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A Note on Affine Motion in a Birecurrent Finsler Space

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Abstract

Several authors discussed affine motion generated by contra, concurrent, special concircular, recurrent, concircular and torse forming vector fields in special spaces such as recurrent, birecurrent and symmetric Riemannian and Finsler spaces. The first author [20-22] for the first time obtained the necessary and sufficient conditions for the above vector fields to generate an affine motion in a general Finsler space. Recently Surendra Pratap Singh [26] discussed affine motion in a birecurrent Finsler space. The aim of this paper is to generalize the results of Surendra Pratap Singh.

Keywords and Phrases : Recurrent Finsler space, Birecurrent Finsler space, Contra vector field, Concurrent vector field, Affine motion.
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1. Introduction

Takano [27-28] studied certain types of affine motion generated by contra, concurrent, special concircular, torse forming and birecurrent vectors in non-Riemannian manifold of recurrent curvature. Following the techniques of Takano, the authors Sinha [25], Misra [5-7], Misra and Meher [8-10], Meher [4] and Kumar [1-3] studied the above mentioned types of affine motion in Finsler space of recurrent curvature and obtained various results. The first author obtained the necessary and sufficient conditions for above vector fields to generate an affine motion in a general Finsler space. Surendra Pratap Singh [26] discussed affine motion in birecurrent Finsler space. In the present paper we have generalized certain results of Surendra Pratap Singh and highlighted some results which are either trivial or meaningless in the aforesaid paper.

2. Preliminaries

Let $F_n\,(F, g, G)$ be an $n-$dimensional Finsler space of class at least C^7 equipped with metric function F, corresponding symmetric metric tensor g and Berwald's connection G. Connection coefficients of Berwald satisfy

(2.1) (a) $G^i_{j\,k} = G^i_{k\,j}$, (b) $G^i_{j\,k}\dot{x}^k = G^i_j$, (c) $\dot{\partial}_k G^i_j = G^i_{k\,j}$,

where $\dot{\partial}_k \equiv \frac{\partial}{\partial \dot{x}^k}$.

$G^i_{jk\,h} = \dot{\partial}_h G^i_{jk}$ constitute a tensor which are symmetric in its lower indices and satisfy

$$G^i_{jkh}\dot{x}^h = G^i_{khj}\dot{x}^h = G^i_{hjk}\dot{x}^h = 0. \tag{2.2}$$

The covariant derivative $\mathcal{B}_k\, T^i_j$ of an arbitrary tensor T^i_j for the connection G is given by

$$\mathcal{B}_k\, T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\, G^r_k + T^r_j\, G^i_{rk} - T^i_r\, G^r_{jk}, \tag{2.3}$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$.

The operator $\mathcal{B}_k$ commutes with $\dot{\partial}_k$ and itself as follows

$$(\dot{\partial}_j \mathcal{B}_k - \mathcal{B}_k \dot{\partial}_j)\, T^i_h = T^r_h\, G^i_{jkr} - T^i_r\, G^r_{jkh}, \tag{2.4}$$

$$(\mathcal{B}_j \mathcal{B}_k - \mathcal{B}_k \mathcal{B}_j)\, T^i_h = T^r_h\, H^i_{jkr} - T^i_r\, H^r_{jkh} - (\dot{\partial}_r T^i_h)\, H^r_{jk}, \tag{2.5}$$

where H^i_{jkh} constitute Berwald's curvature tensor given by

$$H^i_{jkh} = \partial_j G^i_{hk} - \partial_k G^i_{hj} + G^r_{hk}\, G^i_{jr} - G^r_{hj}\, G^i_{rk} + G^i_{rhj}\, G^r_k - G^i_{rhk}\, G^r_j. \tag{2.6}$$

This tensor is anti-symmetric in first two lower indices and is positively homogeneous of degree zero in $\dot{x}^i$. The tensor H^i_{jk} appearing in (2.5) is related with the curvature tensor as

(2.7)(a) $H^i_{jkh}\, \dot{x}^h = H^i_{jk}$, (b) $\dot{\partial}_h\, H^i_{jk} = H^i_{jkh}$,

and with deviation tensor H^i_j as

(2.8)(a) $H^i_{jk}\, \dot{x}^k = H^i_j$,

(b) $\frac{1}{3}(\dot{\partial}_k H^i_j - \dot{\partial}_j H^i_k) = H^i_{jk}$.

