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COMMON FIXED POINT THEOREMS FOR C-CLASS FUNCTIONS IN b-METRIC SPACES WITH APPLICATION

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Abstract: In this paper, we establish the existence and uniqueness of common fixed point theorems for self mappings satisfying the common limit range property with respect to mappings S and T and common (E.A)-property via the concept of C-class functions in b-metric spaces. We furnish two examples to validate our results. Our results improve various results appeared in the current literature. As an application, We provide the existence of a solution of integral equations.

1. Introduction

In the development of nonlinear analysis, fixed point theory occupies a renowned place in many aspects. It has been used in different branches of engineering and sciences. In particular, the famous Banach contraction principle is very popular tool of mathematics to solve a problems in several branches of mathematics such as variational and linear inequalities and approximation theory. Sessa S. [22] introduced the concept of weakly commuting and G. Jungck [16] introduced the concept of compatibility, Jungck and Rhoades [17] introduced the notion of weak compatibility. Bakhtin [6] introduced the concept of b-metric spaces which is a generalization of metric spaces. After that, Czerwik [11, 12] defined it such as current structure which is consider a generalization of metric spaces. Several authors have been interested in investigating fixed point and common fixed point theorems for mappings in b-metric spaces [5, 7, 14, 23].

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Ansari [2] introduced the notion of C-class function as a major generalization of Banach contraction principle and obtained some fixed point results. Subsequently, many authors were interested in fixed point theorems for C-class function [4, 20]. Most recently, some authors obtained fixed point and common fixed point for C-class function [3, 8, 13, 21, 24].

2. Preliminaries

We recall some definitions which will be used in the sequel.

[6] Let X be a non-empty set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if for each x, y, $z \in X$, the following conditions are satisfied.

(1) $d(x, y) = 0$ iff $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

[9] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called

(1) b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(2) b-Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.

[10] Let (X, d) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$.

[10] The b-metric space (X, d) is complete if every b-Cauchy sequence in X is b-convergent.

[16] Let f and g be given self mappings on a set X. The pair (f, g) is said to be weakly compatible if f and g commute at their coincidence point (i.e $fgx = gfx$ whenever $fx = gx$).

[2] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:

(1) $F(s, t) \leq s$,
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Let us denote $\mathcal{C}$ the family of C-class functions.

Remark 2.1. [2] Clearly, for some F we have $F(0, 0) = 0$.

Example 2.2. [2] The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:

(1) $F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$;
(2) $F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s \Rightarrow s = 0$;
(3) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ is continuous, $F(s, t) = s \Rightarrow s = 0$;
(4) $F(s, t) = s - \varphi(s)$, $F(s, t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$.

[18] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied.

(1) ψ is nondecreasing and continuous,

(2) $\psi(t) = 0$ if and only if $t = 0$.

[2] An ultra altering distance function is a continuous nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0,\ t > 0$ and $\varphi(0) \geq 0$.

[25] Two self maps f and S of a metric space (X, d) are said to satisfy common limit range property with respect to S, denoted by (CLR_S) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \ where\ t \in S(X).$$

[15] Two pairs (f, S) and (g, T) of self mappings of a metric space (X, d) are said to satisfy common limit range property with respect to S and T, denoted (CLR_{ST}) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t \ where\ t \in S(X) \cap T(X).$$

[1] Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy $(E.A)$-property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,\ for\ some\ t \in X.$$

[19] Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy common $(E.A)$-property if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

In this paper, we establish the existence and uniqueness of common fixed point theorems for self mappings satisfying the common limit range property with respect to mappings S and T and common (E.A)-property via the concept of C-class functions in b-metric spaces. We furnish two examples to validate our results. Our results improve various results appeared in the current literature. We apply our result to the existence of a solution of an integral equation.

