Journal of The Tensor Society (J.T.S.) ISSN: 0974-5424
Vol. 17 (2023), page 54-59

On the Lanczos Potential

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Abstract

For arbitrary geometries of Petrov types III, N and O we construct the Lanczos potential for the corresponding Weyl tensor. This will provide a technique to construct Lanczos potential in non-vacuum cases.

Keywords: Conformal tensor, Lanczos generator, Canonical null tetrad, Petrov classification, Newman-Penrose formalism, 2-spinors.

1 Introduction

The Lanczos potential $K_{\mu\nu\alpha}$ [1, 2, 3, 4, 5, 6, 7, 8] satisfies the algebraic symmetries:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \quad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0, \quad K^{\mu\nu}{}_{\nu} = 0 \tag{1.1}$$

and it generates the Weyl tensor [9, 10] via the expression [11]:

$$\begin{aligned}-C_{\mu\nu\alpha\beta} &= K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\nu;\mu} + \frac{1}{2}[(K_{\mu\beta} + K_{\beta\mu})g_{\nu\alpha} + (K_{\nu\alpha} + K_{\alpha\nu})g_{\mu\beta} \\ &- (K_{\mu\alpha} + K_{\alpha\mu})g_{\nu\beta} - (K_{\nu\beta} + K_{\beta\nu})g_{\mu\alpha}] + \frac{2}{3}K^{\lambda\sigma}{}_{\lambda;\sigma}, \quad \text{where } K_{\mu\nu} = K_{\mu\sigma\nu}{}^{;\sigma} - K_{\mu\sigma}{}^{\sigma}{}_{;\nu}\end{aligned} \tag{1.2}$$

Though the introduction of such a tensor provides an analogy of gravitation and electromagnetism, the possible physical meaning of $K_{\mu\nu\alpha}$ in general relativity, is an open problem. Here we exhibit the structure of the Lanczos generator for arbitrary spacetimes of Petrov types O, N and III [12], via the Newman-Penrose technique [9, 10, 13, 14] and tensor and spinor formalisms [15]. Given a geometry we must construct the corresponding Lanczos potential, which is difficult in the general case except for certain Petrov types, for example, if in (1.2) we use the expression:

$$K_{\mu\nu\alpha} = N_{\mu}g_{\nu\alpha} - N_{\nu}g_{\mu\alpha}, \tag{1.3}$$

in terms of the metric tensor with N_{μ} arbitrary, we obtain $C_{\mu\nu\alpha\beta} = 0$, that is, (1.3) is a Lanczos generator for any spacetime type O.

This paper is dedicated to late Prof. Oscar Chavoya Aceves (1957-2023) for his work in the field of Relativity theory.

2 Algebraic symmetries of the Lanczos potential

If $a_{\mu\nu\alpha}$ is an arbitrary tensor, then the following tensor [16]:

$$\tilde{K}_{\mu\nu\alpha} = a_{\mu\nu\alpha} - a_{\nu\mu\alpha} + a_{\alpha\nu\mu} - a_{\alpha\mu\nu} \tag{2.1}$$

satisfies the symmetries:

$$\tilde{K}_{\mu\nu\alpha} = -\tilde{K}_{\nu\mu\alpha}, \quad \tilde{K}_{\mu\nu\alpha} + \tilde{K}_{\nu\alpha\mu} + \tilde{K}_{\alpha\mu\nu} = 0, \tag{2.2}$$

where we may employ:

$$a_{\mu\nu\alpha} = \frac{1}{3} F_{\mu\nu;\alpha}, \quad F_{\mu\nu} = -F_{\nu\mu}, \tag{2.3}$$

whose application in (4) gives the expression:

$$\tilde{K}_{\mu\nu\alpha} = \frac{1}{3}(2F_{\mu\nu;\alpha} + F_{\alpha\nu;\mu} - F_{\alpha\mu;\nu}), \tag{2.4}$$

and if now we ask the condition $\tilde{K}_{\mu\nu\alpha} = 0$, then from (2.4) we obtain a tensor verifying the algebraic symmetries (1.1) of the Lanczos potential [11]:

$$K^{\mu\nu\alpha} = \frac{1}{3}(2F^{\mu\nu;\alpha} + F^{\alpha\nu;\mu} - F^{\alpha\mu;\nu} + F^{\nu\lambda}{}_{;\lambda} g^{\alpha\mu} - F^{\mu\lambda}{}_{;\lambda} g^{\alpha\nu}), \tag{2.5}$$

which we shall use in our study.

