

Riemannian manifold admitting a new type of semi-symmetric non-metric connection

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Abstract

We define a new type of semi-symmetric non-metric connection on a Riemannian manifold and established its existence. Further, we find some basic results of curvature tensor and Ricci tensor. It is proved that such connection on a Riemannian manifold is projectively invariant under certain conditions. We also studied some properties of submanifolds of the Riemannian manifolds with respect to the semi-symmetric non-metric connection \bar{D} . To validate our findings, we construct two non-trivial examples of 3-dimensional and 5-dimensional Riemannian manifold equipped with a semi-symmetric non-metric connection \bar{D} .

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1 Introduction

Let M^n be an n -dimensional Riemannian manifold and let D denote the Levi-Civita connection corresponding to the Riemannian metric g on M^n . A linear connection \bar{D} defined on M^n is said to be symmetric if its torsion tensor \bar{T} on \bar{D} defined by

$$\bar{T}(X, Y) = \bar{D}_X Y - \bar{D}_Y X - [X, Y],$$

is zero for all X and Y on M^n ; otherwise, it is non-symmetric. In 1924, Friedmann and Schouten [9] considered a differentiable manifold and introduced the

idea of a semi-symmetric linear connection on it. A linear connection on M^n is said to be semi-symmetric if

$$\bar{T}(X, Y) = [\eta(X)Y - \eta(Y)X] - [a(X)Y - a(Y)X], \quad (1.1)$$

holds for all vector fields X, Y on M^n , where η and a are two non-zero 1- forms associated with the vector fields U and V such that

$$\eta(X) = g(X, U) \text{ and } a(X) = g(X, V). \quad (1.2)$$

In 1932, Hayden [10] gave the idea of a metric connection \bar{D} on a Riemannian manifold and later named such connection as a Hayden connection. A linear connection \bar{D} is said to be metric on M^n , if $\bar{D}g = 0$; otherwise, it is non-metric. A systematic study of the semi-symmetric metric connection \bar{D} on Riemannian manifold was initiated by Yano [23]. Various properties of such connection have been studied by Imai [11], Nakao [13], Smaranda [21], Amur and Pujar [3], Barua and Ray [5], Hit [20], De and Biswas [8] and many authors. In 1992, Agashe and Chafle [1] introduced a new class of connection, called the semi-symmetric non-metric connection on a Riemannian manifold and obtained its various geometric properties. This was further developed by Agashe and Chafle [2], Prasad [14], De and Kamilya [7], Tripathi and Kakkar [22] and several geometers. Binh, De and Sengupta [6], Prasad and Verma [15] defined new types of semi-symmetric non-metric connections on Riemannian manifold in which they generalized the Yano's [23] and Agashe and Chafle's [1] connections and studied some properties of curvature tensor, Ricci tensor and Projective curvature tensor with respect to such connections. In 2008, Prasad, Verma and De [16] introduced the most general form of the semi-symmetric metric and non-metric connections on a Riemannian manifold which includes the known semi-symmetric metric and non-metric connection. Recently, Prasad, Dubey and Yadav [4], Prasad and Haseeb [18], Prasad et al. [19] and Prasad, Kumar and Singh [17] defined and studied a new type of semi-symmetric non-metric connection on Riemannian manifolds. Motivated by the above studies, in the present paper, we define a new type of semi-symmetric non-metric connection on a Riemannian manifold and then prove its existence.

We organize our present work as follows: After an introduction in section 1, we define a new type of semi-symmetric non-metric connection on a Riemannian manifold and proved its existence in Section 2. In Section 3, we established the relation between curvature tensors of the Levi-Civita D and semi-symmetric non-metric connections \bar{D} and prove some algebraic properties of the curvature tensors and Ricci tensor of \bar{D} . The necessary and sufficient conditions for projectively invariant curvature tensors are proved in section 4. Section 5, deals with the submanifolds and its relation with Riemannian manifold. In the last Section 6, we construct two non-trivial examples of 3-

dimensional and 5-dimensional Riemannian manifold with a semi-symmetric non-metric connection and prove some results.

2 Semi-symmetric non-metric connection \bar{D}

Let (M^n, g) be a Riemannian manifold of dimension n endowed with a Levi-Civita connection D corresponding to the Riemannian metric g . A linear connection \bar{D} on (M^n, g) defined by

$$\bar{D}_X Y = D_X Y + \eta(X)Y - a(X)Y, \quad (2.1)$$

for arbitrary vector fields X and Y on M^n is said to be a semi-symmetric connection if the torsion tensor \bar{T} on M^n with respect to \bar{D} satisfies equation (1.1) and (1.2). In view of equation (2.1), the metric g holds the relation

$$(\bar{D}_X g)(X, Y) = -2\eta(X)g(X, Y) + 2a(X)g(X, Y) \neq 0, \quad (2.2)$$

for all vector fields X, Y and Z on M^n and called semi-symmetric non-metric connection.

Remark 1 If in particular $a = 0$ then our connection becomes $\bar{D}_X Y = D_X Y + \eta(X)Y$.

This is defined Melhrotra [12].

Remark 2 If we take $\eta = a$, then the connection is trivial. So, it must said $\eta \neq a$.

Now, we prove the existence of such connection on an n -dimensional Riemannian manifold in the following theorem.

Theorem 2.1 *Let (M^n, g) be an n -dimensional Riemannian manifold endowed with the Levi-Civita connection D . Then there exist a unique linear connection on M^n called a semi-symmetric non-metric connection given by (2.1) and it satisfies equation (1.1) and (2.2).*

Proof: We suppose that (M^n, g) is a Riemannian manifold of dimension n and equipped with a linear connection. Then the Levi-Civita connection D are connected by the relation

$$\bar{D}_X Y = D_X Y + H(X, Y), \quad (2.3)$$

for arbitrary vector fields X and Y on M^n , where H is a tensor of type (1,2). By definition of the torsion \bar{T} and equation (2.3), we conclude that

$$\bar{T}(X, Y) = H(X, Y) - H(Y, X), \quad (2.4)$$

which gives

$$g(\bar{T}(X, Y), Z) = g(H(X, Y), Z) - g(H(Y, X), Z). \quad (2.5)$$

From (1.1) and (2.5), we have

$$g(H(X, Y), Z) - g(H(Y, X), Z) = \eta(X)g(Y, X) - \eta(Y)g(X, Z) - a(X)g(Y, Z) + a(Y)g(X, Z). \quad (2.6)$$

