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RESEARCH ARTICLE

Geometry of contact and complex Structures with statistical manifolds

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Abstract
This paper aims to introduce the concept of a contact structure with statistical manifolds and to explore the properties of lightlike hypersurfaces within such manifolds. It also examines the relationships among induced geometric objects with respect to dual connections. Finally, the paper demonstrates that an invariant lightlike submanifold of an indefinite quasi-Sasakian statis- tical manifold retains the structure of an indefinite quasi-Sasakian statistical manifold.

Keywords: Statistical manifold, Lightlike submanifolds, Affine connections.

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1. Introduction

A statistical manifold is an extension of a Riemannian manifold. It is utilized for modeling information, studying statistical inference, information loss, and estimation through differential geometry. These manifolds find applications in neural networks, machine learning, and artificial intelligence.

An important area in statistical studies is information geometry. From this perspective, statistical manifolds are defined as follows:

Vos [16] derived the fundamental equations of submanifolds of statistical manifolds, Aydin[2] obtained generalized Weingarten inequalities for submanifolds of constant curvature statistical manifolds, Furuhata[13] [14] studied hypersurfaces of statistical manifolds, Balgeshir[3] studied submanifolds of Sasakian statistical manifolds, and Bahadir[4] explored the lightlike geometry of indefinite Sasakian sta- tistical manifolds. Motivated by these studies, we define indefinite quasi-Sasakian statistical manifold and investigate their lightlike geometry. The paper is organized as follows: Section 2 defines statistical manifolds from a differential geometry viewpoint and introduces an indefinite quasi-Sasakian statistical manifolds, providing some preliminary results and a characterization theorem. Section 3 examines warped product manifolds of indefinite quasi-Sasakian statistical manifolds. In section 4, we define golden statistical manifolds to prove the condition for the integrability.

2. Definition and preliminaries

We follow [7] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold M^m immersed in a seami-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\bar{g}$ and the radical distribution $\mathrm{Rad}TM$ is of rank r , where $1 \le r \le m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}TM$ in TM , that is

$$TM = \mathrm{Rad}TM \perp S(TM)$$

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}TM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $\mathrm{Rad}TM$, there exists a local null frame $\{N_i\}$ of section with values in the orthogonal complement of $S\left(TM^{\perp}\right)$ in $[S(TM)]^{\perp}$ such that $\bar{g}\left(\xi_i, N_j\right)=\delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $\operatorname{ltr}(TM)$ locally spanned by $\{N_i\}$[[7], pg144]. Let $\operatorname{tr}(TM)$ be a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then

$$\operatorname{tr}(TM)=\operatorname{ltr}(TM)\perp S\left(TM^{\perp}\right)$$
$$T\bar{M}\big|_M=S(TM)\perp\left[\operatorname{Rad}(TM)\oplus\operatorname{ltr}(TM)\right]\perp S\left(TM^{\perp}\right)$$

The following are four the subcases of a lightlike submanifold ($M, g, S(TM), S\left(TM^{\perp}\right)$).

Case 1: r-lightlike if $r<\min\{m,n\}$.

Case 2:Co-isotropic if $r=n<m; S\left(TM^{\perp}\right)=0$.

Case 3: Isotropic if $r=m<n; S(TM)=0$.

Case 4: Totally lightlike if $r=m=n; S(TM)=0=S\left(TM^{\perp}\right)$.

The Gauss and Weingarten equations are

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y), \forall X,Y\in\Gamma(TM) \# (2.1)$$
$$\bar{\nabla}_X U=-A_U X+\nabla_X^t U, U\in\Gamma(tr(TM)) \# (2.2)$$

where $\{\nabla_X Y, A_U X\}$ and $\left\{h(X,Y), \nabla_X^t U\right\}$ belongs to $\tilde{\mathrm{A}}(TM)$ and $\tilde{\mathrm{A}}(\operatorname{tr}(TM))$, respectively, ∇ and ∇^t are linear connections on M and on the vector bundle $\operatorname{tr}(TM)$, respectively. Moreover, we have

$$\bar{\nabla}_X Y=\nabla_X Y+h^l(X,Y)+h^s(X,Y) \# (2.3)$$
$$\bar{\nabla}_X N=-A_N X+\nabla_X^l N+D^s(X,N) \# (2.4)$$
$$\bar{\nabla}_X W=-A_W X+\nabla_X^s W+D^l(X,W) \# (2.5)$$

