for the Ricci tensor to be symmetric and skew-symmetric under certain condition. Bianchi first identity associated with the connection, Einstein Manifold, Weyl conformal curvature tensor of the same connection were found.

Keywords and Phrases : Semi-symmetric non-metric connection, Ricci tensor, conformal curvature tensor, cosymplectic manifold.
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1. Introduction

Let (M^n, g) be a $n-$dimensional differentiable manifold on which there are defined a tensor field ϕ of type (1,1) a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\,\xi \tag{1.1}$$

$$\eta(\xi) = -1 \tag{1.2}$$

$$g(\phi X,\, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{1.3}$$

$$g(X,\, \xi) = \eta(X) \tag{1.4}$$

Then M^n is called a Lorentzian Para-contact Manifold (or LP-contact Manifold) and the structure(ϕ, ξ, η, g) is called an LP-contact structure (Matsumoto 1989).

In an LP-Contact Manifold, we have

(1.5)(a) $\phi\xi = 0$

(b) $\eta(\phi X) = 0$

(c) rank $\phi = n - 1$.

Let us put

$$F(X,Y) = g(\phi X, Y) \tag{1.6}$$

Then the tensor field F is symmetric (0,2) tensor field

$$F(X,Y) = F(Y,X) \tag{1.7}$$

An LP-contact manifold is said to be an LP-cosympletic manifold (Prasad & Ojha 1994) if

$$D_X\phi = 0 \Rightarrow D_X F(Y,Z) = 0 \tag{1.8}$$

On this manifold, we have

$$(D_X\eta)(Y) = 0 \tag{1.9}$$

and

$$D_X\xi = 0 \tag{1.10}$$

For vector field X, Y and Z where D_X denotes covariant differentiation with respect to g.

2. Semi- Symmetric Non Metric Connection in an LP-Cosympletic Manifold

Let (M^n, g) be an LP-cosympletic manifold with Levi-Civita connection D. We define a linear connection $\overline{D}$ on M^n by

$$\overline{D}_X Y = D_X Y + \eta(Y)X + a(X)Y \tag{2.1}$$

where η and a are 1-form associated with vector field ξ and A on M^n given by

$$g(X,\xi) = \eta(X) \tag{2.2}$$

and

$$g(X,A) = a(X) \tag{2.3}$$

for all vector field $X \,\epsilon \chi(M^n)$ where $\chi(M^n)$ is the set of all differentiable vector field on M^n.

Using (2.1) the torsion tensor $\overline{T}$ of M^n with respect to the connection $\overline{D}$ is given by

$$\overline{T}(X,Y) = \eta(Y)X - \eta(X)Y + a(X)Y - a(Y)X \tag{2.4}$$

A linear connection satisfying (2.4) is called a semi-symmetric connection. Further using (2.1) we have

$$(\overline{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y) - 2a(X)g(Y,Z). \tag{2.5}$$

A linear connection $\overline{D}$ defined by (2.1) and satisfying (2.4) and (2.5) is called a semi-symmetric non metric connection.

Let $\overline{D}$ be a linear connection in M^n given by

$$\overline{D}_X Y = D_X Y + H(X,Y). \tag{2.6}$$

Now we shall determine the tensor field H such that $\overline{D}$ satisfies (2.4) and (2.5)

From (2.6), we have

$$\overline{T}(X,Y) = H(X,Y) - H(Y,X), \tag{2.7}$$

Denote

$$G(X,Y,Z) = (\overline{D}_X g)(Y,Z). \tag{2.8}$$

From (2.6) and (2.8), we have

$$g(H(X,Y),Z) + g(H(X,Z),Y) = -G(X,Y,Z). \tag{2.9}$$

From (2.6), (2.8), (2.9) and (2.5) we have

$$g(\overline{T}(X,Y),Z) + g(\overline{T}(Z,X),Y) + g(\overline{T}(Z,Y),X) = g(H(X,Y),Z)$$
$$-g(H(Y,X),Z)+g(H(Z,X),Y-g(H(X,Z),Y)+g(H(Z,Y),X)-g(H(Y,Z),X)$$
$$= 2g(H(X,Y),Z) + G(X,Y,Z) + G(Y,X,Z) - G(Z,X,Y)$$
$$= 2g(H(X,Y),Z)-2\eta(Z)g(X,Y)-2a(X)g(Y,Z)-2a(Y)g(X,Z)+2a(Z)g(X,Y)$$