In the view of equation (2.1), we conclude that

$$\begin{aligned} g(H(X, Y), Z) + g(H(X, Z), Y) &= 2\eta(X)g(Y, Z) - 2a(X)g(Y, Z), \\ \Rightarrow g(H(X, Y), Z) + g(H(X, Z), Y) &= -(\bar{D}_X g)(Y, Z), \\ \Rightarrow (\bar{D}_X g)(Y, Z) &= -H'(X, Y, Z), \end{aligned} \quad (2.7)$$

where

$$H'(X, Y, Z) = g(H(X, Y), Z) + g(H(Y, X), Z).$$

Further, we have

$$\begin{aligned} g(\bar{T}(X, Y), Z) + g(\bar{T}(Z, X), Y) + g(\bar{T}(Z, Y), X) \\ = g(H(X, Y), Z) - H'(X, Y, Z) + H'(Z, X, Y) - H'(Y, X, Z), \end{aligned} \quad (2.8)$$

where equations (2.4), (2.5) and (2.7) are used. In consequences of equations (2.2) and (2.7), the equation (2.8) assumes the form

$$\begin{aligned} 2g(H(X, Y), Z) &= g(T(X, Y), Z) + g(T'(X, Y), Z) + g(T'(Y, X), Z) \\ &= 2\eta(X)g(Y, Z) + 2\eta(Y)g(X, Z) - 2\eta(Z)g(X, Y) \\ &\quad - 2a(X)g(Y, Z) - 2a(Y)g(X, Z) + 2a(Z)g(X, Y), \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} g(\bar{T}'(X, Y), Z) = g(\bar{T}(Z, X), Y) &= \eta(Z)g(X, Y) - \\ \eta(X)g(Z, Y) - a(Z)g(X, Y) + a(X)g(Z, Y), \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} g(\bar{T}'(Y, X), Z) = g(\bar{T}(Z, Y), X) &= \eta(Z)g(Y, X) - \\ \eta(Y)g(Z, X) - a(Z)g(Y, X) + a(Y)g(Z, X), \end{aligned} \quad (2.11)$$

for all vector fields X, Y and Z on M^n . By using equation (2.10) and (2.11) in equation (2.9), we have

$$\begin{aligned} g(H(X, Y), Z) &= \eta(X)g(Y, Z) - a(X)g(Y, Z), \\ \Rightarrow H(X, Y) &= \eta(X)Y - a(X)Y. \end{aligned} \quad (2.12)$$

Thus, equations (2.3) and (2.12) gives (2.1). This proves the existence of such connection.

Theorem 2.2 *On an n -dimensional Riemannian manifold (M^n, g) endowed with a semi-symmetric non-metric connection \bar{D} , the torsion tensor \bar{T} satisfies the following algebraic properties:*

$${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, X, Z) = 0,$$

$${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, Z, X) + {}'\bar{T}(Z, X, Y) = 0.$$

Proof : We define ${}'\bar{T}(X, Y, Z) = g(\bar{T}(X, Y), Z)$ on (M^n, g) . Therefore, equation (1.1) gives

$$\begin{aligned} {}'\bar{T}(X, Y, Z) = & \eta(X)g(Y, X) - \eta(Y)g(X, Z) - \\ & a(X)g(Y, Z) + a(Y)g(X, Z), \end{aligned} \quad (2.13)$$

with the help of equation (2.13), we can easily prove the statement of Theorem 2.2.

Theorem 2.3 *If (M^n, g) is an n -dimensional Riemannian manifold equipped with a semi-symmetric non-metric connection \bar{D} , then \bar{T} is cyclically parallel if and only if 1-form η and a are closed.*

Proof: Taking the covariant derivative of (1.1) with respect to the semi-symmetric non-metric connection \bar{D} , we find that

$$\begin{aligned} (\bar{D}_X \bar{T})(Y, Z) = & [(\bar{D}_X \bar{T})\eta(Y)Z - (\bar{D}_X \bar{T})\eta(Z)Y] \\ & - [(\bar{D}_X \bar{T})a(Y)Z - (\bar{D}_X \bar{T})a(Z)Y]. \end{aligned} \quad (2.14)$$

The cyclic sum of (2.14) for vector fields X, Y and Z , we get

$$\begin{aligned} & (\bar{D}_X \bar{T})(Y, Z) + (\bar{D}_Y \bar{T})(Z, X) + (\bar{D}_Z \bar{T})(X, Y) = \\ & [(\bar{D}_X \eta)(Y) - (\bar{D}_Y \eta)(X) - (\bar{D}_X a)(Y) + (\bar{D}_Y a)]Z \\ & + [(\bar{D}_Z \eta)(X) - (\bar{D}_X \eta)(Z) - (\bar{D}_Z a)(X) + (\bar{D}_X a)(Z)]Y \\ & + [(\bar{D}_Y \eta)(Z) - (\bar{D}_Z \eta)(Y) - (\bar{D}_Y a)(Z) + (\bar{D}_Z \eta)(Y)]X. \end{aligned} \quad (2.15)$$

From equation (2.15), we can easily show that $(\bar{D}_X \bar{T})(Y, Z) + (\bar{D}_Y \bar{T})(Z, X) + (\bar{D}_Z \bar{T})(X, Y) = 0$ if and only if 1-form η and a are closed. Hence Theorem 2.3 is proved.

Proposition 2.4 *If an n -dimensional Riemannian manifold (M^n, g) admits a semi-symmetric non-metric connection \bar{D} , then the Lie derivatives along the vector fields U and V corresponding to \bar{D} is equal to Lie derivative along the vector fields U and V with respect to D if and only if $\eta(X)\eta(Y) = a(X)a(Y)$.*

Proof : It is well known that

$$\begin{aligned} (L_U g)(X, Y) &= g(D_X U, Y) + g(X, D_Y U) \\ \text{and} \quad (L_V g)(X, Y) &= g(D_X V, Y) + g(X, D_Y V), \end{aligned} \quad (2.16)$$

holds for arbitrary vector fields X and Y on M^n , where L_U and L_V denote the Lie derivatives along the vector fields U and V corresponding to D respectively.

Analogous to the above definition of \bar{D} , we define

$$\begin{aligned} (\bar{L}_U g)(X, Y) &= g(\bar{D}_X U, Y) + g(X, \bar{D}_Y U) \\ \text{and} \quad (\bar{L}_V g)(X, Y) &= g(\bar{D}_X V, Y) + g(X, \bar{D}_Y V), \end{aligned} \quad (2.17)$$

holds for arbitrary vector fields X and Y on M^n , where \bar{L}_U and \bar{L}_V denote the Lie derivatives along the vector fields U and V corresponding to \bar{D} respectively.