$\forall X,Y\in\Gamma(TM)$ and $N\in\tilde{\mathrm{A}}(l\operatorname{tr}(TM))$ and $W\in\tilde{\mathrm{A}}\left(S\left(TM^{\perp}\right)\right)$. Then, by using (2.1), (2.3)-(2.5) and the fact that $\bar{\nabla}$ is a metric connection, we get

$$\bar{g}\left(h^s(X,Y),W\right)+\bar{g}\left(Y,D^l(X,W)\right)=g\left(A_W X,Y\right) \# (2.6)$$

[12] In general, the induced connection ∇ on M is not a metric connection, by using (2.3), we have

$$\left(\nabla_X g\right)(Y,Z)=\bar{g}\left(h^l(X,Y),Z\right)+\bar{g}\left(h^l(X,Z),Y\right) \# (2.7)$$

for any $X,Y\in\Gamma(TM)$, where $\{\nabla_X Y\tilde{A} A_N X, A_W X\}\in, (TM)$ $\left\{h^l(X\ Y)\ \nabla_X^l N\right\}\in$ $\tilde{\mathrm{A}}(l\operatorname{tr}(TM))$ and $\{\nabla_X Y, A_N X, A_W X\}\in\Gamma(TM), \left\{h^l(X,Y), \nabla_X^l N\right\}\in$. If we set $B^l(X,Y)=$

$\bar{g}\left(h^l(X,Y),\xi\right)$, $B^s(X,Y)=\bar{g}\left(h^s(X,Y),\xi\right)$, $\tau^l(X)=\bar{g}\left(\nabla_X^l N,\xi\right)$ and $\tau^s(X)=$ $\bar{g}\left(\nabla_X^s N,\xi\right)$. Then equation (2.3), (2.4) and (2.5) become

$$\overline{\nabla}_X Y = \nabla_X Y + B^l(X,Y)N + B^s(X,Y)N \#(2.8)$$
$$\overline{\nabla}_X N = -A_N X + \tau^l(X)N + E^s(X,N)\#(2.9)$$
$$\overline{\nabla}_X W = -A_W X + \tau^s(X)W + E^l(X,W)\#(2.10)$$

respectively. Here, B and A are called second fundamental form and shape operator of the lightlike submanifold M . On the other hand, if we take the vector field $\xi \in \tilde{A}(\mathrm{Rad}TM)$ and $X \in \tilde{A}(TM)$, we have the following relation like Weingarten formula

$$D_X \xi = -A_\xi^* X + \nabla_X \xi \#(2.11)$$
$$D_X^* \xi = -A_\xi X + \nabla_X^* \xi \#(2.12)$$

where $\{D_X \xi, D_X^* \xi\}$. and $\{A_\xi X, A_\xi^* X\}$ are the shape operators on $\tilde{A}(S(TM))$ and linear connections on $\tilde{A}(\mathrm{Rad}(TM))$, respectively [4].

Now, we define some statistical basic concepts.

Definition 2.1 [11] Let $\tilde{M}$ be a smooth manifold. Let $\tilde{D}$ be an affine connection with the torsion tensor $T^{\tilde{D}}$ and $\tilde{g}$ a semi-Riemannian metric on $\tilde{M}$. Then the pair ($\tilde{D}, \tilde{g}$) is called statistical structure on $\tilde{M}$ if

(1) $(\tilde{D}_X \tilde{g})(Y,Z) - (\tilde{D}_Y \tilde{g})(X,Z) = \tilde{g}(T^{\tilde{D}}(X,Y),Z)$, for all $X,Y,Z \in \Gamma(T\tilde{M})$ and

(2) $T^{\tilde{D}} = 0$.

Definition 2.2 [11] Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold. Two affine connections $\tilde{D}$ and $\tilde{D}^*$ on $\tilde{M}$ are said to be dual with respect to the metric $\tilde{g}$, if

$$Z\tilde{g}(X,Y) = \tilde{g}(\tilde{D}_Z X, Y) + \tilde{g}(X, \tilde{D}_Z^* Y)\#(2.13)$$

for all $X,Y,Z \in \Gamma(T\tilde{M})$.