The associate vector y_i of $\dot{x}^i$ satisfies the relations [18]

(2.9)(a)
$$y_i \dot{x}^i = F^2, \qquad (b) \qquad y_i H^i_{jk} = 0, \qquad (c) \qquad g_{ik} H^i_{mj} + y_i H^i_{mjk} = 0,$$

where g_{ij} are components of metric tensor g.

The curvature tensor fields satisfy the following Bianchi identities [23]

(2.10) $$\mathcal{B}_l H^i_{jkh} + \mathcal{B}_j H^i_{klh} + \mathcal{B}_k H^i_{ljh} + H^r_{jk} G^i_{rlh} + H^r_{kl} G^i_{rjh} + H^r_{lj} G^i_{rkh} = 0,$$

(2.11) $$\mathcal{B}_l H^i_{jk} + \mathcal{B}_j H^i_{kl} + \mathcal{B}_k H^i_{lj} = 0,$$

(2.12) $$\mathcal{B}_l H^i_k - \mathcal{B}_k H^i_l + (\mathcal{B}_r H^i_{kl}) \dot{x}^r = 0.$$

Let us consider the infinitesimal transformation

(2.13) $$\bar{x}^i = x^i + \varepsilon v^i(x^j),$$

generated by a vector field $v^i(x^j)$, ε being an infinitesimal constant. The Lie derivatives of an arbitrary tensor T^i_j and the connection coefficients G^i_{jk} with respect to (2.13) are given by [29]

(2.14) $$\pounds T^i_j = v^r \mathcal{B}_r T^i_j - T^r_j \mathcal{B}_r v^i + T^i_r \mathcal{B}_j v^r + (\dot{\partial}_r T^i_j) \mathcal{B}_s v^r \dot{x}^s,$$

(2.15) $$\pounds G^i_{jk} = \mathcal{B}_j \mathcal{B}_k v^i + H^i_{mjk} v^m + G^i_{jkr} \mathcal{B}_s v^r \dot{x}^s.$$

The operator $\pounds$ commutes with the operators $\mathcal{B}_k$ and $\dot{\partial}_k$ according as

(2.16) $$(\pounds \mathcal{B}_k - \mathcal{B}_k \pounds) T^i_j = T^r_j \pounds G^i_{rk} - T^i_r \pounds G^r_{jk} - (\dot{\partial}_r T^i_j) \pounds G^r_k,$$

(2.17) $$(\dot{\partial}_k \pounds - \pounds \dot{\partial}_k)\Omega = 0,$$

where Ω is a vector, tensor or connection coefficients.

The infinitesimal transformation (2.13) defines an affine motion if it preserves parallelism of pair of vectors. The necessary and sufficient condition for the vector $v^i(x^j)$ to generate an affine motion is that [29]

(2.18) $$\pounds G^i_{jk} = 0.$$

Since the curvature tensor is Lie invariant with respect to an affine motion, in this case we have

(2.19) $$\pounds H^i_{jkh} = 0.$$

The vector field v^i is called contra and concurrent vector field according as it satisfies [27]

$$(2.20)(a) \qquad \mathcal{B}_k v^i = 0, \qquad (b) \qquad \mathcal{B}_k v^i = \lambda \delta^i_k,$$

λ being a constant.

The affine motion generated by the above vector fields is called a contra affine motion and a concurrent affine motion, respectively.

3. Special Finsler Spaces

A non-flat Finsler space F_n is called a recurrent Finsler space if the curvature tensor satisfies

$$\mathcal{B}_l H^i_{jkh} = K_l H^i_{jkh}, \tag{3.1}$$

where K_l is a non-zero vector field [2-4, 6-9, 16, 17, 25]. Pandey [17] proved that the recurrence vector K_l is independent of $\dot{x}^i$, in general.

Following identities are satisfied in a recurrent space [17]:

$$K_l H^i_{jkh} + K_k H^i_{ljh} + K_j H^i_{klh} = 0, \tag{3.2}$$

$$K_l H^i_{jk} + K_k H^i_{lj} + K_j H^i_{kl} = 0, \tag{3.3}$$

$$H^r_{[jk} G^i_{l]m\,r} = 0, \tag{3.4}$$

where square bracket shows the skew-symmetric part with respect to the indices enclosed in it.

A non-flat Finsler space F_n is called a birecurrent Finsler space if the curvature tensor satisfies the relation

$$\mathcal{B}_l \mathcal{B}_m H^i_{jkh} = A_{l\,m} H^i_{jkh}, \tag{3.5}$$

where $A_{l\,m}$ is a non-zero tensor field, called birecurrence tensor field [1, 5, 12].