On adding (2.16) and (2.17) and using equation (1.1), we find

$$\begin{aligned} (\bar{L}_U g)(X, Y) + (\bar{L}_V g)(X, Y) &= (L_U g)(X, Y) + (L_V g)(X, Y) + \\ &2[\eta(X)\eta(Y) - a(X)a(Y)]. \end{aligned}$$

Hence, the statement of Proposition 2.4 is proved.

If the vector fields U and V are killing on (M^n, g) , then $(L_U g)(X, Y) = 0$ and $(L_V g)(X, Y) = 0$. Thus, we can state the following proposition:

Proposition 2.5 *If an n -dimensional Riemannian manifold (M^n, g) admits a semi-symmetric non-metric connection and U and V are killing vector fields with respect \bar{D} , then Lie derivative with respect to D is also killing if and only if $\eta(X)\eta(Y) = a(X)a(Y)$.*

3 Curvature tensor with respect to the semi-symmetric non-metric connection \bar{D}

Let (M^n, g) be an n -dimensional Riemannian manifold admitting a semi-symmetric non-metric connection \bar{D} . The curvature tensor \bar{R} corresponding to \bar{D} is defined by

$$\bar{R}(X, Y)Z = \bar{D}_X \bar{D}_Y Z - \bar{D}_Y \bar{D}_X Z - \bar{D}_{[X, Y]}Z, \quad (3.1)$$

for arbitrary vector fields X, Y and Z on (M^n, g) and the Riemannian curvature R of the Levi-Civita connection D defined by

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z, \quad (3.2)$$

for all vector fields X, Y and Z on (M^n, g) . In view of equation (1.1), (3.1) and (3.2), we get

$$\overline{R}(X, Y)Z = R(X, Y)Z + d\eta(X, Y)Z - da(X, Y)Z, \quad (3.3)$$

where η and a are tensor field of type $(0,2)$ and given by

$$(\overline{D}_X\eta)Y - (\overline{D}_Y\eta)X = d\eta(X, Y) \quad \text{and} \quad (\overline{D}_Xa)Y - (\overline{D}_Ya)X = da(X, Y). \quad (3.4)$$

Contracting (3.3) with respect to X , we have

$$\overline{Ric}(Y, Z) = Ric(Y, Z) + d\eta(Y, Z) - da(Y, Z), \quad (3.5)$$

Again, contracting (3.5) with respect to Y and Z , we get

$$\overline{r} = r, \quad (3.6)$$

where \overline{r} and r are scalar curvature with respect to \overline{D} and D respectively.

Proposition 3.1 *Let (M^n, g) denote an n - dimensional Riemannian manifold endowed with a semi-symmetric non-metric connection \overline{D} . Then the scalar curvatures \overline{r} of \overline{D} is equal to scalar curvature of D .*

Interchanging Y and Z in (3.5) , we have

$$\overline{Ric}(Z, Y) = Ric(Z, Y) + d\eta(Z, Y) - da(Z, Y), \quad (3.7)$$

Subtracting (3.7) from the equation (3.5) and then using the symmetric property of the Ricci tensor in it, we conclude that

$$\overline{Ric}(Y, Z) - \overline{Ric}(Z, Y) = 2[d\eta(Y, Z) - da(Y, Z)]. \quad (3.8)$$

In view of (3.8) , we are in a position to state the following proposition:

Proposition 3.2 *If an n -dimensional Riemannian manifold (M^n, g) admits a semi-symmetric non-metric connection \overline{D} , then the Ricci tensor \overline{Ric} corresponding to the connection \overline{D} is symmetric if and only if $d\eta(Y, Z) = da(Y, Z)$.*

Theorem 3.3 *Let (M^n, g) be a n -dimensional Riemannian manifold for equipped with a semi-symmetric non-metric connection \overline{D} , then the following relations hold for all the vector fields X, Y, Z and W on M^n :*

- (i) $\overline{R}(X, Y)Z + \overline{R}(Y, X)Z = 0,$
- (ii) $\overline{R}(X, Y)Z + \overline{R}(Y, Z)X + \overline{R}(Z, X)Y = 0,$
- (iii) $(\overline{D}_X\overline{R})(Y, Z)W + (\overline{D}_Y\overline{R})(Z, X)W + (\overline{D}_Z\overline{R})(X, Y)W = -2[\{\eta(X) - a(X)\}\overline{R}(Y, Z)W + \{\eta(Y) - a(Y)\}\overline{R}(Z, X)W + \{\eta(Z) - a(Z)\}\overline{R}(X, Y)W],$

$$(iv) \quad ' \bar{R}(X, Y, Z, W) + ' \bar{R}(X, Y, W, Z) = 2[d\eta(X, Y) - da(X, Y)]g(Z, W),$$

$$(v) \quad ' \bar{R}(X, Y, Z, W) - ' \bar{R}(Z, W, X, Y) = [d\eta(X, Y) - da(X, Y)]g(Z, W) - [d\eta(Z, W) - da(Z, W)]g(X, Y).$$

Proof: Interchanging X and Y in equation (3.3) and then adding with (3.3), we obtain (i). Again from (3.3), we find

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

This expression shows that the Riemannian manifold (M^n, g) equipped with a semi-symmetric non-metric connection \bar{D} satisfies Bianchi's first identity. Thus, result (ii). Bianchi's second identity for a semi-symmetric non-metric connection \bar{D} is given by the expression

$$\begin{aligned} (\bar{D}_X \bar{R})(Y, Z)W + (\bar{D}_Y \bar{R})(Z, X)W + (\bar{D}_Z \bar{R})(X, Y)W = \\ - \bar{R}(\bar{T}(X, Y), Z)W - \bar{R}(\bar{T}(Y, Z), X)W - \bar{R}(\bar{T}(Z, X), Y)W, \end{aligned}$$

for arbitrary vector fields X, Y, Z and W on M^n .

With the help of equation (1.1), (i) and the last expressions, we can easily find (iii). If we define $' \bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W)$ and $' R(X, Y, Z, W) = g(R(X, Y)Z, W)$, then equation (3.4) becomes

$$' \bar{R}(X, Y, Z, W) = ' R(X, Y, Z, W) + d\eta(X, Y)g(Z, W) - da(X, Y)g(Z, W), \quad (3.9)$$

or all vector fields X, Y, Z and W on M^n . Expressions (iv) and (v) are obvious from equations (3.4) and (3.9) and the symmetric properties of the curvature tensor. Hence, the proof is complete.