A statistical manifold will be represented by $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$. If $\tilde{\nabla}$ is Levi-Civita connection of $\tilde{g}$, then

$$\tilde{\nabla} = \frac{1}{2}(\tilde{D} + \tilde{D}^*).\#(2.14)$$

In (2.13), if we choose $\tilde{D}^* = \tilde{D}$ then Levi-Civita connection is obtained.

Theorem 2.3 [14] For statistical manifold ($\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*$), we set $\mathbb{K} = \tilde{D} - \tilde{\nabla}$. Then we have

$$\overline{\mathbb{K}}(X,Y) = \overline{\mathbb{K}}(Y,X), \tilde{g}(\overline{\mathbb{K}}((X,Y),Z) = \tilde{g}(\overline{\mathbb{K}}((X,Z),Y)\#(2.15)$$

for any $X,Y,Z \in \Gamma(TM)$. Conversely, for a Riemannian metric g , if $\overline{\mathbb{K}}$ satisfies (2.15) the pair $(\tilde{D} = \tilde{\nabla} + \tilde{\mathbb{K}}, \tilde{g})$ is statistical structure on .

Let (M,g) be a submanifold of $(\tilde{M}, \tilde{g})$. If (M,g,D,D^*) is statistical manifold, then (M,g,D,D^*) is called a statistical submanifold of $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$, where D, D^* are affine dual connection on M and $\tilde{D}, \tilde{D}^*$ are affine dual connections on $\tilde{M}$ ([1], [16], [11]) Let (M,g) be a lightlike submanifold of a statistical manifold $(\tilde{M},\tilde{g},\tilde{D},\tilde{D}^*$) then Gauss and Wiengarten formulas wrt to dual connections are given by

$$\tilde{D}_X Y = D_X Y + B^l(X,Y)N + B^s(X,Y)N, \#(2.16)$$
$$\tilde{D}_X N = -A_N X + \tau^l(X)N + E^s(X,N), \#(2.17)$$
$$\tilde{D}_X W = -A_W X + \tau^s(X)W + E^l(X,W), \#(2.18)$$
$$\tilde{D}^*{}_X Y = D^*_X Y + B^{l^*}(X,Y)N + B^{s^*}(X,Y)N, \#(2.19)$$
$$\tilde{D}^*_X N = -A_N X + \tau^l(X)N + E^s(X,N), \#(2.20)$$
$$\tilde{D}^*_X W = -A^*_W X + \tau^{s^*}(X)W + E^{l^*}(X,W) \#(2.21)$$

for all $X,Y \in \Gamma(TM), N \in \Gamma(ltrTM)$ and $W \in \tilde{A}(S(TM^{\perp}))$, Here, $D, D^*, B, B^{l^*}, B^s, B^{s^*}, A_N$, and A^*_N are called the induced connections on M, the second fundamental forms and the Weingarten mappings with respect to $\tilde{D}$ and $\tilde{D}^*$, respectively.

Using Gauss formulas and the equation (2.17), we obtain

$$Xg(Y,Z) = g(\tilde{D}_X Y, Z) + g(Y, \tilde{D}^*_X Z) = g(D_X Y, Z) + g(Y, D^*_X Z) +$$
$$B(X,Y)\eta(Z) + B^*(X,Z)\eta(Y) \#(2.2)$$

From the equation (2.22), we have the follwing result.

[4] A differentiable semi-Riemannian manifold $(\tilde{M},\tilde{g})$ of dimension $n = 2m+1$, a $(1,1)$ tensor field $\tilde{\phi}$, a contravariant vector field v, a 1-form η and a Riemannian metric $\tilde{g}$ should be admitted, which satisfy

$$\tilde{\phi}v = 0, \eta(\tilde{\phi}X) = 0, \eta(v) = \in, \#(2.23)$$
$$\tilde{\phi}^2(X) = -X + \eta(X)v, \tilde{g}(X,v) = \in \eta(X), \#(2.24)$$
$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \tilde{g}(X,Y) - \int\eta(X)\eta(Y), \in = \mp 1 \#(2.25)$$

for all the vector fields X,Y on $\tilde{M}$. When a practically contact metric manifold performs

$$(\tilde{\nabla}_X \tilde{\phi})Y = -\grave{o}\eta(Y)AX - \tilde{g}(AX,Y)v, \#(2.26)$$
$$\tilde{\nabla}_X v = A\tilde{\phi}X \#(2.27)$$

$\tilde{M}$ is regarded as an indefinite quasi sasakian manifold [15]. In this study, we assume that the vector field v is spacelike.