A birecurrent Finsler space satisfies the following:

$$A_{lm} H^i_{jk} + A_{lk} H^i_{mj} + A_{lj} H^i_{km} = 0. \tag{3.6}$$

We may also define an r-recurrent Finsler space characterized by the condition

$$\mathcal{B}_{l_1} \mathcal{B}_{l_2} \cdots \mathcal{B}_{l_r} H^i_{j\,k\,h} = A_{l_1\,l_2 \cdots l_r} H^i_{jkh}. \tag{3.7}$$

In view of Bianchi identities, the tensor field H^i_{jk} satisfies

$$A_{l_1 l_2 \cdots l_{r-1} l_r} H^i_{jk} + A_{l_1 l_2 \cdots l_{r-1} k} H^i_{l_r j} + \cdots = 0. \tag{3.8}$$

4. Affine Motion in a Birecurrent Finsler Space F_n

Let us consider a Finsler space F_n admitting the affine motion (2.13). Then, we have (2.18) and (2.19). In view of the commutation formula exhibited by (2.16) and the equation (2.18), we find that the operators of covariant differentiation $\mathcal{B}_k$ and Lie-differentiation $\pounds$ are commutative for an arbitrary tensor $T^{\cdots}_{\cdots}$ of any order, i.e.

$$\pounds\, \mathcal{B}_m T^{\cdots}_{\cdots} = \mathcal{B}_m \pounds\, T^{\cdots}_{\cdots}. \tag{4.1}$$

In particular,

$$\begin{aligned} &\pounds\, \mathcal{B}_m H^i_{jkh} = \mathcal{B}_m \pounds\, H^i_{jkh}, \\ &\pounds\, \mathcal{B}_l \mathcal{B}_m H^i_{jkh} = \mathcal{B}_l \pounds\, \mathcal{B}_m H^i_{jkh} = \mathcal{B}_l \mathcal{B}_m \pounds H^i_{jkh}, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ &\pounds\, \mathcal{B}_{m_1} \mathcal{B}_{m_2} \cdots \mathcal{B}_{m_r} H^i_{jkh} = \mathcal{B}_{m_1} \mathcal{B}_{m_2} \cdots \mathcal{B}_{m_r} \pounds\, H^i_{jkh} \end{aligned} \tag{4.2}$$

which, in view of (2.19), give

$$\begin{aligned} &\pounds\, \mathcal{B}_m H^i_{jkh} = 0, \\ &\pounds\, \mathcal{B}_l \mathcal{B}_m H^i_{jkh} = 0, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ &\pounds\, \mathcal{B}_{m_1} \mathcal{B}_{m_2} \cdots \mathcal{B}_{m_r} H^i_{jkh} = 0. \end{aligned} \tag{4.3}$$

In view of (4.3), for a recurrent space, a birecurrent space and an r - recurrent space, we have

$$\pounds\, K_m = 0, \tag{4.4}$$

$$\pounds\, A_{lm} = 0 \tag{4.5}$$

and

$$\pounds\, A_{m_1 m_2 \ldots m_r} = 0, \tag{4.6}$$

respectively.

Singh [26] considered a special birecurrent Finsler space (though he did not use the word "special") whose recurrence tensor A_{lm} is of the form

$$A_{lm} = \mathcal{B}_m K_l + K_m K_l. \tag{4.7}$$

He discussed affine motion in such space and obtained the following theorems :

Theorem 1. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the Lie-derivative of the recurrence tensor field A_{lm} satisfies the relation $\pounds A_{lm} = \pounds \mathcal{B}_m K_l$.

Theorem 2. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the recurrence tensor A_{lm} satisfies the identity $\pounds \mathcal{B}_n A_{[lm]} + \pounds \mathcal{B}_l A_{[mn]} + \pounds \mathcal{B}_m A_{[nl]} = 0$.

Theorem 3. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the recurrence tensor A_{lm} satisfies $\pounds (\dot{\partial}_r A_{[lm]}) = 0$.

Theorem 4. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion, the Bianchi identities satisfied by curvature tensor H^i_{jkh}, H^i_{jk} and H^i_k take the forms

$$(\pounds A_{ls}) \dot{x}^s H^i_{jkh} + (\pounds A_{ks}) \dot{x}^s H^i_{jhl} + (\pounds A_{hs}) \dot{x}^s H^i_{jlk} = 0,$$
$$(\pounds A_{ls}) H^i_{jk} + (\pounds A_{js}) H^i_{kl} + (\pounds A_{ks}) H^i_{lj} = 0$$

and

$$(\pounds A_{ls}) H^i_k - (\pounds A_{ks}) H^i_l + (\pounds A_{rs}) H^i_{kl} \dot{x}^r = 0,$$

respectively.