4 Projective Curvature tensor with respect to \bar{D}

Theorem 4.1 *Let (M^n, g) be a Riemannian manifold of dimension $(n > 2)$ equipped with a semi-symmetric non-metric connection defined in equation (2.1). Then the projective curvature tensor with respect to \bar{D} is equal to the Projective curvature tensor with respect to D if and only if $d\eta(Y, Z) = da(Y, Z)$.*

Proof: The Projective curvature tensor \bar{P} with respect to \bar{D} is defined by

$$\bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{n-1}[\bar{Ric}(Y, Z)X - \bar{Ric}(X, Z)Y]. \quad (4.1)$$

By using equation (3.3) and (3.5) in equation (4.1), we have

$$\begin{aligned} \bar{P}(X, Y)Z = & P(X, Y)Z - \frac{1}{n-1}[d\eta(Y, Z) - da(Y, Z)X] + \\ & \frac{1}{n-1}[d\eta(X, Z) - da(X, Z)]Y + [d\eta(X, Y) - da(X, Y)]Z, \end{aligned} \quad (4.2)$$

where

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[Ric(Y, Z)X - Ric(X, Z)Y], \quad (4.3)$$

for all the arbitrary vector fields X, Y and Z on (M^n, g) .

From equation (4.2), we get

$$\bar{P}(X, Y)Z = P(X, Y)Z, \text{ if and only if } d\eta(Y, Z) = da(Y, Z).$$

Thus, Theorem 4.1 is proved.

Let M^n be the Riemannian manifold satisfying

$$\bar{R}(X, Y)Z = 0. \quad (4.4)$$

Therefore, by contracting equation (4.4), we get

$$\bar{Ric}(Y, Z) = 0. \quad (4.5)$$

Using (4.4) and (4.5) in equation (4.1), we get

$$\bar{P}(X, Y)Z = 0. \quad (4.6)$$

In view of (4.6) and (4.2), we get

$$\begin{aligned} P(X, Y)Z = & \frac{1}{n-1}[d\eta(Y, Z) - da(Y, Z)X] - \\ & \frac{1}{n-1}[d\eta(X, Z) - da(X, Z)]Y - [d\eta(X, Y) - da(X, Y)]Z. \end{aligned} \quad (4.7)$$

Hence in the view of (4.7), we have the following proposition:

Proposition 4.2 *If in a Riemannian manifold the curvature tensor of semi-symmetric non-metric connection \bar{D} vanishes then the manifold with respect to D is projectively flat if and only if $d\eta(Y, Z) = da(Y, Z)$.*

Theorem 4.3 *Let (M^n, g) be an n -dimensional Riemannian manifold equipped with a semi-symmetric non-metric connection \bar{D} , then the projective curvature tensor for all the vector fields X, Y and Z on (M^n, g) holds the following relations:*

$$(i) \quad \overline{P}(X, Y)Z + \overline{P}(Y, X)Z = 0,$$

$$(ii) \quad \overline{P}(X, Y)Z + \overline{R}(Y, Z)X + \overline{R}(Z, X)Y = 0, \text{ if and only if } d\eta(X, Y) = da(X, Y).$$

Proof Interchanging X and Y in equation (4.2) and then adding with (4.2), we obtain

$$\overline{P}(X, Y)Z + \overline{P}(Y, X)Z = 0.$$

Again, from equation (4.2), we have

$$\overline{P}(X, Y)Z + \overline{R}(Y, Z)X + \overline{R}(Z, X)Y = 0, \text{ if and only if } d\eta(X, Y) = da(X, Y).$$

5 Submanifold of a Riemannian manifold with respect to the semi-symmetric non-metric connection \overline{D}

Let (M^{n-2}, g) be an $(n-2)$ -dimensional submanifold of an n -dimensional Riemannian manifold M^n . Suppose $i : M^{n-2} \rightarrow M^n$ is an inclusion map such that $p \in M^{n-2} \rightarrow pi \in M^n$. The inclusion map i induces a Jacobian map $K : T(M^{n-2}) \rightarrow T(M^n)$, where $T(M^{n-2})$ and $T(M^n)$ denote the tangent spaces to M^{n-2} at i and M^n at p , respectively. Let L be a metric tensor of M^n and g be an induced metric tensor of the submanifold M^{n-2} at p and i , respectively. Then we have

$$L(KX, KY)op = g(X, Y), \text{ for every } X, Y \in T(M^{n-2}).$$

Let S_1 and S_2 and to be two mutually orthogonal unit normal vector fields to the submanifold M^{n-2} satisfying the following relations:

$$(i) \quad L(KX, S_1) = L(KX, S_2) = L(S_1, S_2) = 0,$$

$$(ii) \quad L(S_1, S_1) = L(S_2, S_2) = 1. \quad (5.1)$$

Let \overline{D} be the induced connection on M^{n-2} corresponding to the Levi-Civita connection D on M^n . Then we can write

$$D_{KX}KY = K(D_XY) + q_1(X, Y)S_1 + q_2(X, Y)S_2, \quad (5.2)$$

for all vector fields X and Y on M^{n-2} . Here q_1 and q_2 denote the second fundamental tensors of the submanifold M^{n-2} . Let \overline{D} be the induced connection of the submanifold M^{n-2} corresponding to the semi-symmetric non-metric connection \overline{D} on M^n defined as (2.1). Then for the unit normal vectors S_1 and S_2 , we have

$$\overline{D}_{KX}KY = K(\overline{D}_XY) + \phi_1(X, Y)S_1 + \phi_2(X, Y)S_2, \quad (5.3)$$

for arbitrary vector fields X and Y of M^{n-2} and ϕ_1 and ϕ_2 denote the tensor fields of type $(0,2)$ of the submanifold M^{n-2} .

Theorem 5.1 *The induced connection \bar{D} on the submanifold M^{n-2} of the Riemannian manifold M^n endowed with a semi-symmetric non-metric connection \bar{D} is also a semi-symmetric non-metric connection.*