Definition 2.4 Let $(\tilde{g},\tilde{\phi},v)$ be a contact structure on $\tilde{M}$. A quadruplet ($\tilde{D} = \tilde{\nabla} + \mathbb{K}, \tilde{g}, \tilde{\phi}, v$) is called a contact statistical structure on $\tilde{M}$ if $(\tilde{D},\tilde{g})$ is a statistical structure on $\tilde{M}$ and the formula

$$\overline{\mathbb{K}}(X,\tilde{\phi}Y) = -\tilde{\phi}\overline{\mathbb{K}}(X,Y) \#(2.28)$$

holds for any $X,Y \in \Gamma(T\tilde{M})$. Then $(\tilde{M},\tilde{D},\tilde{g},\tilde{\phi},v)$ is said to a contact statistical manifold.

Theorem 2.5 Let $(\tilde{M},\tilde{D},\tilde{g}\ ,)$ be a statistical manifold and ($\tilde{g},\tilde{\phi},v)$ an almost contact metric structure on $\tilde{M}$. $\left(\tilde{D},\tilde{g},\tilde{\phi},v\right)$ is an indefinite quasi-Sasakian statistical structure if and only if the following conditions hold:

$$\tilde{D}_X\tilde{\phi}Y-\tilde{\phi}\tilde{D}_X^*Y=\tilde{g}\left(Y,v\right)AX-\tilde{g}\left(AX,Y\right)v,\#\left(2.29\right)$$

$$\tilde{D}_Xv=\tilde{\phi}\left(AX\right)+\tilde{g}\left(\tilde{D}_Xv,v\right)v\#\left(2.30\right)$$

for all the vector field X,Y on $\tilde{M}$.

3. Warped product submanifolds of Contact statistical manifolds

Definition 3.1 For a warped product submanifolds of a statistical manifolds $N_1\times_f\ \ N_2=\left(N_1\times N_2,g=g_1+f^2g_2\right)$ we recall the following well known identity [6].

$$D_XY=D_YX=\left(XInvf\right)Y\#\left(3.1\right)$$

$$D_X^*Y=D_Y^*X=\left(XInv\right)Y\#\left(3.2\right)$$

for any $X\in\tilde{A}\left(TN_1\right),Y\in\tilde{A}\left(TN_2\right)$.

Therem 3.1 If $N=N_1\times_fN_2$ is a warped product submanifold of a quasi-sasakian statistical manifold $\tilde{N}$ such that ξ is tangent to N_2 then it becomes statistical product manifold

Proof. Suppose $\xi\in TN_2$, putting $Y=\xi$ in equation (3.1), we obtain

$$D_X\xi=\left(XInvf\right)\xi\#\left(3.3\right)$$

$$D_X^*\xi=\left(XInvf\right)\xi\#\left(3.4\right)$$

Now, From (3.3)

$$\phi FX=\tilde{D}_X\xi=D_X\xi+B^l\left(X,\xi\right)+B^s\left(X,\xi\right)=\left(XInvf\right)\xi\#\left(3.5\right)$$

Since $g\left(\phi FX,\xi\right)=-g\left(FX,\phi\xi\right)=0$ and $g\left(B^l\left(X,\xi\right),\xi\right)=g\left(B^s\left(X,\xi\right),\xi\right)=0$

Again, From (3.4)

$$\phi FX=\widetilde{D^*}_X\xi=D_X^*\xi+B^{l^*}\left(X,\xi\right)+B^{s^*}\left(X,\xi\right)=\left(XInvf\right)\xi.\#\left(3.6\right)$$

Since $g\left(\phi FX,\xi\right)=-g\left(FX,\phi\xi\right)=0$ and $g\left(B^{l^*}\left(X,\xi\right),\xi\right)=g\left(B^{s^*}\left(X,\xi\right),\xi\right)=0$ From (3.5) and (3.6), we obtain XinvF $=0\forall X\in\Gamma\left(TN\right)$.

