

EXPLICIT CHARACTERIZATION OF THE RADIUS OF CURVATURE OF CURVES LYING IN A SURFACE OF \mathbb{E}^3 OR \mathbb{E}_1^3

ATHOUMANE NIANG AND AMETH NDIAYE

ABSTRACT. The differential equation characterizing a spherical curve in \mathbb{R}^3 expresses the radius of curvature of the curve in terms of its torsion. In this paper we show that the differential equation characterizing a curve lying in an oriented surface \mathbb{E}^3 or \mathbb{E}_1^3 allows to express the radius of the curvature of the curve in terms of its torsions. We give two examples for illustrate our main result.

1. INTRODUCTION

We denote by \mathbb{E}^3 the Euclidean 3-space \mathbb{R}^3 with its canonical metric $dx_1^2 + dx_2^2 + dx_3^2$, and by \mathbb{E}_1^3 the 3-space \mathbb{R}^3 with the Minkowski metric $dx_1^2 + dx_2^2 - dx_3^2$. We will denote the scalar product of \mathbb{E}^3 and \mathbb{E}_1^3 by $\langle \cdot, \cdot \rangle$.

The curves to be considered here are the unit speed curves in \mathbb{E}^3 (respectively \mathbb{E}_1^3) of the form $\alpha = \alpha(s)$, $s \in [0, L]$ and having non-degenerate principal normal. Recall that a curve α in \mathbb{E}^3 (respectively in \mathbb{E}_1^3) is a unit speed curve if $\langle \alpha', \alpha' \rangle = 1$ (respectively $\langle \alpha', \alpha' \rangle = \pm 1$). For such a curve (in \mathbb{E}^3), the following facts are well known.

There exist two functions κ, τ defined on $[0, L]$ that determine completely the shape of the curve in \mathbb{R}^3 . The functions κ and τ are respectively the curvature and the torsion of the curve. Such a curve $\alpha : [0, L] \rightarrow \mathbb{R}^3$ has a Frenet frame (T, N, B) which is a map on $[0, L]$, $s \mapsto (T(s), N(s), B(s))$ that satisfies the Frenet equations

$$\begin{cases} T' &= \kappa N \\ N' &= -\kappa T - \tau B \\ B' &= \tau N \end{cases}, \quad (1.1)$$

where the prime ($'$) denotes the differentiation with respect to arc length. For more information see [1, 3].

The condition for a curve to be a spherical curve, (i.e) it lies on a sphere, is usually given by the relation

$$\left[\frac{1}{\tau} \left(\frac{1}{\kappa} \right)' \right]' + \frac{\tau}{\kappa} = 0. \quad (1.2)$$

Clearly, condition (1.2) has a meaning only if κ and τ are nowhere zero, and it is only under this hypothesis that (1.2) is a necessary and sufficient condition for a curve to be a spherical curve.

2010 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42, 53C50.
Key words and phrases. Curve, Surface, curvature, Torsion.

In this paper we deal with an oriented surface. Let Σ be a surface on \mathbb{E}^3 . We will assume that Σ is oriented by choice of a unit normal field

$$\xi : \Sigma \longrightarrow S^2. \quad (1.3)$$

Recently, the authors in [4] gives the analogous of (1.2) when the curve α lies in an arbitrary oriented surface $\Sigma \subset \mathbb{E}^3$. Their result is obtained by using the second trihedron $(T, \xi \times T, \xi)$ (which is positively oriented), where $T(s) = \alpha'(s)$, $\xi(s)$ is the unit normal $\xi : \Sigma \longrightarrow S^2$ of the surface restricted on α and the \times is the vector product in \mathbb{E}^3 .

One can ask what is the analogous of the equation (1.2) when the curve is assumed to be in an arbitrary oriented surface in \mathbb{E}_1^3 ? One of the aim of this work, is to give an answer to this question.

When a curve consider as above is assumed to lie in a given oriented surface $\Sigma \subset \mathbb{R}^3$, then there exist two other invariants κ_n and τ_g defined on $[0, L]$ which are unique except for the sign (depending on the orientation of Σ). The functions κ_n and τ_g defined on $[0, L]$ are the normal curvature and the geodesic torsion of the curve.