Proof: In view of equation (2.1), we have

$$\bar{D}_{KX}KY = D_{KX}KY + \eta(KX)KY - a(KX)KY, \quad (5.4)$$

for all arbitrary vector fields X and Y . In consequences of equation (5.1), (5.2) and (5.3), equation (5.4) assumes the form

$$K(\bar{D}_X Y) + \phi_1(X, Y)S_1 + \phi_2(X, Y)S_2 = K(D_X Y) + q_1(X, Y)S_1 + q_2(X, Y)S_2 + \eta(KX)KY - a(KX)KY,$$

which gives

$$\begin{aligned} K(\bar{D}_X Y) &= K(D_X Y) + \eta(KX)KY - a(KX)KY, \\ \Rightarrow \bar{D}_X Y &= D_X Y + \eta(X)Y - a(X)Y, \end{aligned} \quad (5.5)$$

and

$$q_1(X, Y) = \phi_1(X, Y) \text{ and } q_2(X, Y) = \phi_2(X, Y). \quad (5.6)$$

Thus, the induced connection \bar{D} and D on M^{n-2} corresponding to the semi-symmetric non-metric connection and Levi-Civita connection of the Riemannian manifold M^n are connected by (5.5). The torsion \bar{T} of \bar{D} is defined by

$$\bar{T}(X, Y) = \bar{D}_X Y - \bar{D}_Y X - [X, Y] = \eta(X)Y - \eta(Y)X - [a(X)Y - a(Y)X],$$

where equations (5.5) is used. Thus, the induced connection \bar{D} of the submanifold M^{n-2} is semi-symmetric. Next, we have to prove that the connection \bar{D} is non-metric; i.e., $\bar{D}g \neq 0$. We have

$$Xg(Y, Z) = (\bar{D}_X g)(Y, Z) + g(\bar{D}_X Y, Z) + g(Y, \bar{D}_X Z) = g(\bar{D}_X Y, Z) + g(Y, \bar{D}_X Z).$$

This shows that the induced connection \bar{D} of the submanifold M^{n-2} corresponding to the semi-symmetric non-metric connection \bar{D} is also semi-symmetric non-metric. Hence, the statement of the theorem is proved.

Theorem 5.2 *Let M^{n-2} be a submanifold of the Riemannian manifold M^n . Then*

- (i) The mean curvature of M^{n-2} corresponding to the induced connections \bar{D} and D coincide.
- (ii) The submanifold M^{n-2} will be totally geodesic with respect to \bar{D} if and only if it is totally geodesic for D .
- (iii) The submanifold M^{n-2} is totally umbilical with respect to \bar{D} if and only if it is totally umbilical for D .
- (iv) The submanifold M^{n-2} is minimal corresponding to \bar{D} if and only if it is also minimal for D .

Proof: We define

$$DK(X, Y) = (D_X K)(Y) = D_K XKY - K(D_X Y),$$

$$(\bar{D}K)(X, Y) = (\bar{D}_X K)(Y) = (D_K XKY) - K(\bar{D}_X Y).$$

In the view of equation (5.2) and (5.3) the above equations are considered in the forms

$$(D_X K)(Y) = q_1(X, Y)S_1 + q_2(X, Y)S_2,$$

$$(\bar{D}_X K)(Y) = \phi_1(X, Y)S_1 + \phi_2(X, Y)S_2.$$

Let $\{e_1, e_2, \dots, e_{n-2}\}$ be a set of $(n-2)$ orthonormal local vector fields in M^{n-2} . Then the mean curvature tensor ϕ of the submanifold M^{n-2} with respect to the connection D is a function defined by $\phi = \frac{1}{n-2} \sum_{i=1}^{n-2} q(e_i, e_i)$.

Let $\bar{\phi} = \frac{1}{n-2} \sum_{i=1}^{n-2} q(e_i, e_i)$ denote the mean curvature of M^{n-2} with respect to the semi-symmetric non-metric induced connection \bar{D} . In particular, if $\bar{\phi} = 0$ on M^{n-2} , then the submanifold is said to be a minimal submanifold for \bar{D} . Also, if $\phi = 0$ on M^{n-2} , then the submanifold is said to be a minimal for D .

On the other hand, the submanifold M^{n-2} is said to be totally geodesic with respect to the Levi-Civita connection D if and only if q_1 and q_2 vanish identically on M^{n-2} . If q_1 and q_2 are proportional to the metric g , i.e., $q_1 = \phi g$ and $q_2 = \phi g$, then the submanifold M^{n-2} is said to be totally umbilical with respect to the Levi-Civita connection D . In a similar fashion, we can say that the submanifold M^{n-2} is said to be totally umbilical with respect to the semi-symmetric non-metric induced connection \bar{D} if ϕ_1 and ϕ_2 are proportional to g ($\phi_1 = \bar{\phi}_1 g$ and $\phi_2 = \bar{\phi}_2 g$). The statements of Theorem 5.2 are obvious from the above discussion and equation (5.6).

6 Examples

Example 6.1 Let us consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of R^3 .

We choose the vector fields

$$e_1 = e^{-3z} \frac{\partial}{\partial x}, \quad e_2 = e^{-3z} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad (6.1)$$

which is linearly independently at each point of M and therefore it forms a basis for the tangent space $T(M^3)$.

Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (6.2)$$

Let D be the Levi-Civita connection with respect to metric g . Then from equation (6.1), we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 3e_1, \quad [e_2, e_3] = 3e_2. \quad (6.3)$$

The Riemannian connection D of the metric g is given by

$$2g(D_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \quad (6.4)$$

which is known as Koszul's formula. Thus, we obtain

$$\begin{aligned} D_{e_1} e_1 &= 3e_3, \quad D_{e_1} e_2 = 0, \quad D_{e_1} e_3 = 3e_1, \\ D_{e_2} e_1 &= 0, \quad D_{e_2} e_2 = 3e_3, \quad D_{e_2} e_3 = 3e_2, \\ D_{e_3} e_1 &= 0, \quad D_{e_3} e_2 = 0, \quad D_{e_3} e_3 = 0, \end{aligned} \quad (6.5)$$

where D denotes the Levi-Civita connection corresponding to the metric g . The non-vanishing components of the Riemannian curvature tensor can be calculated by the using the equation (3.2), (6.3) and (6.5), we have The curvature tensor is given by

$$\begin{aligned} R(e_1, e_2) e_1 &= 9e_1, \quad R(e_1, e_2) e_2 = 9e_1, \quad R(e_1, e_2) e_3 = 0, \\ R(e_2, e_3) e_1 &= 0, \quad R(e_2, e_3) e_2 = -9e_3, \quad R(e_2, e_3) e_3 = -9e_2, \\ R(e_1, e_3) e_1 &= -9e_3, \quad R(e_1, e_3) e_2 = 0, \quad R(e_1, e_3) e_3 = -9e_1, \\ R(e_1, e_1) e_1 &= R(e_1, e_1) e_2 = R(e_1, e_1) e_3 = 0, \\ R(e_2, e_2) e_1 &= R(e_2, e_2) e_2 = R(e_2, e_2) e_3 = 0, \\ R(e_3, e_3) e_1 &= R(e_3, e_3) e_2 = R(e_3, e_3) e_3 = 0. \end{aligned} \quad (6.6)$$

The Ricci tensor can be calculated by the following expression

$$\text{Ric}(X, Y) = \sum_{i=1}^3 g(R(e_i, X)Y, e_i). \quad (6.7)$$

From (6.6) and (6.7), we get

$$\begin{aligned} \text{Ric}(e_1, e_1) &= 9, & \text{Ric}(e_1, e_2) &= 0, & \text{Ric}(e_1, e_3) &= 0, \\ \text{Ric}(e_2, e_1) &= 18, & \text{Ric}(e_2, e_2) &= -9, & \text{Ric}(e_2, e_3) &= 0, \\ \text{Ric}(e_3, e_1) &= -18, & \text{Ric}(e_3, e_2) &= 0, & \text{Ric}(e_3, e_3) &= -9. \end{aligned} \quad (6.8)$$

It is obvious that the scalar curvature is $r = -9$.

