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RESEARCH ARTICLE

On Pointwise Bi-Slant Riemannian Maps

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Abstract
In this article, we study Pointwise bi-slant Riemannian maps (PBSRM) from almost Hermitian manifolds to Riemannian manifolds. The current study aims to establish the various results satisfied by these maps from Kähler manifolds to Riemannian manifolds. To check the existence of such maps, we provide an example. We derive some important results for these maps including the necessary and sufficient conditions for integrability of distributions related to these maps.

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1. Introductions

In differential geometry, smooth maps have an important role in study the geometrical properties of a manifold in order to compare with another manifold. It is well known that the Riemannian maps are the most important type of maps in Riemannian geometry and these maps are the generalization of isometric immersion, Riemannian submersion and an isometry. In 1992, the notion of Riemannian maps between Riemannian manifold was introduced by Fischer [7] as a generalization of isometric immersions and Riemannian submersions.

The concept of Riemannian submersion was initiated by O' Neill [17] and Gray [8] in 1966-67. The theory of almost Hermitian submersions was introduced by Watson [27] in 1976. The vital role and capability of Riemannian submersions in present science era can be seen in ([2], [4], [5], [6], [18]). In 1995, Riemannian submersion has been used as an application to robotics by Bedrossian and Spong [3]. Both of them showed it in the presence of a special category of robotic chains with zero Riemannian curvature when potential energy and friction phenomena are totally disregarded. Further, in 2004, an interesting application of Riemannian submersions for the theory of modeling and control of inessential robotic chain by Altafini [1]. Different types of Riemannian submersions are discussed by several geometers in ([13],[19], [22], [26]).

On the other hand, In 2010, Sahin [23] introduced Riemannian maps between almost Hermitian manifolds and Riemannian manifolds. In 2014, Park and Sahin [20] discussed semi-slant Riemannian maps into almost Hermitian manifolds and Kumar et al [21] also investigated various results about similar maps from almost contact manifolds into Riemannian manifolds in 2018. In recent past, many authors have broadly studied various types of Riemannian maps ([9], [11], [14], [15], [24], [25]). Recently, Kumar et al [12] have studied Clairaut semi-invariant- from Kenmotsu manifolds to Riemannian manifolds. Slant Riemannian maps have many applications in various field of science. Therefore, it is very enticing different types

of slant Riemannian maps on various structures in complex as well as contact geometry. So, it is interesting to study pointwise bi-slant Riemannian maps. A succint summary of the article is provided below.

In this paper, we investigate pointwise bi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. The paper is divided into four sections. In section 2, we recall all the basic definitions and terminologies which are needed throughout the paper. In section 3, we study pointwise bi-slant Riemannian maps from Kähler manifolds to Riemannian manifolds. We investigate the integrability of distributions and derive the conditions for horizontal and vertical distributions to be totally geodesic. In section 4, we construct an example to show the existence of such maps.

2. Preliminaries

Let M_1 be an even-dimensional differentiable manifold and J be a $(1,1)$ tensor field on M_1 such that

$$J^2 = -I \#(2.1).$$

where I is identity operator. Then J is called an almost complex structure on M_1. The manifold M_1 with an almost complex structure J is called an almost complex manifold [28]. It is well known that an almost complex manifold is necessarily orientable. Nijenhuis tensor N of an almost complex structure is defined as:

$$N(Z_1,V_2) = [JZ_1,JV_2] - [Z_1,V_2] - J[JZ_1,V_2] - J[Z_1,JV_2], \text{ for all } Z_1,V_2 \in \Gamma(TM_1).$$

If Nijenhuis tensor field N on an almost complex manifold M_1 is zero, then the almost complex manifold M_1 is called a complex manifold.

Let g_{M_1} be a Riemannian metric on M_1 such that

$$g_{M_1}(JZ_1,JV_2) = g_{M_1}(Z_1,V_2), g_{M_1}(Z_1,JV_2) = -g_{M_1}(JZ_1,V_2), \#(2.2)$$

for all $Z_1,V_2 \in \Gamma(TM_1)$.

Then g_{M_1} is called an almost Hermitian metric on M_1 and manifold M_1 with Hermitian metric g_{M_1} is called almost Hermitian manifold. The Riemannian connection ∇ of the almost Hermitian manifold M_1 can be extended to the whole tensor algebra on M_1. Tensor fields $(\nabla_{Z_1}J)$ is defined as

$$(\nabla_{Z_1}J)V_2 = \nabla_{Z_1}JV_2 - J\nabla_{Z_1}V_2,$$

for all $Z_1,V_2 \in \Gamma(TM_1)$.

An almost Hermitian manifold (M_1, g_{M_1}, J) is called a Kähler manifold [28] if

$$(\nabla_{Z_1}J)V_2 = 0 \#(2.3)$$

for all $Z_1,V_2 \in \Gamma(TM_1)$.

Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be a Riemannian manifold, where g_{M_1} and g_{M_2} are Riemannian metrics on C^∞-manifolds M_1 and M_2 respectively.

Let $F:(M_1, g_{M_1}) \to (M_2, g_{M_2})$ be a Riemannian maps. Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by [17]

$$\mathcal{A}_E F = H\nabla_{\mathcal{H}E}VF + V\nabla_{\mathcal{H}E}HF \#(2.4)$$
$$\mathcal{T}_E F = H\nabla_{\mathcal{V}_E}VF + V\nabla_{\mathcal{V}_E}HF \#(2.5)$$

for any vector fields E,F on M_1, where ∇ is the Levi-Civita connection of g_{M_1}. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M_1 reversing the vertical and the horizontal distributions.