Or

$$H(X,Y) = \frac{1}{2}\{'\overline{T}(X,Y) + {}'\overline{T}(X,Y) + {}'\overline{T}(Y,X)\} + a(X)Y + a(Y)X$$
$$+g(X,Y)\xi - g(X,Y)A$$

Where $'\overline{T}$ be a tensor field of type (1, 2) defined by

$$g('\overline{T}(X,Y),Z) = g(\overline{T}(Z,X),Y)$$

Or

$$H(X,Y) = \eta(Y)X + a(X)Y$$

This implies

$$\overline{D}_X Y = D_X Y + \eta(Y)\,X + a(X)\,Y.$$

Thus we have the following theorem :

Theorem (2.1) : Let (M^n, g) be an LP-cosympletic manifold with almost Lorentzian para contact metric structure (ϕ, ξ, η, g) admitting a semi-symmetric non metric connection $\overline{D}$ which satisfies (2.4) and (2.5) then the semi-symmetric non metric connection is given by

$$\overline{D}_X\, Y = D_X\, Y + \eta(Y)\,X + a(X)Y.$$

3. Existence of semi-symmetric non metric connection $\overline{D}$ in an LP-cosympletic manifold

Let X, Y, Z be any three vector fields on an $LP-$cosympletic manifold (M^n, g) with almost Lorentzian para contact metric structure (ϕ, ξ, η, g). We define a connection $\overline{D}$ by the following equation :

(3.1)

$$\begin{aligned}2g(\overline{D}_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) \\ & + g([Z,X],Y) + g(\eta(Y)X - \eta(X)Y + a(X)Y - a(Y)X, Z) \\ & + g(\eta(X)z - \eta(Z)X + a(Z)X - a(X)Z, Y) \\ & + g(\eta(Y)Z - \eta(Z)Y + a(Y)Z - a(Z)Y, X)\end{aligned}$$

Which holds for all vector fields $X, Y, Z \in \chi(M^n)$.

It can easily be verified that the mapping

$$\overline{D}\ :\ (X,Y) \to \overline{D}_X\, Y$$

satisfying the following identities

$$\overline{D}_X(Y+Z) = \overline{D}_X Y + \overline{D}_X Z \tag{3.2}$$

$$\overline{D}_{X+Y} Z = \overline{D}_X Z + \overline{D}_Y Z \tag{3.3}$$

$$\overline{D}_{fX} Y = f\overline{D}_X Y \tag{3.4}$$

$$\overline{D}_X fY = f\overline{D}_X + (Xf)Y \tag{3.5}$$

for all $X, Y, Z \in \chi(M^n)$ and for all $f \in F(M^n)$, the set of all differentiable mapping over M^n. From (3.2), (3.3), (3.4) and (3.5) we conclude that $\overline{D}$ determines a linear connection on M^n. Now from (3.1) we have

$$\overline{D}_X Y - \overline{D}_Y X - [X,Y] = \eta(Y)\, X - \eta(X)\, Y + a(X)\, Y - a(Y)\, X \tag{3.6}$$

Or

$$\overline{T}(X,Y) = \eta(Y)\, X - \eta(X)\, Y + a(X)\, Y - a(Y)\, X$$

Also, we have from (3.1)

$$\begin{aligned} 2g(\overline{D}_X Y, Z) + 2g(\overline{D}_X Z, Y) =& 2Xg(Y,Z) + 2\eta(Y)g(X,Z) \\ &+ 2\eta(Z)g(X,Y) + 4a(X)g(Y,Z) \end{aligned}$$

i.e.

$$(\overline{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - 2\eta(Z)g(X,Y) - 2a(X)g(Y,Z) \tag{3.7}$$

From (3.6) and (3.7) it follows that $\overline{D}$ determines a semi-symmetric non metric connection on (M^n, g). it can be easily verified that $\overline{D}$ determines a unique semi-symmetric non metric connection on (M^n, g).