Hence F is a constant and the warped product is nothing but a simply statistical product.

4. Screen semi invariant lightlike submanifold of an Indefinite golden statistical manifold

Definition 4.1 A polynomial structure on manifold M is called an indefinite Golden structure if it is determined by tensor field ϕ of type $\left(1,1\right)$, which satisfies

$$\tilde{\phi}^2=\tilde{\phi}+I\#\left(4.1\right)$$

where I is the identity map. Also If

$$\tilde{g}\left(\tilde{\phi}W,U\right)=\tilde{g}\left(W\tilde{\phi}U\right)\#(4.2)$$

holds, then the semi-Riemannian metric $\tilde{g}$ is called $\tilde{\phi}$ -compatible, for every $U,W\in$ Ã$\left(T\tilde{M}\right)$. In this case $\left(\tilde{M},\tilde{g},\tilde{\phi}\right)$ is named an indefinite Golden semi-Riemannian manifolds. Also an indefinite Golden semi-Riemannian structure $\tilde{\phi}$ is called a locally Golden structure if $\tilde{\phi}$ is parallel with respect to two affine connections $\tilde{D}$ and $\tilde{D}^{*}$, that is

$$\tilde{D}_{W}^{*}\tilde{\phi}U=\tilde{\phi}\tilde{D}_{W}^{*}U\,\#(4.3)$$

and

$$\tilde{D}_{W}\tilde{\phi}U=\tilde{\phi}\tilde{D}_{W}U\,\#(4.4)$$

If $\tilde{\phi}$ be a Golden structure, then (4.2) is equivalent to

$$\tilde{g}\left(\tilde{\phi}W,\tilde{\phi}U\right)=\tilde{g}\left(\tilde{\phi}W,U\right)+\tilde{g}\left(W,U\right)\#(4.5)$$

for any $W,U\in\Gamma\left(T\tilde{M}\right)$.

Definition 4.2 Let $\left(M,g\right)$ be a lightlike submanifold of Golden statistical manifold $\left(\tilde{M},\tilde{g}\right)$, then if the following conditions are satisfied then we can say that M is a screen semi-invariant lightlike submanifold

$$\phi\left(Rad\left(TM\right)\right)\subseteq S\left(TM\right)\#(4.6)$$
$$\phi\left(ltr\left(TM\right)\right)\subseteq S\left(TM\right)\#(4.7)$$

Theorem 4.1 Let N be a screen semi-invariant lightlike submanifold of Golden statistical manifold $\left(\tilde{M},\tilde{g}\right)$, then invariant distribution M is integrable for any $X,Y\in\Gamma\left(M\right)$ iff

$$B^{l}\left(\phi X,\phi Y\right)=B^{l}\left(X,\phi Y\right)+B^{l}\left(X,Y\right)\#(4.8)$$
$$B^{l^{*}}\left(\phi X,\phi Y\right)=B^{l^{*}}\left(X,\phi Y\right)+B^{l^{*}}\left(X,Y\right).\#(4.9)$$

Proof. We know that M is integrable iff $\left[X,Y\right]\in\Gamma\left(M\right)\forall X,Y\in\Gamma\left(M\right)$.

That is $\tilde{g}([\phi Y,X],\phi\xi=0\forall X,Y\in\Gamma\left(M\right)\xi\in\Gamma\left(M\right)$. By using (4.2), (4.3) and (2.11)

$$\tilde{g}\left(\tilde{D}_{\phi Y}\phi X,\xi\right)-\tilde{g}\left(\tilde{D}_{X}Y,\phi\xi\right)-\tilde{g}\left(\tilde{D}_{X}Y,\xi\right)=0$$

Finally using equation (2.16).