Taking $U = e_3$ and $V = e_2$. In consequences of the above discussion and equation (2.1), we have

$$\left. \begin{aligned} \bar{D}_{e_1}e_1 &= 3e_3, & \bar{D}_{e_1}e_2 &= 2e_2, & \bar{D}_{e_1}e_3 &= 3e_1, \\ \bar{D}_{e_2}e_1 &= -e_1, & \bar{D}_{e_2}e_2 &= 3e_3 - e_1, & \bar{D}_{e_2}e_3 &= e_3, \\ \bar{D}_{e_3}e_1 &= e_1, & \bar{D}_{e_3}e_2 &= e_2, & \bar{D}_{e_3}e_3 &= e_3. \end{aligned} \right\} \quad (6.9)$$

In view of (6.9), we can easily prove that equation (1.1) holds for all vector fields $e_i (i = 1, 2, 3)$, e.g.,

$$\left. \begin{aligned} \bar{T}(e_1, e_1) &= \bar{T}(e_2, e_2) = \bar{T}(e_3, e_3) = 0, \\ \bar{T}(e_1, e_2) &= e_1, & \bar{T}(e_1, e_3) &= -e_1, & \bar{T}(e_2, e_3) &= -(e_2 + e_3). \end{aligned} \right\} \quad (6.10)$$

This shows that the linear connection \bar{D} defined as (2.1) is a semi-symmetric connection on (M^3, g) , also

$$(\bar{D}_{e_1}g)(e_3, e_3) = 2 \neq 0, \quad (\bar{D}_{e_2}g)(e_1, e_1) = 2 \neq 0, \quad (\bar{D}_{e_3}g)(e_1, e_1) = 2 \neq 0.$$

Similarly, we can verify this for other components.

Hence, the semi-symmetric connection \bar{D} is non-metric on (M^3, g) . Let X, Y and Z be vector fields on M^3 . Then it can be expressed as a linear combination of e_1, e_2 and e_3 , that is,

$$X = X^1e_1 + X^2e_2 + X^3e_3, \quad Y = Y^1e_1 + Y^2e_2 + Y^3e_3, \quad Z = Z^1e_1 + Z^2e_2 + Z^3e_3,$$

where X^i, Y^i and $Z^i, i = 1, 2, 3$ are real constants, we have

$$\begin{aligned} \bar{T}(X, Y, Z) &= g(\bar{T}(X, Y), Z), \\ \bar{T}(X, Y, Z) &= (X^1Y^2 - X^2Y^1)Z^1 - (X^1Y^3 - X^3Y^1)Z^1 - \\ &\quad (X^2Y^3 - X^3Y^2)Z^2 - (X^2Y^3 - X^3Y^2)Z^3, \end{aligned} \quad (6.11)$$

$$\begin{aligned} {}'\bar{T}(Y, X, Z) = & (Y^1 X^2 - Y^2 X^1)Z^1 - (Y^1 X^3 - Y^3 X^1)Z^1 - \\ & (Y^2 X^3 - Y^3 X^2)Z^2 - (Y^2 X^3 - Y^3 X^2)Z^3, \end{aligned} \quad (6.12)$$

$$\begin{aligned} {}'\bar{T}(Y, Z, X) = & (Y^1 Z^2 - Y^2 Z^1)X^1 - (Y^1 Z^3 - Y^3 Z^1)X^1 - \\ & (Y^2 Z^3 - Y^3 Z^2)X^2 - (Y^2 Z^3 - Y^3 Z^2)X^3, \end{aligned} \quad (6.13)$$

$$\begin{aligned} {}'\bar{T}(Z, X, Y) = & (Z^1 X^2 - Z^2 X^1)Y^1 - (Z^1 X^3 - Z^3 X^1)Y^1 - \\ & (Z^2 X^3 - Z^3 X^2)Y^2 - (Z^2 X^3 - Z^3 X^2)Y^3. \end{aligned} \quad (6.14)$$

Hence, from the equation (6.11), (6.12), (6.13) and (6.14), we have

$${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, X, Z) = 0,$$

and

$${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, Z, X) + {}'\bar{T}(Z, X, Y) = 0.$$

Therefore, Theorem 2.2 is verified.

we have also

$$\bar{R}(e_i, e_j)e_k \neq 0, \text{ for all } i, j, k = 1, 2, 3. \quad (6.15)$$

Hence, the Riemannian equipped with a semi-symmetric non-metric connection \bar{D} is not flat.

Example 6.2 Let us consider a 5-dimensional manifold

$M^5 = (z_1, z_2, z_3, z_4, z_5) \in R^5, z \neq 0$, where z_1, z_2, z_3, z_4, z_5 are standard co-ordinate in R^5 .

We choose the vector fields

$$\begin{aligned} e_1 = e^{-z_5} \frac{\partial}{\partial z_1}, \quad e_2 = e^{-z_5} \frac{\partial}{\partial z_2}, \quad e_3 = e^{-z_5} \frac{\partial}{\partial z_3}, \\ e_4 = e^{-z_5} \frac{\partial}{\partial z_4}, \quad e_5 = e^{-z_5} \frac{\partial}{\partial z_5}. \end{aligned} \quad (6.16)$$

which are linearly independent at each point of M^5 and therefore it forms a basis for the tangent space $T(M^5)$.