Thus we have

Theorem (3.1) : Let (M^n, g) be an $LP-$Cosympletic manifold with an almost Lorentzian para-contact metric structure (ϕ, ξ, η, g) on it. Then there exist a unique linear connection $\overline{D}$ satisfying (2.4) and (2.5).

The above theorem proves the existence of a semi-symmetric non metric connection in an LP cosympletic manifold.

4. Curvature tensor of an $LP-$Cosympletic manifold with respect to the semi symmetric non metric connection $\overline{D}$

Let $\overline{R}$ and R be the curvature tensor of the connections $\overline{D}$ and D respectively then

$$\overline{R}(X,Y)Z = \overline{D}_X \overline{D}_Y Z - \overline{D}_Y \overline{D}_X Z - \overline{D}_{[X,Y]} Z. \tag{4.1}$$

From (2.1) and (4.1) we get

$$\begin{aligned} \overline{R}(X,Y,Z) = & \overline{D}_X(D_Y Z + \eta(Z)Y + a(Y)Z) - \overline{D}_Y(D_X Z + \eta(Z)X - a(X)Z) \\ & - D_{[X,Y]}Z - \eta(Z)[X,Y] - a([X,Y])Z. \end{aligned} \tag{4.2}$$

Using (1.9) in (4.2), we get

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + da(X,Y)Z \tag{4.3}$$

where

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

is the curvature tensor of D with respect to Riemannian connection. Contracting (4.3) we find

$$\overline{S}(Y,Z) = S(Y,Z) + \eta(Y)\eta(Z)n - \eta(Y)\eta(Z) + da(Z,Y) \tag{4.4}$$

Contracting with respect to Z we get

$$\overline{Q}_Y = Q_Y + n\eta(Y)\xi - \eta(Y)\xi - da(Y).$$

Again contracting w.r.t. Y

$$\overline{r} = (r+1) - n + \lambda. \tag{4.5}$$

Theorem (4.1) : The curvature tensor $\overline{R}(X,Y)Z$, the Ricci tensor $\overline{S}(Y,Z)$ and the scalar curvature $\overline{r}$ of an LP-Cosympletic manifold with respect to the semi-symmetric non metric connection $\overline{D}$ is given by(4.3),(4.4) and (4.5) respectively.

Let us assume that $\overline{R}(X,Y)Z = 0$ in (4.3) and contracting w.r.t. X we get

$$S(Y,Z) = \eta(Y)\eta(Z) - \eta(Y0\eta(Z)n - da(Z,Y).$$

Which again on contracting gives

$$r = 1 + n - \lambda \tag{4.6}$$

Hence we have

Theorem (4.2) : If the curvature tensor of an LP-Cosympletic manifold M^n admitting semi-symmetric non metric connection vanishes, then its scalar curvature is given by(4.6).

5. Symmetric and skew-symmetric condition of Ricci tensor of $\overline{D}$ in an LP-Cosympletic manifold

From (4.4) we have

$$\overline{S}(Z,Y) = S(Z,Y) + \eta(Y)\eta(Z)n - \eta(Y)\eta(Z) + da(Y,Z). \tag{5.1}$$

From (4.4) and (5.1) we have

$$\overline{S}(Y,Z) - \overline{S}(Z,Y) = da(Y,Z) - da(Z,Y). \tag{5.2}$$

If $\overline{S}(Y,Z)$ is symmetric, then the L.H.S. of (5.2) vanishes and we have

$$da(Y,Z) = da(Z,Y). \tag{5.3}$$

More over, if the relation (5.3) holds, then from (5.2) $\overline{S}(Y,Z)$ is symmetric. Hence we have

Theorem (5.1) : The Ricci tensor $\overline{S}(Y,Z)$ of the manifold with respect to the semi-symmetric non metric connection in an LP-cosympletic manifold is symmetric if and only if the relation (5.3) holds.