From equations (2.4) and (2.5), we have

$$\nabla_X V = \mathcal{T}_X V + V\nabla_X V \#(2.6)$$
$$\nabla_X Z = \mathcal{T}_X Z + H\nabla_X Z \#(2.7)$$
$$\nabla_Z X = \mathcal{A}_Z X + V\nabla_Z X \#(2.8)$$
$$\nabla_Z W = H\nabla_Z W + \mathcal{A}_Z W \#(2.9)$$

for $X, V \in \Gamma(\ker F_*)$ and $Z, W \in \Gamma(\ker F_*)^{\perp}$, where $\mathcal{H}\nabla_X W = \mathcal{A}_W X$, if W is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution [2].

We recall that the notation of second fundamental form of a map between two Riemannian manifolds. Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be Riemannian manifolds and $F:(M_1, g_{M_1}) \to (M_2, g_{M_2})$ be a C^{∞} map then the second fundamental form of F is given by [24]

$$(\nabla F_*)(Z,U) = \nabla_Z^F (F_* U) - F_*(\nabla_Z U) \#(2.10)$$

for $Z, U \in \Gamma(TM_1)$, where ∇^F is the pullback connection and we denote for convenience by ∇ the Riemannian connections of the metrics g_{M_1} and g_{M_2} .

Finally we also recall that a differentiable map F between two Riemannian manifolds is totally geodesic if [16]

$$(\nabla F_*)(Z,U) = 0, \text{ for all } Z, U \in \Gamma(TM_1) \#(2.11)$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

3. Pointwise bi-slant Riemannian maps(PBSRM)

In this section, we discuss some results satisfied by pointwise bi-slant Riemannian maps from a Kähler manifold (M_1, g_{M_1}, J) to a Riemannian manifold (M_2, g_{M_2}).

Definition 1. Let (M_1, g_{M_1}, J) be an almost Hermitian manifold and (M_2, g_{M_2}) be a Riemannian manifold. A Riemannian maps $F:(M_1, g_{M_1}, J) \to (M_2, g_{M_2})$ is called a pointwise bi-slant Riemannian maps (PBSRM) if for $i = 1,2$ the angles θ_i between JX_i and a pair of orthogonal distributions D_i, respectively, are independent of the choice of the nonzero vector $X_i \in \Gamma(\ker F_*)$ such that $\ker F_* = D_1 \oplus D_2$ and $JD_1 \perp D_2, D_1 \perp JD_2$. Then the angle θ_i is called the slant function of the PBSRM.

Let F be pointwise PBSRM from an almost Hermitian manifold (M_1, g_{M_1}, J) to a Riemannian manifold (M_2, g_{M_2}). Then, we have

$$TM_1 = ker F_* \oplus (\ker F_*)^{\perp} \#(3.1)$$

Now, for any vector field $X_1 \in \Gamma(\ker F_*)$, we put

$$X_1 = PX_1 + QX_1 \#(3.2)$$

where P and Q are projection morphisms of $\ker F_*$ onto D_1 and D_2, respectively.

For $Z_1 \in (\Gamma \ker F_*)$, we set

$$JZ_1 = \phi Z_1 + \omega Z_1 \#(3.3)$$

where $\phi Z_1 \in (\Gamma \ker F_*)$ and $\omega Z_1 \in (\Gamma \ker F_*)^{\perp}$.

Also for any non-zero vector field $U_1 \in \Gamma(\ker F)^{\perp}$, we have

$$JU_1 = BU_1 + CU_1 \#(3.4)$$

where $BU_1 \in \Gamma(\ker F_*)$ and $CU_1 \in \Gamma(\ker F_*)^{\perp}$.

Lemma 1. Let F be a PBSRM from an almost Hermitian manifold (M_1, g_{M_1}, J) to a Riemannian manifold (M_2, g_{M_2}). Then, we have

$$\begin{aligned} \phi^2 V_1 + B\omega V_1 &= -V_1, \omega\phi V_1 + C\omega V_1 = 0 \\ \omega B Z_1 + C^2 Z_1 &= -Z_1, \phi B Z_1 + BC Z_1 = 0 \end{aligned}$$

for all $V_1 \in \Gamma(\ker F_*)$ and $U_1 \in \Gamma(\ker F_*)^{\perp}$.

Proof. Using equations (3.3), (3.4) and $J^2 = -I$, we have Lemma 3.2.

The proof of the following result is the same as given in [10], therefore, we omit its proof

Lemma 2. Let F be a pointwise bi-slant Riemannian map from an almost Hermitian manifold (M_1, g_{M_1}, J) to a Riemannian manifold (M_2, g_{M_2}). Then, we have

(i) $\phi^2 Z_1 = -(\cos^2\theta_1) Z_1$, for $Z_1 \in \Gamma(D_1)$, where θ_1 is the slant function,

(ii) $B\omega Z_1 = -(\sin^2\theta_1) Z_1$, for $Z_1 \in \Gamma(D_1)$, where θ_1 is the slant function,

(iii) $\phi^2 X_1 = -(\cos^2\theta_2) X_1$, for $X_1 \in \Gamma(D_2)$, where θ_2 is the slant function,

(iv) $B\omega X_1 = -(\sin^2\theta_2) X_1$, for $X_1 \in \Gamma(D_2)$, where θ_2 is the slant function,