Let g be the Riemannian metric defined by (6.2). Let D be the Levi-Civita connection with respect to metric g . Then from equation (6.16), we have

$$\begin{aligned} [e_1, e_2] = 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_2, e_3] = 0, [e_2, e_4] = 0, [e_3, e_4] = 0, \\ [e_1, e_5] = e^{-Z_5} e_1, [e_2, e_5] = e^{-Z_5} e_2, [e_3, e_5] = e^{-Z_5} e_3, [e_4, e_5] = e^{-Z_5} e_4. \end{aligned} \quad (6.17)$$

The Riemannian connection D of the metric g is given by (6.4). With help of (6.2), (6.4) and (6.17), we obtain

$$\begin{aligned} D_{e_1} e_1 = e^{-Z_5} e_5, \quad D_{e_1} e_2 = 0, \quad D_{e_1} e_3 = 0, \quad D_{e_1} e_4 = 0, \quad D_{e_1} e_5 = e^{-Z_5} e_1, \\ D_{e_2} e_1 = 0, \quad D_{e_2} e_2 = e^{-Z_5} e_5, \quad D_{e_2} e_3 = 0, \quad D_{e_2} e_4 = 0, \quad D_{e_2} e_5 = e^{-Z_5} e_2, \\ D_{e_3} e_1 = 0, \quad D_{e_3} e_2 = 0, \quad D_{e_3} e_3 = e^{-Z_5} e_5, \quad D_{e_3} e_4 = 0, \quad D_{e_3} e_5 = e^{-Z_5} e_3, \\ D_{e_4} e_1 = 0, \quad D_{e_4} e_2 = 0, \quad D_{e_4} e_3 = 0, \quad D_{e_4} e_4 = e^{-Z_5} e_5, \quad D_{e_4} e_5 = e^{-Z_5} e_4, \\ D_{e_5} e_1 = 0, \quad D_{e_5} e_2 = 0, \quad D_{e_5} e_3 = 0, \quad D_{e_5} e_4 = 0, \quad D_{e_5} e_5 = 0. \end{aligned} \quad (6.18)$$

The non-vanishing components of the Riemannian curvature tensor can be calculated by the using the equation (3.2), (6.17) and (6.18), we have The curvature tensor is given by

$$\begin{aligned}
R(e_1, e_2)e_1 &= e^{-2Z_5}e_2, \quad R(e_1, e_2)e_2 = 0, \quad R(e_1, e_2)e_3 = 0, \quad R(e_1, e_2)e_4 = 0, \\
R(e_1, e_2)e_5 &= 0, \quad R(e_1, e_3)e_1 = e^{-2Z_5}e_5, \quad R(e_1, e_3)e_2 = -9e_3, \quad R(e_1, e_3)e_3 = e^{-2Z_5}e_1, \\
R(e_1, e_3)e_4 &= 0, \quad R(e_1, e_3)e_5 = 0, \quad R(e_1, e_5)e_1 = 0, \quad R(e_1, e_5)e_2 = 0, \\
R(e_1, e_5)e_3 &= 0, \quad R(e_1, e_5)e_4 = 0, \quad R(e_1, e_5)e_5 = 0, \quad R(e_2, e_3)e_1 = 0, \\
R(e_2, e_3)e_2 &= 0, \quad R(e_2, e_3)e_3 = -e^{-2Z_5}e_2, \quad R(e_2, e_3)e_4 = 0, \quad R(e_2, e_3)e_5 = 0, \\
R(e_2, e_4)e_1 &= 0, \quad R(e_2, e_4)e_2 = e^{-2Z_5}e_4, \quad R(e_2, e_4)e_3 = 0, \quad R(e_2, e_4)e_4 = e^{-2Z_5}e_2, \\
R(e_2, e_4)e_5 &= 0, \quad R(e_2, e_5)e_1 = 0, \quad R(e_2, e_5)e_2 = 0, \quad R(e_2, e_5)e_3 = 0, \\
R(e_2, e_5)e_4 &= 0, \quad R(e_2, e_5)e_5 = e^{-2Z_5}e_2, \quad R(e_3, e_4)e_1 = 0, \quad R(e_3, e_4)e_2 = 0, \\
R(e_3, e_4)e_3 &= -e^{-2Z_5}e_3, \quad R(e_3, e_4)e_4 = 0, \quad R(e_3, e_4)e_5 = 0, \quad R(e_3, e_5)e_1 = 0, \\
R(e_3, e_5)e_2 &= 0, \quad R(e_3, e_5)e_3 = -e^{-2Z_5}e_5, \quad R(e_3, e_5)e_4 = 0, \quad R(e_3, e_5)e_5 = -e^{-2Z_5}e_3, \\
R(e_4, e_5)e_1 &= 0, \quad R(e_4, e_5)e_2 = 0, \quad R(e_4, e_5)e_3 = 0, \quad R(e_4, e_5)e_4 = 0, \quad R(e_4, e_5)e_5 = 0.
\end{aligned} \tag{6.19}$$

The Ricci tensor can be calculated by the following expression

$$Ric(X, Y) = \sum_{i=1}^5 g(R(e_i, X)Y, e_i). \tag{6.20}$$

From (6.19) and (6.20), we get

$$\begin{aligned}
Ric(e_1, e_1) &= 0, \quad Ric(e_1, e_2) = 0, \quad Ric(e_1, e_3) = 0, \quad Ric(e_1, e_4) = 0, \\
Ric(e_1, e_5) &= 0, \quad Ric(e_2, e_1) = 0, \quad Ric(e_2, e_2) = -e^{-2Z_5}, \quad Ric(e_2, e_3) = 0, \\
Ric(e_2, e_4) &= -9, \quad Ric(e_2, e_5) = 0, \quad Ric(e_3, e_1) = 0, \quad Ric(e_3, e_2) = 0, \\
Ric(e_3, e_3) &= e^{-2Z_5}, \quad Ric(e_3, e_4) = 0, \quad Ric(e_3, e_5) = 0, \quad Ric(e_4, e_1) = 0, \\
Ric(e_4, e_2) &= 0, \quad Ric(e_4, e_3) = 0, \quad Ric(e_4, e_4) = e^{-2Z_5}, \quad Ric(e_4, e_5) = 0, \\
Ric(e_3, e_1) &= Ric(e_3, e_2) = Ric(e_3, e_3) = Ric(e_5, e_4) = Ric(e_5, e_5) = 0.
\end{aligned} \tag{6.21}$$

It is obvious that the scalar curvature is $r = e^{-2Z_5}$.