Theorem 5. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion in order that the vector field $v^i(x^j)$ spans a contra field, the relations $H^i_{sjk} v^s = 0$ and $H^i_{sjk} \pounds v^s = 0$ hold good.

Theorem 6. In a birecurrent Finsler space $\bar{F}_n$, which admits an affine motion in order that the vector filed $v^i(x^j)$ determines concurrent field the relations $H^i_{sjk} v^s = 0$ and $H^i_{sjk} \pounds v^s = 0$ are necessarily true.

In view of (4.5), Theorem 1 is not correct while the next three theorems (Theorem 2, Theorem 3 and Theorem 4) reduce to $0 = 0$.

The Lie-derivative of a tensor field T^i_j with respect to the infinitesimal transformation (2.13) is given by (2.14).

In particular,

$$\pounds v^i = v^r \mathcal{B}_r v^i + (\dot{\partial}_r v^i) \mathcal{B}_s v^r \dot{x}^s - v^r \mathcal{B}_r v^i = 0. \tag{4.8}$$

The main finding in Theorem 5 and Theorem 6 of Singh [26] is that a contra or concurrent vector field $v^i(x^j)$ generating an affine motion in the so called birecurrent Finsler space satisfies

(4.9) $$H^i_{sjk}\, \pounds\, v^s \;=\; 0.$$

In view of (4.8), it is trivial.

Pandey [20] proved that an infinitesimal transformation, generated by a contra vector field, is necessarily an affine motion in a general Finsler space. Therefore, it is an affine motion in a birecurrent Finsler space.

If a birecurrent Finsler space admits an infinitesimal transformation generated by a contra vector field $v^i(x^j)$, then the recurrence tensor A_{lm} satisfies (vide Pandey [20]):

(4.10)(a) $$A_{lm}\, v^m \;=\; 0, \qquad (b) \qquad A_{lm}\, v^l \;=\; 0.$$

In case of recurrence tensor $A_{l\,m}$ considered by Singh [26] above conditions become

(4.11)(a) $$(\mathcal{B}_m K_l \;+\; K_m\, K_l)\, v^m \;=\; 0,$$

(b) $$(\mathcal{B}_m K_l \;+\; K_m\, K_l)\, v^l \;=\; 0.$$

In view of (4.11a) and (4.11b), we have

(4.12)(a) $$v^m\, \mathcal{B}_m\, K_l \;=\; -(K_m\, v^m) K_l,$$

(b) $$\mathcal{B}_m\, (K_l\, v^l) \;=\; -(K_l\, v^l) K_m.$$

If we put $K_l\, v^l \;=\; L$, then (4.12a) and (4.12b) reduce to

(4.13)(a) $$v^m\, \mathcal{B}_m\, K_{\,l} \;=\; -L\, K_{\,l},$$

(b) $$\mathcal{B}_m\, L \;=\; -L\, K_m.$$

Using (2.14) for K_l and applying (2.20a), we have

(4.14) $$\pounds\, K_l = v^m\, \mathcal{B}_m\, K_l.$$

From (4.13a) and (4.14), we obtain

(4.15) $$\pounds\, K_l = \,-\, L K_l.$$

Thus, we have

Theorem 7. In a birecurrent Finsler space admitting an infinitesimal transformation generated by a contra vector field $v^i(x^j)$, if the birecurrence tensor A_{lm} is characterized by (4.7), then the vector K_l is Lie-recurrent.

Again, from (4.15), we observe that $\pounds K_l = 0$ if and only if $L = K_l v^l = 0$. Thus, we conclude that

Theorem 8. In a birecurrent Finsler space admitting an infinitesimal transformation generated by a contra vector field $v^i(x^j)$, if the birecurrence tensor A_{lm} is characterized by (4.7), then the necessary and sufficient condition for the vector K_l to be Lie-invariant is that K_l is orthogonal to the contra vector $v^i(x^j)$.

Pandey [20] proved that a birecurrent Finsler space does not admit any infinitesimal transformation generated by a concurrent vector field. Therefore, the study of a birecurrent Finsler space admitting a concurrent affine motion is wastage of precious time and is to indulge in unnecessary mechanical labour.

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