Lemma 3. Let F be a PBSRM from a Kähler manifold (M_1, g_{M_1}, J) onto a Riemannian manifold (M_2, g_{M_2}). Then, we have

$$\mathcal{V}\nabla_{U_1}\phi X_1 + \mathcal{T}_{U_1}\omega X_1 = \phi\mathcal{V}\nabla_{U_1}X_1 + B\mathcal{T}_{U_1}X_1$$

$$\mathcal{T}_{U_1}\phi X_1 + \mathcal{H}\nabla_{U_1}\omega X_1 = \omega\mathcal{V}\nabla_{U_1}X_1 + C\mathcal{T}_{U_1}X_1,$$

$$\mathcal{V}\nabla_{Z_1}BX_2 + \mathcal{A}_{Z_1}CX_2 = \phi\mathcal{A}_{Z_1}X_2 + B\mathcal{H}\nabla_{Z_1}X_2$$

$$\mathcal{A}_{Z_1}BX_2 + \mathcal{H}\nabla_{Z_1}CX_2 = \omega\mathcal{A}_{Z_1}X_2 + C\mathcal{H}\nabla_{Z_1}X_2,$$

$$\mathcal{V}\nabla_{U_1}BZ_1 + \mathcal{T}_{U_1}CZ_1 = \phi\mathcal{T}_{U_1}Z_1 + B\mathcal{H}\nabla_{U_1}Z_1,$$

$$\mathcal{T}_{U_1}BZ_1 + \mathcal{H}\nabla_{U_1}CZ_1 = \omega\mathcal{T}_{U_1}Z_1 + C\mathcal{H}\nabla_{U_1}Z_1,$$

$$\mathcal{V}\nabla_{Z_1}\phi U_1+\mathcal{A}_{Z_1}\omega U_1=B\mathcal{A}_{Z_1}U_1+\phi\mathcal{V}\nabla_{Z_1}U_1$$

$$\mathcal{A}_{Z_1}\phi U_1+\mathcal{H}\nabla_{Z_1}\omega U_1=\omega\mathcal{V}_{Z_1}U_1+C\mathcal{A}_{Z_1}U_1$$

for any $U_1, X_1 \in \Gamma(\ker F_*)$ and $Z_1, X_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. Using equations (2.3), (2.6)-(2.9), (3.3) and (3.4), we get equations (3.5)(3.12).

Now, we define

$$\left(\nabla_{X_1}\phi\right)U_1=V\nabla_{X_1}\phi U_1-\phi V\nabla_{X_1}U_1 \#(3.13)$$
$$\left(\nabla_{X_1}\omega\right)U_1=H\nabla_{X_1}\omega U_1-\omega V\nabla_{X_1}U_1 \#(3.14)$$
$$\left(\nabla_{Z_1}C\right)X_2=H\nabla_{Z_1}CX_2-CH\nabla_{Z_1}X_2 \#(3.15)$$
$$\left(\nabla_{Z_1}B\right)X_2=V\nabla_{Z_1}BX_2-BH\nabla_{Z_1}X_2 \#(3.16)$$

for any $X_1, U_1 \in \Gamma(\ker F_*)$ and $Z_1, X_2 \in \Gamma(\ker F_*)^{\perp}$.

Lemma 4. Let F be a PBSRM from a Kähler manifold (M_1, g_{M_1}, J) onto a Riemannian manifold (M_2, g_{M_2}). Then, we have

$$\begin{aligned}
\left(\nabla_{U_1}\phi\right)X_1 &= B\mathcal{T}_{U_1}X_1-\mathcal{T}_{U_1}\omega X_1,\\
\left(\nabla_{U_1}\omega\right)X_1 &= C\mathcal{T}_{U_1}X_1-\mathcal{T}_{U_1}\phi X_1,\\
\left(\nabla_{Z_1}C\right)X_2 &= \omega\mathcal{A}_{Z_1}X_2-\mathcal{A}_{Z_1}BX_2,\\
\left(\nabla_{Z_1}B\right)X_2 &= \phi\mathcal{A}_{Z_1}X_2-\mathcal{A}_{Z_1}CX_2,
\end{aligned}$$

for any vectors $U_1, X_1 \in \Gamma(\ker F_*)$ and $U_1, X_1 \in \Gamma(\ker F_*)$.

Proof. Using equations (3.5)-(3.8) and (3.13)-(3.16), we get all equations of Lemma 3.5.

Theorem 1. Let F be a PBSRM from a Kähler manifold (M_1, g_{M_1}, J) to a Riemannian manifold (M_2, g_{M_2}) with the slant function θ_1. Then, the slant distribution D_1 is integrable if and only if

$$\begin{aligned}
& g_{M_1}\left(\mathcal{T}_{Z_1}\omega\phi Z_2-\mathcal{T}_{Z_2}\omega\phi Z_1, W_1\right)\\
= \; & g_{M_1}\left(\mathcal{T}_{Z_1}\omega Z_2-\mathcal{T}_{Z_2}\omega Z_1, \phi W_1\right)+g_{M_1}\left(\mathcal{H}\nabla_{Z_1}\omega Z_2-\mathcal{H}\nabla_{Z_2}\omega Z_1, \omega W_1\right),
\end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D_1)$ and $W_1 \in \Gamma(D_2)$.