Again from (4.4) and (5.1), we find

(5.4)

$$\overline{S}(Y,Z)+\overline{S}(Z,Y)=2S(Y,Z)+2\eta(Y)\eta(Z)n-2\eta(Y)\eta(Z)+da(Z,Y)+da(Y,Z).$$

If $\overline{S}(Y,Z)$ is skew-symmetric then the L.H.S. of (5.4) vanishes and we get

$$S(Y,Z)=\eta(Y)\eta(Z)-n\,\eta(Y)\eta(Z)-\frac{1}{2}da(Z,Y)-\frac{1}{2}da(Y,Z). \tag{5.5}$$

More over, if $S(Y,Z)$ is given by (5.5) then from (5.4) we get

$$\overline{S}(Y,Z)\;+\;\overline{S}(Z,Y)=0$$

i.e. the Ricci tensor of $\overline{D}$ is skew-symmetric. Hence, we have

Theorem (5.2) : If an LP-cosympletic manifold admits a semi-symmetric non-metric connection $\overline{D}$ then a necessary and sufficient condition for the Ricci tensor of $\overline{D}$ to be skew-symmetric, that is the Ricci tensor of the Levi-civita connection D is given by (5.5).

6. Bianchi first identity associated with semi-symmetric non-metric connection $\overline{D}$ in an LP-cosympletic manifold

From (2.4), we have

$$\overline{T}(X,Y,Z)+\overline{T}(Y,Z,X)+\overline{T}(Z,X,Y)=0, \tag{6.1}$$

where

$$\overline{T}(X,Y,Z)=g(\overline{T}(X,Y),Z).$$

Again from (2.4) we have

$$\begin{aligned}\overline{T}(\overline{T}(X,Y),Z)+\overline{T}(\overline{T}(Y,Z),X)+\overline{T}(\overline{T}(Z,X),Y)\\ =\eta(Y)a(X)Z-\eta(X)a(Y)Z+a(X)a(Y)Z-a(Y)a(X)Z\\ +\eta(Z)a(Y)X-\eta(Y)a(Z)X+a(Y)a(Z)X-a(Z)a(Y)X\\ +\eta(X)a(Z)Y-\eta(Z)a(X)Y+a(Z)a(X)Y-a(X)a(Z)Y\end{aligned} \tag{6.2}$$

and

$$(6.3)\quad (\overline{D}_X\overline{T})(Y,Z) + (\overline{D}_Y\overline{T})(Z,X) + (\overline{D}_Z\overline{T})(X,Y)$$
$$= da(X,Y)Z + da(Y,Z)X + da(Z,X)Y + a(Z)\eta(Y)X - a(Y)\eta(Z)X$$
$$- a(X)\eta(Y)Z - a(X)a(Y)Z + a(X)\eta(Z)Y + a(X)a(Z)Y + a(X)\eta(Z)Y$$
$$- a(Z)\eta(X)Y - a(Y)\eta(Z)X - a(Y)a(Z)X + a(Y)\eta(X)Z + a(Y)a(X)Z$$
$$+ a(Y)\eta(X)Z - a(X)\eta(Y)Z - a(Z)\eta(X)Y - a(Z)a(X)Y$$
$$+ a(Z)\eta(Y)X + a(Z)a(Y)X$$

Bianchi first identity for a linear connection on M^n is given by (Sinha 1982)

$$(6.4)\quad \overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = \overline{T}(\overline{T}(X,Y),Z) + \overline{T}(\overline{T}(Y,Z),X)$$
$$+\overline{T}(\overline{T}(Z,X),Y) + (\overline{D}_X\overline{T})(Y,Z) + (\overline{D}_Y\overline{T})(Z,X) + (\overline{D}_Z\overline{T})(X,Y).$$

Using (6.2) and (6.3) and(6.4) we get

$$(6.5)\quad \overline{R}(X,Y)Z+\overline{R}(Y,Z)X+\overline{R}(Z,X)Y = da(X,Y)Z+da(Y,Z)X+da(Z,X)Y$$
$$+ a(X)\eta(Z)Y - a(X)\eta(Y)Z + a(Y)\eta(X)Z$$
$$- a(Y)\eta(Z)X + a(Z)\eta(Y)X - a(Z)\eta(X)Y.$$

We call (6.5) as the first Bianchi's identity with respect to semi-symmetric non-metric connection $\overline{D}$ in an LP-cosympletic manifold.