In Minkowski space the characterization of curve lying on pseudohyperbolic space and Lorentzian hypersphere are stated both depending on curvature functions and character of Serret-Frenet frame of the curve, respectively. For detail see [5, 6, 8]. The following theorem and corollary are proved in [4].

Theorem 1.1. [4]

Under the assumptions and notations above, we have the following

- i) *the trihedron $(T, \xi, T \times \xi)$ and the functions κ , τ , κ_n and τ_g satisfy the following equation*

$$\begin{cases} T' &= \kappa_n \xi + \sqrt{\kappa^2 - \kappa_n^2} (\xi \times T) \\ \xi' &= -\kappa_n T + \tau_g (\xi \times T) \\ (T \times \xi)' &= -\sqrt{\kappa^2 - \kappa_n^2} T - \tau_g (\xi \times T) \end{cases}, \quad (1.4)$$

- ii)

$$\left(\frac{\kappa_n}{\kappa}\right)' = -(\tau - \tau_g) \sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2} \quad (1.5)$$

- iii)

$$\tau_g^2 = -(K - 2H\kappa_n + \kappa_n^2) \quad (1.6)$$

where H and K are respectively the restriction of mean curvature and the Gauss curvature of Σ to α .

Corollary 1.2. [4]

If the curve α is lying in a sphere with τ and κ' are nowhere zero in $[0, L]$, then equation (1.5) implies (1.2).

The aim of this paper is to give the analog of the theorem 1.1 for curve lying in an oriented spacelike surface $\Sigma \subset \mathbb{E}_1^3$. More precisely the equation (1.6) and its analog can be solved to express the ratio $\frac{\kappa_n}{\kappa}$ in term of the torsion τ and the geodesic torsion τ_g . Then we obtain the following theorem

Theorem 1.3. *In the above notations we have the following results.*

- (1) Let $\alpha = \alpha(s)$ be a curve lying in an oriented spacelike surface Σ of the Minkowski space \mathbb{E}_1^3 . Then the ratio $\frac{\kappa_n}{\kappa}$ satisfies

$$\frac{\kappa_n}{\kappa} = A_1 \cosh \int (\tau - \tau_g) ds + A_2 \sinh \int (\tau - \tau_g) ds,$$

where A_1 and A_2 are constants in some interval $[s_1, s_2] \subset [0, L]$ on which $\tau - \tau_g > 0$.

- (2) Let $\alpha = \alpha(s)$ be a curve lying in an oriented surface Σ of the Euclidean space \mathbb{E}^3 . Then the ratio $\frac{\kappa_n}{\kappa}$ satisfies

$$\frac{\kappa_n}{\kappa} = A_1 \cos \int (\tau - \tau_g) ds + A_2 \sin \int (\tau - \tau_g) ds,$$

where A_1 and A_2 are constants in some interval $[s_1, s_2] \subset [0, L]$ on which $\tau - \tau_g > 0$.

The paper is organised as follow: in section 2 we recall some results and definitions which we use for the proof of our main results. In section 3, we show the analog of the theorme 1.1 for curve lying in a spacelike surface in \mathbb{E}_1^3 and the expression of the ratio $\frac{\kappa_n}{\kappa}$ in term of $\tau - \tau_g$.

2. PRELIMINARIES

Let Σ be a surface in a three dimensional Minkowski space \mathbb{E}_1^3 oriented by a choice of a unit normal vector $\xi : \Sigma \rightarrow H^2$, where

$$H^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / x_1^2 + x_2^2 - x_3^2 = -1; x_3 > 0\}$$

is a hyperbolic space.

The canonical 3-volume form in \mathbb{E}_1^3 is $dx_1 \wedge dx_2 \wedge dx_3$ and the canonical basis (e_1, e_2, e_3) associated to the coordonates (x_1, x_2, x_3) is positively oriented. If u and v are two vectors in \mathbb{R}^3 , the vectors product denoted by $u \times v$ is defined by

$$\langle u \times v, w \rangle = \det(u, v, w), \quad w \in \mathbb{E}_1^3. \quad (2.1)$$

From (2.1) one gets

$$u \times v = (u_2 v_3 - u_3 v_2) e_1 - (u_1 v_3 - u_3 v_1) e_2 - (u_1 v_2 - u_2 v_1) e_3, \quad (2.2)$$

where $u = \sum_{i=1}^3 u_i e_i$ et $v = \sum_{i=1}^3 v_i e_i$.