Proof. For $Z_1, Z_2 \in \Gamma(D_1)$, and $W_1 \in \Gamma(D_2)$, using equations (2.2), (2.3), (3.3), (3.4) and Lemma 3.3, we have

$$\begin{aligned}
& g_{M_1}\left(\left[Z_1, Z_2\right], W_1\right)\\
= \; & g_{M_1}\left(\nabla_{Z_1}JZ_2, JW_1\right)-g_{M_1}\left(\nabla_{Z_2}JZ_1, RW_1\right),\\
= \; & \left(\cos^2\theta_1\right)g_{M_1}\left(\left[Z_1, Z_2\right], W_1\right)-g_{M_1}\left(\nabla_{Z_1}\omega\phi Z_2, W_1\right)+g_{M_1}\left(\nabla_{Z_2}\omega\phi Z_1, W_1\right)+\\
& g_{M_1}\left(\nabla_{Z_1}\omega Z_2, JW_1\right)-g_{M_1}\left(\nabla_{Z_2}\omega Z_1, JW_1\right).
\end{aligned}$$

By equation (2.7), we have

$$=\left(\sin^2\theta_1\right)g_{M_1}\left(\left[Z_1, Z_2\right], W_1\right)-g_{M_1}\left(\mathcal{T}_{Z_1}\omega\phi Z_2, W_1\right)+g_{M_1}\left(\mathcal{T}_{Z_2}\omega\phi Z_1, W_1\right)+g_{M_1}\left(\mathcal{T}_{Z_1}\omega Z_2, \phi W_1\right)$$
$$-g_{M_1}\left(\mathcal{T}_{Z_2}\omega Z_1, \phi W_1\right)+g_{M_1}\left(\mathcal{H}\nabla_{Z_1}\omega Z_2, \omega W_1\right)-g_{M_1}\left(\mathcal{H}\nabla_{Z_2}\omega Z_1, \omega W_1\right)$$

which prove.

In a similar way in as above, we obtain the following Theorem.

Theorem 2. Let F be a PBSRM from a Kähler manifold $\left(M_1, g_{M_1}, J\right)$ to a Riemannian manifold (M_2, g_{M_2}) with the slant function θ_2. Then, the slant distribution D_2 is integrable if and only if

$$g_{M_1}\left(\mathcal{T}_{X_1}\omega\phi X_2-\mathcal{T}_{X_2}\omega\phi X_1, Y_1\right)$$

$$= \quad g_{M_1}\left(\mathcal{T}_{X_1}\omega X_2-\mathcal{T}_{X_2}\omega X_1, \phi Y_1\right)+g_{M_1}\left(\mathcal{H}\nabla_{X_1}\omega X_2-\mathcal{H}\nabla_{X_2}\omega X_1, \omega Y_1\right),$$

for all $X_1, X_2 \in \Gamma\left(D_2\right)$ and $Y_1 \in \Gamma\left(D_1\right)$.

Theorem 3. Let F be a PBSRM from a Kähler manifold $\left(M_1, g_{M_1}, J\right)$ to a Riemannian manifold $\left(M_2, g_{M_2}\right)$ with the slant functions θ_1, θ_2. Then, $\left(\ker F_*\right)^{\perp}$ is integrable if and only if

$$\left(\cos^2\theta_2-\cos^2\theta_1\right)g_{M_1}\left(\left[V_1, V_2\right], QU\right)$$

$$= \quad g_{M_1}\left(\left[V_1, V_2\right], \omega\phi U\right)-g_{M_1}\left(\mathcal{A}_{V_1}BV_2-\mathcal{A}_{V_2}BV_1, \omega U\right)-g_{M_1}\left(\mathcal{H}\nabla_{V_1}CV_2-\mathcal{H}\nabla_{V_2}CV_1, \omega U\right)$$

for $V_1, V_2 \in \left(\ker F_*\right)^{\perp}$ and $U \in \left(\ker F_*\right)$.

Proof. For $V_1, V_2 \in \left(\ker F_*\right)^{\perp}$ and $U \in \left(\ker F_*\right)$, we have

$$g_{M_1}\left(\left[V_1, V_2\right], U\right)=g_{M_1}\left(\nabla_{V_1}V_2, U\right)-g_{M_1}\left(\nabla_{V_2}V_1, U\right)$$

Using equations $(2.2),(2.3),(2.8),(2.9),(3.2),(3.3),(3.4)$ and Lemma 3.3, we get

$$g_{M_1}\left(\left[V_1, V_2\right], U\right)$$

$$= g_{M_1}\left(J\nabla_{V_1}V_2, \phi PU\right)+g_{M_1}\left(J\nabla_{V_1}V_2, \phi QU\right)+g_{M_1}\left(J\nabla_{V_1}V_2, \omega U\right)-$$

$$g_{M_1}\left(J\nabla_{V_2}V_1, \phi PU\right)-g_{M_1}\left(J\nabla_{V_2}V_1, \phi QU\right)-g_{M_1}\left(J\nabla_{V_2}V_1, \omega U\right)$$

$$=\cos^2\theta_1 g_{M_1}\left(\left[V_1, V_2\right], U\right)+\left(\cos^2\theta_2-\cos^2\theta_1\right)g_{M_1}\left(\left[V_1, V_2\right], QU\right)-g_{M_1}\left(\nabla_{V_1}V_2, \omega\phi U\right)+g_{M_1}$$

$$\left(\nabla_{V_2}V_1, \omega\phi U\right)+g_{M_1}\left(\mathcal{A}_{V_1}BV_2, \omega U\right)+g_{M_1}\left(\mathcal{H}\nabla_{V_1}CV_2, \omega U\right)-g_{M_1}\left(\mathcal{A}_{V_2}BV_1, \omega U\right)-g_{M_1}\left(\mathcal{H}\nabla_{V_2}CV_1, \omega U\right)$$