7. Einstein Manifold with respect to semi-symmetric non-metric connection on LP-cosympletic manifold

A Riemannian manifold Mn is called an Einstein manifold with respect to Riemannian connection if

$$(7.1)\quad S(X,Y) = \frac{r}{n}g(X,Y).$$

Analogous to this definition, we define Einstein manifold with respect to semi-symmetric non metric connection $\overline{D}$

$$(7.2)\quad \overline{S}(X,Y) = \frac{\overline{r}}{n}\, g(X,Y).$$

From (4.4) ,(4.5) and(7.2) we have

$$\overline{S}(X,Y) - \frac{\overline{r}}{n}\, g(X,Y) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) - da(Z,Y) - \frac{r+1-n+\lambda}{n} g(X,Y)$$

$$(7.3)\quad \overline{S}(X,Y) - \frac{\overline{r}}{n}g(X,Y) = S(Y,Z) - \frac{r}{n}g(X,Y) + (n-1)\eta(Y)\eta(Z)$$

$$-da(Z,Y)+\frac{\lambda+1-n}{n}g(X,Y).$$

If

(7.4) $$n(n-1)\eta(Y)\eta(Z)+(\lambda+1-n)\,g(X,Y)=n.da(Z,Y)$$

then from (7.3), we get

$$\overline{S}(X,Y)-\frac{\overline{r}}{n}g(X,Y)=S(X,Y)-\frac{r}{n}g(X,Y).$$

Hence we have

Theorem (7.1) : If the relation(7.4) holds in an LP-cosympletic manifold M^n with semi-symmetric non metric connection, then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\overline{D}$.

8. Weyl Projective Curvature Tensor

If $\overline{P}$ and P denote the projective curvature tensor with respect to $\overline{D}$ and D respectively, then we have

(8.1) $$\overline{P}(X,Y)Z=\overline{R}(X,Y)Z-\frac{1}{n-1}[\overline{S}(Y,Z)X-\overline{S}(X,Z)Y]$$

(8.2) $$P(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}[S(Y,Z)X-S(X,Z)Y]$$

Using (4.4) and (4.3) in equation (8.1), we have

$$\begin{aligned}\overline{P}(X,Y)Z&=R(X,Y,Z)+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y+da(X,Y)Z\\&-\frac{1}{n-1}[S(Y,Z)X+(n-1)\eta(Y)\eta(Z)X-da(Z,Y)X-S(X,Z)Y\\&-(n-1)\eta(X)\eta(Z)Y-da(X,Z)Y]\\&=R(X,Y,Z)-\frac{1}{n-1}[S(Y,Z)X-S(X,Z)Y]+\frac{1}{n-1}[(n-1)da(X,Y)Z-\\&da(Z,Y)X+da(X,Z)Y].\end{aligned}$$

(8.3)
$$\overline{P}(X,Y)Z=P(X,Y)Z+\frac{1}{n-1}[(n-1)da(X,Y)Z-da(Z,Y)X+da(X,Z)Y]$$

It is clear that if 1-form a is closed i.e. $da=0$. Then from (8.3) we get

$$\overline{P}(X,Y)Z=P(X,Y)Z.$$

Hence we have

Theorem (8.1) : If in an LP-cosympletic manifold M^n admits a semi-symmetric non metric connection $\overline{D}$ then the Weyl projective curvature tensor of $\overline{D}$ is equal to the Weyl projective tensor of D if 1-form is closed.

$$\overline{P}(X, Y)Z = 0 \tag{8.4}$$

Which implies $\overline{S}(Y, Z) = 0$.

Then from (8.3), we have

$$P(X, Y)Z = \frac{1}{n-1}[da(Z, Y) - (n-1)da(X, Y)Z - da(X, Z)Y]. \tag{8.5}$$

If 1-form a is closed i.e. $da = 0$.

Then from (8.5) we get

$$P(X, Y)\,Z = 0.$$

Hence we have

Theorem (8.2) : If in an Lorentzian Para cosympletic manifold M^n the curvature tensor of semi-symmetric non-metric connection $\overline{D}$ vanish and 1-form a is closed, then the manifold is projectively flat.

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