In particular we have $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$. By using (2.1), one can prove the following useful formulas in \mathbb{E}_1^3

$$(u \times v) \times w = \langle v, w \rangle u - \langle u, w \rangle v. \quad (2.3)$$

3. PROOF OF THE MAIN THEOREM

Now let $\alpha : [0, L] \rightarrow \Sigma \subset \mathbb{E}_1^3$ be a curve parametrised by its arc-length $s \in [0, L]$. Let $\xi(s)$ be the restriction of ξ on α . We put $T(s) = \alpha'(s)$ and we assume that $T'(s) = \alpha''(s)$ is not isotropic and non zero vector. We will consider two cases:

Case 1: If T' is timelike, we put

$$T' = -\kappa N, \quad (3.1)$$

where N is the unit normal principal (then N is timelike).

Case 2: If T' is spacelike, we put

$$T' = \kappa N. \quad (3.2)$$

In the formulas (3.1) and (3.2), $\kappa = \kappa(s)$ is the curvature of α at s .

In the case 1 as in the case 2, we define the torsion τ of α and the geodesic torsion τ_g of α by

$$\begin{aligned}\tau &= \langle B', T \rangle, \\ \tau_g &= \langle \xi', \xi \times T \rangle,\end{aligned}\tag{3.3}$$

where B is the binormal vector defined such that (T, N, B) is a positively oriented basis of \mathbb{R}^3 .

In the case 1, we put

$$B = T \times N\tag{3.4}$$

and for the case 2 we take

$$B = N \times T.\tag{3.5}$$

- Case 1:

we have $T' = -\kappa N$. There is angle $\varphi = \varphi(s)$ such that

$$N = \cosh \varphi \xi + \sinh \varphi (\xi \times T).\tag{3.6}$$

By $\langle N, N \rangle = -1$, we get that

$$N' = aT + bB,$$

where a and b are smooth functions of s .

By $B = T \times N$ and $T' = -\kappa N$, we get

$$B' = T \times N.$$

Since (T, N, B) is positively oriented then (B, T, N) also. So we have

$$\begin{cases} B \times T &= -N \\ T \times N &= B \\ N \times B &= T. \end{cases}$$

Thus we have $B' = bN$. And we have also

$$\begin{aligned}\langle N', B \rangle &= b \\ &= -\langle N, B' \rangle \\ &= -\tau,\end{aligned}$$

and

$$\begin{aligned}\langle N', T \rangle &= a \\ &= -\langle N, T' \rangle \\ &= -\langle N, -\kappa N \rangle \\ &= -\kappa.\end{aligned}$$

Thus we get the Frenet equations

$$\begin{cases} T' &= -\kappa N \\ N' &= -\kappa T - \tau B \\ B' &= -\tau N. \end{cases}\tag{3.7}$$

Now let us find the analogous equation for the trihedron $(T, \xi \times T, \xi)$.

From (3.2), (3.3), (3.6) and $\langle \xi, \xi \rangle = -1$, we have

$$\xi' = xT + \tau_g(\xi \times T).$$

But

$$\begin{aligned}
 \langle \xi', T \rangle &= x \\
 &= -\langle \xi, T' \rangle \\
 &= -\kappa_n \\
 &= -\langle \xi, -\kappa N \rangle \\
 &= \kappa \langle \xi, N \rangle \\
 &= -\kappa \cosh \varphi.
 \end{aligned}$$

Therefore we have

$$\xi = -\kappa \cosh \varphi T + \tau_g (\xi \times T).$$

An easy computation shows that

$$(\xi \times T)' = \kappa \sinh \varphi T + \tau_g \xi.$$

We summarize this by

$$\begin{cases} T' &= -\kappa N \\ \xi' &= -\kappa_n T + \tau(\xi \times T) \\ (\xi \times T)' &= \kappa \sinh \varphi T + \tau_g \xi. \end{cases} \quad (3.8)$$

We have that

$$\begin{cases} N = \cosh \varphi \xi + \sinh \varphi (\xi \times T) \\ N' = -\kappa T - \tau B. \end{cases} \quad (3.9)$$

With the equation $B = T \times N$ we get

(i)

$$\begin{aligned}
 N' &= -\kappa T - \tau(T \times (\cosh \varphi \xi + \sinh \varphi (\xi \times T))) \\
 &= -\kappa T + \tau \cosh \varphi \xi \times T + \tau \sinh \varphi \xi,
 \end{aligned}$$

by $T \times (\xi \times T) = -\xi$.