If in (4) we employ the option $a_{\mu\nu\alpha} = F_{\mu\nu} k_\alpha, F_{\mu\nu} = -F_{\nu\mu}$ with the properties:

$$F_{\mu\nu} k_\alpha + F_{\nu\alpha} k_\mu + F_{\alpha\mu} k_\nu = 0, \quad \text{where } F_{\mu\nu} k^\nu = 0, \tag{2.6}$$

and the trace is eliminated, then we obtain that $K_{\mu\nu\alpha} \propto F_{\mu\nu} k_\alpha$, which allows construction of the Lanczos generator for plane gravitational waves [spacetime of Petrov type N] [17].

3 Lanczos potential for arbitrary 4-geometries with Petrov types O, N and III

If we select (2.5) for arbitrary $F_{\mu\nu} = -F_{\nu\mu}$, then (1.2) implies the relation:

$$C_{\mu\nu\alpha\beta} = C_{\sigma\nu\alpha\beta} F^\sigma{}_\mu - C_{\sigma\mu\alpha\beta} F^\sigma{}_\nu + C_{\sigma\beta\mu\nu} F^\sigma{}_\alpha - C_{\sigma\alpha\mu\nu} F^\sigma{}_\beta \tag{3.1}$$

which identically vanishes for any conformally flat space, that is, (2.5) is a Lanczos potential for any spacetime with Petrov type O.

Now we consider two Petrov types in the canonical null tetrad [9, 13, 18], for certain $F_{\mu\nu}$:

(a) Type N:

$$C_{\mu\nu\alpha\beta} = \psi_4 V_{\mu\nu} V_{\alpha\beta} + \bar{\psi}_4 \bar{V}_{\mu\nu} \bar{V}_{\alpha\beta}, \tag{3.2}$$

where $V_{\mu\nu} = l_\mu m_\nu - l_\nu m_\mu$ and

$$F_{\alpha\beta} = q(n_\alpha l_\beta - n_\beta l_\alpha). \tag{3.3}$$

Thus, (3.1) takes the form $C_{\mu\nu\alpha\beta} = 2q C_{\mu\nu\alpha\beta}$, therefore (2.5) is a Lanczos potential with (3.3) for $q = 1/2$.

(b) Type III:

$$C_{\mu\nu\alpha\beta} = \psi_3(V_{\mu\nu}M_{\alpha\beta} + M_{\mu\nu}V_{\alpha\beta}) + \bar{\psi}_3(\bar{V}_{\mu\nu}\bar{M}_{\alpha\beta} + \bar{M}_{\mu\nu}\bar{V}_{\alpha\beta}), \tag{3.4}$$

where $M_{\mu\nu} = m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu + n_\mu l_\nu - n_\nu l_\mu$. Then (3.1), (3.3) and (3.4) give the relation $C_{\mu\nu\alpha\beta} = qC_{\mu\nu\alpha\beta}$, that is, (2.5) is a Lanczos generator with (3.3) for $q = 1$.

Hence, the Lanczos potential for arbitrary Petrov types N and III spacetimes has the structure (2.5) if we employ the corresponding canonical null tetrad and $F_{\mu\nu}$ is given by (3.3) with $q = 1/2$ and $q = 1$, respectively; in Petrov type O geometries, we can use (2.5) with any $F_{\alpha\beta}$. The construction of $K_{\mu\nu\alpha}$ for arbitrary 4-spaces of types I, II, and D, is an open problem.

4 About Lanczos Spinor

From (2.5):

$$S_{\mu\nu\alpha} = K_{\mu\nu\alpha} + i\ {}^*K_{\mu\nu\alpha} = \frac{1}{3}(2S_{\mu\nu;\alpha} + S_{\alpha\nu;\mu} - S_{\alpha\mu;\nu} + S_{\nu\lambda}{}^{;\lambda} g_{\alpha\mu} - S_{\mu\lambda}{}^{;\lambda} g_{\alpha\nu}), \tag{4.1}$$

such that $S_{\mu\nu} = F_{\mu\nu} + i\ {}^*F_{\mu\nu}$, with the participation of the dual tensor:

$${}^*K_{\mu\nu\alpha} = \frac{1}{2}\eta_{\mu\nu\lambda\beta}K^{\lambda\beta}{}_{\alpha} \quad \text{and} \quad {}^*F_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu\lambda\beta}F^{\lambda\beta} \tag{4.2}$$