3. Main Result

Now, we state and prove our main results as follows: Let (X, d) be a b-metric space and $f, g, S, T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that

$$\psi(sd(fx, gy)) \leq F(\psi(M_s(x, y)), \varphi(M_s(x, y))),\ for\ all\ x, y \in X, \tag{3.1}$$

$$where\ M_s(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(fx, Ty) + d(Sx, gy)}{2s}\}.$$

Suppose that the pairs (f, S) and (g, T) satisfy the CLR_{ST}-property then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. If the pairs (f,S) and (g,T) satisfy the CLR_{ST}-property, then there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in S(X)\cap T(X)$. Since $z \in S(X)$, then there exists a point $p \in X$ such that $Sp = z$. Now, we show that $fp = z$. We put $x = p$ and $y = y_n$ in (3.1), we get

$$\begin{aligned}\psi(d(fp, gy_n)) &\le \psi(sd(fp, gy_n))\\ &\le F(\psi(M_s(p, y_n)), \varphi(M_s(p, y_n))),\end{aligned} \tag{3.2}$$

where

$$M_s(p, y_n) = \max\{d(Sp, Ty_n), d(fp, Sp), d(gy_n, Ty_n), \frac{d(fp, Ty_n) + d(Sp, gy_n)}{2s}\}$$

taking limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} M_s(p, y_n) &= \max\{d(z, z), d(fp, z), d(z, z), \frac{d(fp, z) + d(z, z)}{2s}\}\\ &= d(fp, z).\end{aligned}$$

Taking limit as $n \to \infty$ in (3.2) and definition of ψ and φ, we get

$$\psi(d(fp, z)) \le F(\psi(d(fp, z)), \varphi(d(fp, z))),$$

which implies $\psi(d(fp, z)) = 0$ or $\varphi(d(fp, z)) = 0$ which gives $fp = z$. Thus p is a coincidence point of the pair (f,S). Since the pair (f,S) is weakly compatible and $fp = Sp$,
therefore $fSp = Sfp$ which implies that $fz = Sz$.

Since $z \in T(X)$, then there exists a point $q \in X$ such that $Tq = z$. Now, we show that $gq = z$. We put $x = p$ and $y = q$ in (3.1), we get

$$\begin{aligned}\psi(d(fp, gq)) &\le \psi(sd(fp, gq))\\ &\le F(\psi(M_s(p, q)), \varphi(M_s(p, q))),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}M_s(p, q) &= \max\{d(Sp, Tq), d(fp, Sp), d(gq, Tq), \frac{d(fp, Tq) + d(Sp, gq)}{2s}\}\\ &= \max\{d(z, z), d(z, z), d(gq, z), \frac{d(z, z) + d(z, gq)}{2s}\}\\ &= d(z, gq).\end{aligned}$$

Thus, from (3.3) and definition of ψ and φ, we get

$$\psi(d(z, gq)) \le F(\psi(d(z, gq)), \varphi(d(z, gq))),$$

which implies $\psi(d(z, gq)) = 0$ or $\varphi(d(z, gq)) = 0$ which gives $gq = z$. Thus q is a coincidence point of the pair (g,T). Since the pair (g,T) is weakly compatible and $gq = Tq$, therefore $gTq = Tgq$ which implies that $gz = Tz$.

Now we show that z is a common fixed point of the pair (f, S). Putting $x = z$, $y = q$ in (3.1), we get

$$\begin{aligned}\psi(d(fz, gq)) &\leq \psi(sd(fz, gq))\\ &\leq F(\psi(M_s(z, q)), \varphi(M_s(z, q))),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}M_s(z, q) &= \max\{d(Sz, Tq), d(fz, Sz), d(gq, Tq), \frac{d(fz, Tq) + d(Sz, gq)}{2s}\}\\ &= \max\{d(fz, z), d(fz, Sz), d(z, z), \frac{d(fz, z) + d(fz, z)}{2s}\}\\ &= d(fz, z).\end{aligned}$$

Thus, from (3.4) and definition of ψ and φ, we get

$$\psi(d(fz, z)) \leq F(\psi(d(fz, z)), \varphi(d(fz, z))),$$

which implies $\psi(d(fz, z)) = 0$ or $\varphi(d(fz, z)) = 0$ which gives $fz = z$. Hence $fz = z = Sz$. Thus z is a common fixed point of the pair (f, S). Similarly, we can show that $gz = z = Tz$. Hence z is a common fixed point of f, g, S and T.