By differentiating the first equation of (3.9) and using (3.8) we get

(ii)

$$N' = -\kappa T + \sinh \varphi (\varphi' + \tau_g) \xi + \cosh \varphi (\varphi' + \tau_g) (\xi \times T)$$

By differentiating (i) and (ii) we get

$$\varphi' = \tau - \tau_g. \quad (3.10)$$

We have found that the normal curvature $\kappa_n = \kappa \cosh \varphi$. Then

$$\left(\frac{\kappa_n}{\kappa} \right) = \cosh \varphi. \quad (3.11)$$

Differentiating this equation, we get $\left(\frac{\kappa_n}{\kappa} \right)' = \varphi' \sinh \varphi$; and by (3.10), we obtain

$$\left(\frac{\kappa_n}{\kappa} \right)' = (\tau - \tau_g) \left(\pm \sqrt{\cosh^2 \varphi - 1} \right) \implies \left(\frac{\kappa_n}{\kappa} \right)' = (\tau - \tau_g) \left(\sqrt{\left(\frac{\kappa_n}{\kappa} \right)^2 - 1} \right).$$

From this last equation, we get

$$\left[\left(\frac{1}{\tau - \tau_g} \right) \left(\frac{\kappa_n}{\kappa} \right)' \right]' - (\tau - \tau_g) \left(\frac{\kappa_n}{\kappa} \right) = 0. \quad (3.12)$$

We will soon come back to equation (3.12).

- **Case 2:** T' is spacelike.

Thus we have

$$\begin{cases} N = \sinh \varphi \xi + \cosh \varphi (\xi \times \tau) \\ T' = \kappa N \end{cases} \quad (3.13)$$

and as before we put $N' = a\tau + bB$ and $B = N \times T$, for some smooth functions a, b of s .

We get $B' = N' \times T = bB$. The fact B is timelike and the Frenet frame (T, N, B) is positively oriented then $B \times T = N$.

Thus

$$B' = bN; \quad \tau = \langle B', N \rangle = b. \quad (3.14)$$

We have also,

$$a = \langle N, \tau \rangle = -\langle \tau, N \rangle.$$

Thus we have the Frenet equations

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = \tau N \end{cases} \quad (3.15)$$

By the same method as in the case 1, we find that the trihedron $(T, \xi \times T, \xi)$ satisfies the equations

$$\begin{cases} \tau' = \kappa \sinh \varphi \xi + \kappa \cosh \varphi \times T \\ \xi = \kappa \sinh \varphi T + \tau_g (\xi \times T) \\ (\xi \times T)' = -\kappa \cosh \varphi T + \tau_g \xi \end{cases} \quad (3.16)$$

with $\kappa_n = -\kappa \sinh \varphi$.

By using the relation $\frac{\kappa_n}{\kappa} = -\sinh \varphi$, one gets that. And we can obtain also

$$\varphi' = \tau - \tau_g. \quad (3.17)$$

Using (3.17) with $\frac{\kappa_n}{\kappa} = -\sinh \varphi$, we get

$$\left(\frac{1}{\tau - \tau_g} \right) \left(\frac{\kappa_n}{\kappa} \right)' + (\tau - \tau_g) \left(\frac{\kappa_n}{\kappa} \right) = 0. \quad (3.18)$$

The equation (3.12), (3.18) have the following form

$$[p(s)y'(s)]' \pm |q(s)|y = 0. \quad (3.19)$$

This equation has been already studied by the author in [2] and [7].