Taking $U = e_3$ and $V = e_2$ and in consequences of the above discussion and equation (2.1), we have

$$\begin{aligned}
\bar{D}_{e_1}e_1 &= e^{-Z_5}e_5, \quad \bar{D}_{e_1}e_2 = 0, \quad \bar{D}_{e_1}e_3 = 0, \quad \bar{D}_{e_1}e_4 = 0, \quad \bar{D}_{e_1}e_5 = e^{-Z_5}e_1, \\
\bar{D}_{e_2}e_1 &= -e_1, \quad \bar{D}_{e_2}e_2 = e^{-Z_5}e_5 - e_2, \quad \bar{D}_{e_2}e_3 = -e_3, \quad \bar{D}_{e_2}e_4 = -e_4, \\
\bar{D}_{e_2}e_5 &= e^{-Z_5}e_2 - e_5, \quad \bar{D}_{e_3}e_1 = e_1, \quad \bar{D}_{e_3}e_2 = e_2, \quad \bar{D}_{e_3}e_3 = e^{-Z_5}e_5 + e_3, \\
\bar{D}_{e_3}e_4 &= e_5, \quad \bar{D}_{e_3}e_5 = e^{-Z_5}e_3 + e_5, \quad \bar{D}_{e_4}e_1 = e_1, \quad \bar{D}_{e_4}e_2 = e_2, \\
\bar{D}_{e_4}e_3 &= 0, \quad \bar{D}_{e_4}e_4 = e^{-Z_5}e_5, \quad \bar{D}_{e_4}e_5 = e^{-Z_5}e_4, \\
\bar{D}_{e_5}e_1 &= \bar{D}_{e_5}e_2 = \bar{D}_{e_5}e_3 = \bar{D}_{e_5}e_4 = \bar{D}_{e_5}e_5 = 0.
\end{aligned} \tag{6.22}$$

In view of (6.22), we can easily prove that equation (1.1) holds for all vector fields $e_i (i = 1, 2, 3, 4, 5)$, e.g.,

$$\begin{aligned}
\bar{T}(e_1, e_1) &= 0, \bar{T}(e_2, e_2) = 0, \bar{T}(e_3, e_3) = 0, \bar{T}(e_4, e_4) = 0, \bar{T}(e_5, e_5) = 0, \\
\bar{T}(e_1, e_2) &= e_1, \bar{T}(e_1, e_3) = -e_1, \bar{T}(e_1, e_4) = 0, \bar{T}(e_1, e_5) = 0, \\
\bar{T}(e_2, e_1) &= -e_1, \bar{T}(e_2, e_3) = -(e_2 + e_3), \bar{T}(e_2, e_4) = -e_4, \bar{T}(e_2, e_5) = -e_5, \\
\bar{T}(e_3, e_1) &= e_1, \bar{T}(e_3, e_2) = e_2 + e_3, \bar{T}(e_3, e_4) = e_4, \bar{T}(e_3, e_5) = e_5, \\
\bar{T}(e_4, e_1) &= 0, \bar{T}(e_4, e_2) = e_4, \bar{T}(e_4, e_3) = -e_4, \bar{T}(e_4, e_5) = 0, \\
\bar{T}(e_5, e_1) &= 0, \bar{T}(e_5, e_2) = e_5, \bar{T}(e_5, e_3) = -e_5, \bar{T}(e_5, e_4) = 0.
\end{aligned} \tag{6.23}$$

This shows that the linear connection \bar{D} defined as (2.1) is a semi-symmetric connection on (M^5, g) , also

$$(\bar{D}_{e_2}g)(e_3, e_3) = 2 \neq 0, \quad (\bar{D}_{e_3}g)(e_1, e_1) = 2 \neq 0.$$

Similarly, we can verify this for other components.

Hence, the semi-symmetric connection \bar{D} is non-metric on (M^5, g) . Let X, Y and Z be vector fields on M^5 . Then it can be expressed as a linear combination of e_1, e_2, e_3, e_4 and e_5 , that is,

$$\begin{aligned}
X &= X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5, \\
Y &= Y^1 e_1 + Y^2 e_2 + Y^3 e_3 + Y^4 e_4 + Y^5 e_5, \\
Z &= Z^1 e_1 + Z^2 e_2 + Z^3 e_3 + Z^4 e_4 + Z^5 e_5,
\end{aligned}$$

where X^i, Y^i and $Z^i, i = 1, 2, 3, 4, 5$ are real constants, we have

$$\begin{aligned}
{}'\bar{T}(X, Y, Z) &= g(\bar{T}(X, Y), Z), \\
{}'\bar{T}(X, Y, Z) &= (X^1 Y^2 - X^2 Y^1 - X^1 Y^3 - X^3 Y^1) Z^1 - \\
&\quad (X^2 Y^3 - X^3 Y^2) Z^2 - (X^2 Y^3 - X^3 Y^2) Z^3 \\
&\quad - (X^2 Y^4 - X^4 Y^2 + X^3 Y^4 - X^4 Y^3) Z^4 \\
&\quad - (X^2 Y^5 - X^4 Y^2 - X^3 Y^5 + X^5 Y^3) Z^4,
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
{}'\bar{T}(Y, X, Z) &= (Y^1 X^2 - Y^2 X^1 - Y^1 X^3 - Y^3 X^1) Z^1 - \\
&\quad (Y^2 X^3 - Y^3 X^2) Z^2 - (Y^2 X^3 - Y^3 X^2) Z^3 \\
&\quad - (Y^2 X^4 - Y^4 X^2 + Y^3 X^4 - Y^4 X^3) Z^4 \\
&\quad - (Y^2 X^5 - Y^4 X^2 - Y^3 X^5 + Y^5 X^3) Z^4,
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
{}'\bar{T}(Y, Z, X) &= (Y^1 Z^2 - Y^2 Z^1 - Y^1 Z^3 - Y^3 Z^1) X^1 - \\
&\quad (Y^2 Z^3 - Y^3 Z^2) X^2 - (Y^2 Z^3 - Y^3 Z^2) X^3 \\
&\quad - (Y^2 Z^4 - Y^4 Z^2 + Y^3 Z^4 - Y^4 Z^3) X^4 \\
&\quad - (Y^2 Z^5 - Y^4 Z^2 - Y^3 Z^5 + Y^5 Z^3) X^4,
\end{aligned} \tag{6.26}$$

$$\begin{aligned}
{}'\bar{T}(Z, X, Y) = & (Z^1 X^2 - Z^2 X^1 - Z^1 X^3 - Z^3 X^1)Y^1 - \\
& (Z^2 X^3 - Z^3 X^2)Y^2 - (Z^2 X^3 - Z^3 X^2)Y^3 \\
& - (Z^2 X^4 - Z^4 X^2 + Z^3 X^4 - Z^4 X^3)Y^4 \\
& - (Z^2 X^5 - Z^4 X^2 - Z^3 X^5 - Z^5 X^3)Y^4.
\end{aligned} \tag{6.27}$$

Hence, from the equation (6.24), (6.25), (6.26) and (6.27), we have

$${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, X, Z) = 0,$$

and

$${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, Z, X) + {}'\bar{T}(Z, X, Y) = 0.$$

Therefore, Theorem 2.2 is verified.

We have also

$$\bar{R}(e_i, e_j)e_k \neq 0, \text{ for all } i, j, k = 1, 2, 3, 4, 5. \tag{6.28}$$

Hence, the Riemannian equipped with a semi-symmetric non-metric connection \bar{D} is not flat.

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