$$\tilde{g}\left(B^{l}\left(\phi Y,\phi X\right)N,\xi\right)-\tilde{g}\left(B^{l}\left(X,\phi Y\right)N,\xi\right)-\tilde{g}\left(B^{l}\left(X,Y\right)N=0\right.$$
$$B^{l}\left(\phi Y,\phi X\right)-B^{l}\left(X,\phi Y\right)-B^{l}\left(X,Y\right)=0$$
$$B^{l}\left(\phi Y,\phi X\right)=B^{l}\left(X,\phi Y\right)+B^{l}\left(X,Y\right)$$

Again using (4.2), (4.4) and (2.12)

$$\tilde{g}\left(\widetilde{D^{*}}_{\phi Y}\phi X,\xi\right)-\tilde{g}\left(\widetilde{D_{X}^{*}}Y,\phi\xi\right)-\tilde{g}\left(\widetilde{D^{*}}_{X}Y,\xi\right)=0.$$

Finally using equation (2.19).

$$\tilde{g}\left(B^{l^*}(\phi Y,\phi X)N,\xi\right)-\tilde{g}\left(B^{l^*}(X,\phi Y)N,\xi\right)-\tilde{g}\left(B^{l^*}(X,Y)N=0,\right.$$

$$B^{l^*}(\phi Y,\phi X)-B^{l^*}(X,\phi Y)-B^{l^*}(X,Y)=0$$

$$B^{l^*}(\phi Y,\phi X)=B^{l^*}(X,\phi Y)+B^{l^*}(X,Y).$$

References

[1] Amari, S.: Differential Geometry Methods in statistics, in: Lecture Notes in Statistics, Springer, New York (28) (1985).

[2] Aydin, M.E., Mihai, A., Mihai, I.: Some inequalities on submanifolds in statis- tical manifolds of constant curvature, Filomat, 465-476 (29) (2015).

[3] Balgeshir, M.: On submanifolds of Sasakian statistical manifolds, doi:10.5269/bspm.42402 (2018).

[4] Bahadir, O.: On lightlike geometry of indefinite Sasakian statistical manifold AIMS Mathematics, 12845-12862 (11) (2021).

[5] Bahadir, O., Tripathi, M. M.: Geometry of lightlike hypersurfaces of a statis- tical manifold, http://arxiv.org/abs/1901.09251 (2019).

[6] Bishop, R.L., O. Neill, B.: Manifolds of negative cuvature, Transl. Math. Monogr, Amer. Math. Soc. 1-49, (1969).

[7] Duggal, K. L., Bejancu, A.: Lightlike submanifolds of semi-Riemannian man- ifolds and applications. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, (1996).

[8] Duggal, K. L., Jin, D. H.: Null curves and hypersurfaces of semi-Riemannian manifolds. World Scientific Publishing Co. Pte. Ltd., Hackensack, N J, (2007).

[9] Duggal, K. L., Sahin, B.: Differential Geometry of lightlike submanifolds. Fron- tiers in Mathematics. Birkhauser Verlag, Basel, (2010).

[10] Effron, B.: Defining the curvature of a statistical problem (with applications to second order efficiency). With a discussion by Rao, C. R., Don A. Pierce, Cox, D. R., Lindley, D. V., Lucien LeCam., Ghosh, J. K., Pfanzagl, J., Niels Keiding, Dawid, A. P., Jim Reeds and with a reply by the author. Ann. Statist, 1189-1242 (6) (1975).

[11] H. Furuhuta, Hypersurfaces in statistical manifolds. Differential Geom. Appl. 27, no. 3, 420-429, (2009).

[12] M. Balgeshir, On submanifolds of Sasakian statistical manifolds doi:10.5269/bspm.42402, (2018)

[13] Furuhata, H., Hasegawa, I.: Submanifold theory in holomorphic statistical manifolds. Geometry of Cauchy-Riemann submanifolds, Springer, Singapore, 179-215 (2016).

[14] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K., Shahid, M. H.: Sasakian statistical manifolds, J. Geom. Phys. 179-186 (117) (2017).

[15] Kanemaki, S.: Quasi-Sasakian manifolds, Tohoku Math. Journ. 227-233 (29) (1977).

[16] Vos, P.W.: Fundamental equations for statistical submanifolds with applications to the Bartlett correction. Ann. Inst. Statist. Math. 429-450 (41) (1989).