Now, we get

$$=\sin^2\theta_1 g_{M_1}\left(\left[V_1, V_2\right], U\right)$$

$$\left(\cos^2\theta_2-\cos^2\theta_1\right)g_{M_1}\left(\left[V_1, V_2\right], QU\right)-g_{M_1}\left(\left[V_1, V_2\right], \omega\phi U\right)+g_{M_1}\left(\mathcal{A}_{V_1}BV_2-\mathcal{A}_{V_2}BV_1, \omega U\right)+g_{M_1}\left(\mathcal{H}\nabla_{V_1}CV_2-\mathcal{H}\nabla_{V_2}CV_1, \omega U\right).$$

Theorem 4. Let F be a PBSRM from a Kähler manifold $\left(M_1, g_{M_1}, J\right)$ to a Riemannian manifold $\left(M_2, g_{M_2}\right)$ with the slant functions θ_1, θ_2. Then the horizontal distribution ($\ker F_*$) defines a totally geodesic foliation on M_1 if and only if

$$\sin^2\theta_1 g_{M_1}\left(\left[Y_1, Z_1\right], Y_2\right)$$

$$=\left(\cos^2\theta_1-\cos^2\theta_2\right)g_{M_1}\left(\mathcal{V}\nabla_{Z_1}QY_1, Y_2\right)+\sin 2\theta_1 Z_1\left[\theta_1\right]g_{M_1}\left(PY_1, PY_2\right)+\sin 2\theta_2 Z_1\left[\theta_2\right]g_{M_1}\left(QY_1, QY_2\right)-g_{M_1}\left(\mathcal{A}_{Z_1}\omega\phi Y_1, Y_2\right)$$

$$-g_{M_1}\left(\mathcal{H}\nabla_{Z_1}\omega Y_1,\omega Y_2\right)-g_{M_1}\left(\mathcal{A}_{Z_1}\omega Y_1,\phi Y_2\right)$$

for $Y_1,Y_2\in\Gamma(\ker F_*)$ and $Z_1\in\Gamma(\ker F_*)^{\perp}$.

Proof. For $Y_1,Y_2\in\Gamma(\ker F_*)$ and $Z_1\in\Gamma(\ker F_*)^{\perp}$, using equations (2.2), (2.3), (3.2), (3.3) and Lemma 3.3, we have

$$g_{M_1}\left(\nabla_{Y_1}Y_2,Z_1\right)$$

$$=-g_{M_1}\left([Y_1,Z_1],Y_2\right)-g_{M_1}\left(\nabla_{Z_1}Y_1,Y_2\right),$$

$$=-g_{M_1}\left([Y_1,Z_1],Y_2\right)-\cos^2\theta_1 g_{M_1}\left(\nabla_{Z_1}PY_1,Y_2\right)+\sin 2\theta_1 Z_1[\theta_1]g_{M_1}\left(PY_1,Y_2\right)$$

$$-\cos^2\theta_2 g_{M_1}\left(\nabla_{Z_1}QY_1,Y_2\right)+\sin 2\theta_2 Z_1[\theta_2]g_{M_1}\left(QY_1,Y_2\right)+$$

$$g_{M_1}\left(\nabla_{Z_1}\omega\phi Y_1,Y_2\right)-g_{M_1}\left(\nabla_{Z_1}\omega Y_1,JY_2\right)$$

Now, using equations (2.8) and (2.9), we obtains

$$=\sin^2\theta_1 g_{M_1}\left(\nabla_{Y_1}Y_2,Z_1\right)$$

$$-\sin^2\theta_1 g_{M_1}\left([Y_1,Z_1],Y_2\right)+\left(\cos^2\theta_1-\cos^2\theta_2\right)g_{M_1}\left(\mathcal{V}\nabla_{Z_1}QY_1,Y_2\right)+$$

$$\sin 2\theta_1 Z_1[\theta_1]g_{M_1}\left(PY_1,PY_2\right)+\sin 2\theta_2 Z_1[\theta_2]g_{M_1}\left(QY_1,QY_2\right)-$$

$$g_{M_1}\left(\mathcal{A}_{Z_1}\omega\phi Y_1,Y_2\right)-g_{M_1}\left(\mathcal{H}\nabla_{Z_1}\omega Y_1,\omega Y_2\right)-g_{M_1}\left(\mathcal{A}_{Z_1}\omega Y_1,\phi Y_2\right).$$

Theorem 5. Let F be a PBSRM a Kähler manifold $\left(M_1,g_{M_1},J\right)$ to a Riemannian manifold $\left(M_2,g_{M_2}\right)$ with the slant functions θ_1,θ_2. Then the vertical distribution $(\ker F_*)^{\perp}$ defines a totally geodesic foliation on M_1 if and only if

$$g_{M_2}\left(\nabla_{Y_1}F_*(\omega\phi Z_1),F_*(Y_2)\right)=g_{M_1}\left(\mathcal{A}_{Y_1}\omega Z_1,BY_2\right)+g_{M_2}\left(\nabla_{Y_1}F_*(\omega Z_1),F_*(CY_2)\right),$$

$$g_{M_2}\left(\nabla_{Y_1}F_*(\omega\phi Z_2),F_*(Y_2)\right)=g_{M_1}\left(\mathcal{A}_{Y_1}\omega Z_2,BY_2\right)+g_{M_2}\left(\nabla_{Y_1}F_*(\omega Z_2),F_*(CY_2)\right),$$

for $Y_1,Y_2\in\Gamma(\ker F_*)^{\perp},Z_1\in\Gamma(D_1)$ and $Z_2\in\Gamma(D_2)$.