From (4.1) the corresponding Lanczos spinor [19, 20, 21, 22, 23] is given by:

$$3L_{(}ABC\dot{D}) = \nabla_{A\dot{D}}\ \varphi_{BC} + \nabla_{B\dot{D}}\ \varphi_{CA} + \nabla_{C\dot{D}}\ \varphi_{AB}, \tag{4.3}$$

which implies the following equations in the Newman-Penrose (NP) formalism [9, 13, 14, 21, 24]:

$$\begin{aligned}
&\Omega_0 = D\phi_0 + 2(-\epsilon\phi_0 + \kappa\phi_1), && 3\Omega_1 = \bar{\delta}\phi_0 + 2[D\phi_1 - (\alpha+\pi)\phi_0 + \rho\phi_1 + \kappa\phi_2],\\
&\Omega_3 = \bar{\delta}\phi_2 + 2(-\lambda\phi_1 + \alpha\phi_2), && 3\Omega_2 = D\phi_2 + 2[\bar{\delta}\phi_1 - \lambda\phi_0 - \pi\phi_1 + (\rho+\epsilon)\phi_2],\\
&\Omega_4 = \delta\phi_0 + 2(-\beta\phi_0 + \sigma\phi_1), && 3\Omega_5 = \Delta\phi_0 + 2[\delta\phi_1 - (\gamma+\mu)\phi_0 + \tau\phi_1 + \sigma\phi_2],\\
&\Omega_7 = \Delta\phi_2 + 2(-\nu\phi_1 + \gamma\phi_2), && 3\Omega_6 = \delta\phi_2 + 2[\Delta\phi_1 - \nu\phi_0 - \mu\phi_1 + (\beta+\tau)\phi_2],
\end{aligned} \tag{4.4}$$

for the NP components of Lanczos potential in terms of the spin coefficients and the NP projections of $F_{\mu\nu}$.

The work [25] used the canonical null tetrad [9, 18, 26] to determine the NP components of $K_{\mu\nu\alpha}$, that is, a solution of the Weyl-Lanczos equations [8, 20, 21, 25, 27] thus:

$$\begin{aligned}
&\Omega_0 = qk, && \Omega_3 = -q\lambda, && \Omega_4 = q\sigma, && \Omega_7 = -q\nu\\
&\Omega_1 = \frac{q}{3}\rho, && \Omega_2 = -\frac{q}{3}\pi, && \Omega_5 = \frac{q}{3}\tau, && \Omega_6 = -\frac{q}{3}\mu
\end{aligned} \tag{4.5}$$

for arbitrary spacetimes with Petrov types III and N for $q = 1$ and $q = 1/2$, respectively. It is easy to see that the relations (4.4) imply (4.5) if $\phi_0 = \phi_2 = 0$ and $\phi_1 = \frac{q}{2}$, hence $F_{\mu\nu}$ has the structure (3.3) and $S_{\mu\nu} = qM_{\mu\nu}$.

5 Conclusion

Due to non-linearity in Weyl-Lanczos relations, their solution is not unique and it becomes difficult to solve them for spacetimes of general nature. In this work we have derived Lanczos potential for metric of Petrov types III, N and O. It is pending to determine the Lanczos potential for Petrov types I, II and D in the general case, however, for these spacetimes $K_{\mu\nu\alpha}$ has been constructed for specific metrics [28, 29, 4, 5, 6, 8, 30, 31, 32, 33, 34, 35] and in almost all of these situations the NP components Ω_r have turned out to be linear combinations of the spin coefficients if we properly choose the corresponding null tetrad. Equation (2.5) may be seen as interaction between gravitational field in vacuum and electromagnetic field, which requires further investigation. There are various proposals [30, 31, 36, 37, 38] for the possible physical meaning of the Lanczos spintensor, which we consider important because $K_{\mu\nu\alpha}$ is present in all Riemannian 4-geometries and therefore it is necessary to know its role in the description of the gravitational field.

Acknowledgments
RP is thankful to Sardar Patel University for providing financial support under SEED Money Grant.

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