Applying their results we state that the solutions $\left(\frac{\kappa_n}{\kappa} \right)$ of (3.18) and (3.12) have the following solution

$$\frac{\kappa_n}{\kappa} = A_1 \cosh \int (\tau - \tau_g) ds + A_2 \sinh \int (\tau - \tau_g) ds, \quad (3.20)$$

where A_1 and A_2 are constants in some interval $[s_1, s_2] \subset [0, L]$ on which $\tau - \tau_g > 0$.

The case of Euclidean space obtain by *ii*) of the theorem 1.1.

This end the proof.

Example 3.1. The Euclidean case:

Consider the right cylinder H parametrized by

$$f(s, t) = (\cos(s), \sin(s), t)$$

and the helix α in H given by

$$\alpha(s) = \left(\cos\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2}s \right).$$

By computing $T = \alpha'$ and T' , we find $\kappa = \frac{1}{2}$. An easy computation show that

$$\xi = f_s \times f_t = (\cos s, \sin s, 0)$$

and

$$\kappa_n = \langle T', \xi \rangle = -\frac{1}{2} \cos\left(\frac{2 - \sqrt{2}}{2}s\right).$$

Thus $\frac{\kappa_n}{\kappa} = -\cos\left(\frac{2 - \sqrt{2}}{2}s\right)$ which satisfy the differential equation

$$\left(\frac{\kappa_n}{\kappa}\right)'' + \left(\frac{\kappa_n}{\kappa}\right) = 0.$$

Then we have the condition of 2) of the theorem 1.3.

Example 3.2. The Minkowski case:

Consider the ruled surface S parametrized by

$$g(s, t) = (t \cosh(s), \sqrt{2}s, t \sinh(s)).$$

The surface S is spacelike if $-\sqrt{2} < t < \sqrt{2}$. The curve α in S given by

$$\alpha(s) = \left(\cosh(s), \sqrt{2}s, \sinh(s) \right)$$

lies in the surface S . By computing $T = \alpha'$ and T' , we find $\kappa = 1$. An easy computation show that

$$\xi = (\sqrt{2} \sinh s, -1\sqrt{2} \cosh s).$$

The same computation in the Euclidean case shows that $\frac{\kappa_n}{\kappa}$ satisfies the differential equation

$$\left(\frac{\kappa_n}{\kappa}\right)'' - \left(\frac{\kappa_n}{\kappa}\right) = 0.$$

Then we have the condition of 1) of the theorem 1.3.

REFERENCES

- [1] M. Berger et B. Gostiaux, *Géométrie différentielle: variétés, courbes et surfaces*, Edition PUF, (2013).
- [2] S. Breuer and D. Gottlieb, Explicit characterization of spherical curves, *Proc. Amer. Math. Soc.* 27 (1971), 126-127.
- [3] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976. viii+503 pp. 190-191.
- [4] A. Niang and A. Ndiaye, A note for a generalization of the differential equation of the spherical curves, *Facta Univ. Ser. Math. Inform.* 36 (4) 801-808, 2021.
- [5] M. Turgut and A.T. Ali, Some Characterizations of Special Curves in Euclidean Space \mathbb{E}^4 , *Acta Univ. Sapientiae, Mathematica*, 2 (2010), pp:111-122.
- [6] J. Walrave, *Curves and Surfaces in Minkowski Space*, Doctoral Thesis, K.U. Leuven, Fac. of Science, Leuven (1995).
- [7] Y. C. Wong, On an explicit characterization of spherical curves. *Proc. Amer. Math. Soc.* 34 (1972), 239?242.

- [8] S. Yilmaz, M. Turgut, On the Differential Geometry of the Curves in Minkowski Space-time I, Int. J. Contemp. Math. Science, 3(2008), pp:1343-1349.
- [9] M. Spivak, A Comprehensive Introduction to Differential Geometry, Publish or Perish, Inc., Vol 3, 3.Edition (1999).

ATHOUMANE NIANG

DÉPARTEMENT DE MATHÉMATIQUES ET INFORMATIQUE, FST, UCAD, DAKAR, SENEGAL.

Email address: `athoumane.niang@ucad.edu.sn`

AMETH NDIAYE

DÉPARTEMENT DE MATHÉMATIQUES, FASTEF, UCAD, DAKAR, SENEGAL.

Email address: `ameth1.ndiaye@ucad.edu.sn`