Proof. For $Y_1,Y_2\in\Gamma(\ker F_*)^{\perp},Z_1\in\Gamma(D_1)$ and $Z_2\in\Gamma(D_2)$. Using equations (2.2), (2.3), (3.3), (3.4) and Lemma 3.3, we have

$$g_{M_1}\left(\nabla_{Y_1}Y_2,Z_1\right)$$

$$=-g_{M_1}\left(\nabla_{Y_1}Z_1,Y_2\right),$$

$$=\cos^2\theta_1 g_{M_1}\left(\nabla_{Y_1}Z_1,Y_2\right)+g_{M_1}\left(\nabla_{Y_1}\omega\phi Z_1,Y_2\right)-g_{M_1}\left(\nabla_{Y_1}\omega Z_1,BY_2\right)-g_{M_1}\left(\nabla_{Y_1}\omega Z_1,CY_2\right)$$

Now, using equations (2.9), and (2.10), we have

$$\sin^2\theta_1 g_{M_1}\left(\nabla_{Y_1}Y_2,Z_1\right)$$

$$=\quad g_{M_2}\left(\nabla_{Y_1}F_*(\omega\phi Z_1),F_*(Y_2)\right)-g_{M_1}\left(\mathcal{A}_{Y_1}\omega Z_1,BY_2\right)-g_{M_2}\left(\nabla_{Y_1}F_*(\omega Z_1),F_*(CY_2)\right).$$

Similarlly, we

$$\sin^2\theta_2 g_{M_1}\left(\nabla_{Y_1}Y_2, Z_2\right)$$

$$= \quad g_{M_2}\left(\nabla_{Y_1}F_*\left(\omega\phi Z_2\right), F_*\left(Y_2\right)\right) - g_{M_1}\left(A_{Y_1}\omega Z_2, BY_2\right) - g_{M_2}\left(\nabla_{Y_1}F_*\left(\omega Z_2\right), F_*\left(CY_2\right)\right).$$

Theorem 6. Let F be a PBSRM from a Kähler manifold $\left(M_1, g_{M_1}, J\right)$ to a Riemannian manifold $\left(M_2, g_{M_2}\right)$ with the slant functions θ_1, θ_2. Then, the distribution D_1 defines a totally geodesic foliation on M_1 if and only if

$$g_{M_1}\left(\mathcal{T}_{X_1}\omega\phi X_2, V_1\right) = g_{M_1}\left(\mathcal{T}_{X_1}\omega X_2, \phi V_1\right) + g_{M_1}\left(\mathcal{H}\nabla_{X_1}\omega X_2, \omega V_1\right),$$

$$\sin^2\theta_1 g_{M_1}\left(\left[X_1, U_1\right], X_2\right) - \sin 2\theta_1 U_1\left[\theta_1\right] g_{M_1}\left(X_1, X_2\right)$$
$$= g_{M_1}\left(\mathcal{A}_{U_1}\omega\phi X_1, X_2\right) - g_{M_1}\left(\mathcal{A}_{U_1}\omega X_1, \phi X_2\right) - g_{M_1}\left(\mathcal{H}\nabla_{U_1}\omega X_1, \omega X_2\right),$$

for $X_1, X_2 \in \Gamma\left(D_1\right), V_1 \in \Gamma\left(D_2\right)$ and $U_1 \in \Gamma\left(\ker F_*\right)^{\perp}$.

Proof. For $X_1, X_2 \in \Gamma\left(D_1\right), V_1 \in \Gamma\left(D_2\right)$ and $U_1 \in \Gamma\left(\ker F_*\right)^{\perp}$, using equations (2.2), (2.3), (2.7) and (3.3), we have

$$g_{M_1}\left(\nabla_{X_1}X_2, V_1\right)$$
$$= g_{M_1}\left(\nabla_{X_1}JX_2, JV_1\right),$$
$$= -g_{M_1}\left(\nabla_{X_1}\phi^2 X_2, V_1\right) - g_{M_1}\left(\nabla_{X_1}\omega\phi X_2, V_1\right) + g_{M_1}\left(\nabla_{X_1}\omega X_2, JV_1\right),$$
$$= \cos^2\theta_1 g_{M_1}\left(\nabla_{X_1}X_2, V_1\right) - g_{M_1}\left(\mathcal{T}_{X_1}\omega\phi X_2, V_1\right) + g_{M_1}\left(\mathcal{T}_{X_1}\omega X_2, \phi V_1\right) + g_{M_1}\left(\mathcal{H}\nabla_{X_1}\omega X_2, \omega V_1\right).$$

Now, we get

$$\sin^2\theta_1 g_{M_1}\left(\nabla_{X_1}X_2, V_1\right)$$
$$= \quad -g_{M_1}\left(\mathcal{T}_{X_1}\omega\phi X_2, V_1\right) + g_{M_1}\left(\mathcal{T}_{X_1}\omega X_2, \phi V_1\right) + g_{M_1}\left(\mathcal{H}\nabla_{X_1}\omega X_2, \omega V_1\right).$$

Next, using equations (2.2), (2.3), (3.3) and Lemma 3.3, we have

$$g_{M_1}\left(\nabla_{X_1}X_2, U_1\right)$$
$$= -g_{M_1}\left(\left[X_1, U_1\right], X_2\right) - g_{M_1}\left(\nabla_{U_1}X_1, X_2\right),$$
$$= -g_{M_1}\left(\left[X_1, U_1\right], X_2\right) + g_{M_1}\left(\nabla_{U_1}\phi^2 X_1, X_2\right) + g_{M_1}\left(\nabla_{U_1}\omega\phi X_1, X_2\right) - g_{M_1}\left(\nabla_{U_1}\omega X_1, JX_2\right),$$
$$= -\sin^2\theta_1 g_{M_1}\left(\left[X_1, U_1\right], X_2\right) + \sin 2\theta_1 U_1\left[\theta_1\right] g_{M_1}\left(X_1, X_2\right) +$$
$$\cos^2\theta_1 g_{M_1}\left(\nabla_{X_1}X_2, U_1\right) + g_{M_1}\left(\nabla_{U_1}\omega\phi X_1, X_2\right) - g_{M_1}\left(\nabla_{U_1}\omega X_1, JX_2\right)$$

Then form the equation (2.9), we get

$$\sin^2\theta_1 g_{M_1}\left(\nabla_{X_1}X_2, U_1\right)$$
$$= \quad -\sin^2\theta_1 g_{M_1}\left(\left[X_1, U_1\right], X_2\right) + \sin 2\theta_1 U_1\left[\theta_1\right] g_{M_1}\left(X_1, X_2\right) +$$
$$g_{M_1}\left(\mathcal{A}_{U_1}\omega\phi X_1, X_2\right) - g_{M_1}\left(\mathcal{A}_{U_1}\omega X_1, \phi X_2\right) - g_{M_1}\left(\mathcal{H}\nabla_{U_1}\omega X_1, \omega X_2\right),$$

which completes the proof.

In a similar way in as above, we obtain the following Theorem.

Theorem 7. Let F be a PBSRM from a Kähler manifold $\left(M_1, g_{M_1}, J\right)$ to a Riemannian manifold $\left(M_2, g_{M_2}\right)$ with the slant functions θ_1, θ_2. Then, the distribution D_2 defines a totally geodesic foliation on M_1 if and only if

$$g_{M_1}\left(\mathcal{T}_{Y_1}\omega\phi Y_2, V_1\right) = g_{M_1}\left(\mathcal{T}_{Y_1}\omega Y_2, \phi V_1\right) + g_{M_1}\left(\mathcal{H}\nabla_{Y_1}\omega Y_2, \omega V_1\right),$$

$$\sin^2\theta_1 g_{M_1}\left(\left[Y_1, U_1\right], Y_2\right) - \sin 2\theta_1 U_1\left[\theta_1\right] g_{M_1}\left(Y_1, Y_2\right)$$
$$= g_{M_1}\left(\mathcal{A}_{U_1}\omega\phi Y_1, Y_2\right) - g_{M_1}\left(\mathcal{A}_{U_1}\omega Y_1, \phi Y_2\right) - g_{M_1}\left(\mathcal{H}\nabla_{U_1}\omega Y_1, \omega Y_2\right),$$

for $Y_1, Y_2 \in \Gamma\left(D_2\right), V_1 \in \Gamma\left(D_1\right)$ and $U_1 \in \Gamma\left(\ker F_*\right)^{\perp}$.

Theorem 8. Let F be a PBSRM from a Kähler manifold $\left(M_1, g_{M_1}, J\right)$ to a Riemannian manifold $\left(M_2, g_{M_2}\right)$. Then, F is a totally geodesic map if and only if

$$\cos^2\theta_1\mathcal{T}_{Y_1}PY_2 + \cos^2\theta_2\mathcal{T}_{Y_1}QY_2$$
$$= \mathcal{H}\nabla_{Y_1}\omega\phi PY_2 + \mathcal{H}\nabla_{Y_1}\omega\phi PY_2 + \omega\mathcal{T}_{Y_1}\omega Y_2 + C\mathcal{H}\nabla_{Y_1}\omega Y_2,$$

$$\cos^2\theta_1\mathcal{A}_{X_1}PY_1 + \cos^2\theta_2\mathcal{A}_{X_1}QY_1$$
$$= \mathcal{H}\nabla_{X_1}\omega\phi PY_1 + \mathcal{H}\nabla_{X_1}\omega\phi QY_1 + \omega\mathcal{A}_{X_1}\omega Y_1 + C\mathcal{H}\nabla_{X_1}\omega Y_1,$$

for $Y_1, Y_2 \in \Gamma\left(\ker F_*\right)$ and $X_1, X_2 \in \Gamma\left(\ker F_*\right)^{\perp}$.

Proof. Since F is a PBSRM, we have

$$\left(\nabla F_*\right)\left(X_1, X_2\right) = 0,$$

for $X_1, X_2 \in \Gamma\left(\ker F_*\right)^{\perp}$.

For $X_1, X_2 \in \Gamma\left(\ker F_*\right)^{\perp}$, using equations (2.1), (2.3), (2.6), (2.7), (3.2), (3.3), (3.4) and Lemma 3.3, we have

$$\left(\nabla F_*\right)\left(Y_1, Y_2\right)$$
$$= F_*\left(J\nabla_{Y_1}JY_2\right),$$
$$= F_*\left(J\nabla_{Y_1}\phi PY_2 + J\nabla_{Y_1}\phi QY_2 + J\nabla_{Y_1}\omega Y_2\right),$$
$$= F_*\Big(-\cos^2\theta_1\left(\mathcal{T}_{Y_1}PY_2 + \mathcal{V}\nabla_{Y_1}PY_2\right) + \sin 2\theta_1 Y_1\left[\theta_1\right]PY_2 - \cos^2\theta_2\left(\mathcal{T}_{Y_1}QY_2 + \mathcal{V}\nabla_{Y_1}QY_2\right) + \sin 2\theta_2 Y_1\left[\theta_2\right]PY_2$$
$$\mathcal{T}_{Y_1}\omega\phi PY_2 + \mathcal{H}\nabla_{Y_1}\omega\phi PY_2 + \mathcal{T}_{Y_1}\omega\phi PY_2 + \mathcal{H}\nabla_{Y_1}\omega\phi PY_2 + \phi\omega\mathcal{T}_{Y_1}\omega Y_2 + \omega\mathcal{T}_{Y_1}\omega Y_2 + B\mathcal{H}\nabla_{Y_1}\omega Y_2 + C\mathcal{H}\nabla_{Y_1}\omega Y_2\Big)$$

Next, using equations $(2.1),(2.3),(2.6),(2.7),(3.2),(3.3),(3.4)$ and Lemma 3.3, we have

$$\left(\nabla F_*\right)\left(X_1, Y_1\right),$$
$$= -F_*\left(\nabla_{X_1}Y_1\right)$$
$$= F_*\left(J\nabla_{X_1}\phi PY_1 + J\nabla_{X_1}\phi QY_1 + J\nabla_{X_1}\omega Y_1\right),$$

$$= F_*\Big(-\cos^2\theta_1\left(\mathcal{A}_{X_1}PY_1 + \mathcal{V}\nabla_{X_1}PY_1\right) + \sin 2\theta_1 X_1[\theta_1]PY_1 -$$

$$\cos^2\theta_2\left(\mathcal{A}_{X_1}QY_1 + \mathcal{V}\nabla_{X_1}QY_1\right) + \sin 2\theta_2 X_1[\theta_2]QY_1 +$$

$$\mathcal{A}_{X_1}\omega\phi PY_1 + \mathcal{H}\nabla_{X_1}\omega\phi PY_1 + \mathcal{A}_{X_1}\omega\phi QY_1 + \mathcal{H}\nabla_{X_1}\omega\phi QY_1 +$$

$$\phi\mathcal{A}_{X_1}\omega Y_1 + \omega\mathcal{A}_{X_1}\omega Y_1 + B\mathcal{H}\nabla_{X_1}\omega Y_1 + C\mathcal{H}\nabla_{X_1}\omega Y_1\Big)$$

Example

Note that given an Euclidean space R^{2s} with coordinates $(x_1, x_2, \ldots\ldots, x_{2s-1}, x_{2s})$ we can canonically choose an almost complex structure J on R^{2s} as follows:

$$J\left(b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + \ldots\ldots\ldots\ldots + b_{2s-1}\frac{\partial}{\partial x_{2s-1}} + b_{2n}\frac{\partial}{\partial x_{2s}}\right)$$

$$= \quad -b_2\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial x_2} + \ldots\ldots\ldots\ldots - b_{2n}\frac{\partial}{\partial x_{2s-1}} + b_{2s-1}\frac{\partial}{\partial x_{2s}},$$

where $b_1, b_2, \ldots\ldots\ldots\ldots\ldots, b_{2s}$ are C^∞ functions defined on R^{2s}. Throughout this section, we will use this notation.

Example 1. Let $\left(R^8, g_{R^8}, J\right)$ be a Kähler manifold with usual metric g_{R^8} and $\left(R^4, g_{R^4}\right)$ be a Riemannian manifold with Riemannian metric

$$\begin{bmatrix} \frac{1}{\sin^2 x_1 + \cos^2 x_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\cos^2 x_6 + \sin^2 x_8} \end{bmatrix}$$

where $\sin^2 x_1 + \cos^2 x_3 \neq 0, \cos^2 x_6 + \sin^2 x_8 \neq 0$. Define a map $F : R^8 \to R^4$ by

$$F(x_1, x_1, \ldots\ldots\ldots, x_8) = (\cos x_1 + \sin x_3, x_4, x_5, \sin x_6 + \cos x_8),$$

which is a pointwise bi-slant Riemannian map such that

$$X_1 = \quad \cos x_3\frac{\partial}{\partial x_1} + \sin x_1\frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2}$$

$$X_3 = \quad \frac{\partial}{\partial x_7}, X_4 = \sin x_8\frac{\partial}{\partial x_6} + \cos x_6\frac{\partial}{\partial x_8}$$

$$\ker F_* = D_1 \oplus D_2$$

where

$$D_1 = \quad \left\langle X_1 = \cos x_3\frac{\partial}{\partial x_1} + \sin x_1\frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2}\right\rangle$$

$$D_1 = \quad \left\langle X_3 = \frac{\partial}{\partial x_7}, X_4 = \sin x_8\frac{\partial}{\partial x_6} + \cos x_6\frac{\partial}{\partial x_8}\right\rangle$$

$$(\ker F_*)^\perp$$

$$= \quad < H_1 = \sin x_1 \frac{\partial}{\partial x_1} - \cos x_3 \frac{\partial}{\partial x_3}, H_2 = \frac{\partial}{\partial x_4}$$

$$H_3 = \quad \frac{\partial}{\partial x_5}, H_4 = \cos x_6 \frac{\partial}{\partial x_6} - \sin x_8 \frac{\partial}{\partial x_8} >$$

with bi-slant functions x_3 and x_6.

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