Now, we show that z is a unique common fixed point. Let t be another common fixed point of f, g, S and T. Putting $x = z,\ y = t$ in (3.1), we get

$$\begin{aligned}\psi(d(fz, gt)) &\leq \psi(sd(fz, gt))\\ &\leq F(\psi(M_s(z, t)), \varphi(M_s(z, t))),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}M_s(z, t) &= \max\{d(Sz, Tt), d(fz, Sz), d(gt, Tt), \frac{d(fz, Tt) + d(Sz, gt)}{2s}\}\\ &= \max\{d(z, t), d(z, z), d(t, t), \frac{d(z, t) + d(z, t)}{2s}\}\\ &= d(z, t).\end{aligned}$$

Thus, from (3.5) and definition of ψ and φ, we get

$$\psi(d(z, t)) \leq F(\psi(d(z, t)), \varphi(d(z, t))),$$

which implies $\psi(d(z, t)) = 0$ or $\varphi(d(z, t)) = 0$ which gives $z = t$. Hence z is a unique common fixed point of mappings f, g, S and T. □

Remark 3.1. If we put $\psi(t) = t$ in Theorem 3, we get the following result.

Let (X, d) be a b-metric space and $f, g, S, T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that

$$sd(fx, gy) \leq F(M_s(x, y), \varphi(M_s(x, y))),\ \textit{for all}\ x, y \in X,$$

where,

$$M_s(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(fx, Ty) + d(Sx, gy)}{2s}\}.$$

Suppose that the pairs (f, S) and (g, T) satisfy the CLR_{ST}-property then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs

(f,S) and (g,T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Remark 3.2. If we put $g=f$ and $S=T$ in Theorem 3, we get the following result.

Let (X,d) be a b-metric space and $f,T:X\to X$ be mappings with $f(X)\subseteq T(X)$ such that

$$\psi(sd(fx,fy))\le F(\psi(M_s(x,y)),\varphi(M_s(x,y))),\ \textit{for all}\ x,y\in X,$$

where,

$$M_s(x,y)=\max\{d(Tx,Ty),d(fx,Tx),d(fy,Ty),\frac{d(fx,Ty)+d(Tx,fy)}{2s}\}.$$

Suppose that the pair (f,T) satisfy the CLR_T-property then the pair (f,T) has a point of coincidence in X. Moreover, if the pair (f,T) is weakly compatible, then f and T have a unique common fixed point.

Now, we illustrate an example to validate our main Theorem 3.

Example 3.3. Consider $X=[0,12]$, $d(x,y)=\max\{x,y\}$, $\psi(t)=t$, $\varphi(t)=\frac{t}{10}$ and $F(s,t)=\frac{9}{10}s$. Then (X,d) is a b-metric space with constant $s=\frac{7}{5}$. Define mappings f, g, S and T on X such that

$$f(x)=\begin{cases}0, & \text{if } x\in\{0\}\cup(5,12];\\ 3, & \text{if } x\in(0,5];\end{cases}\qquad g(x)=\begin{cases}0, & \text{if } x\in\{0\}\cup(5,12];\\ 4, & \text{if } x\in(0,5];\end{cases}$$

$$S(x)=\begin{cases}0, & \text{if } x=0;\\ 7, & \text{if } x\in(0,5];\\ \frac{x+4}{4}, & \text{if } x\in(5,12];\end{cases}\qquad T(x)=\begin{cases}0, & \text{if } x=0;\\ 6, & \text{if } x\in(0,5];\\ \text{x-5}, & \text{if } x\in(5,12].\end{cases}$$

We take the sequence $x_n=\{0\}$ and $y_n=\{5+\frac{1}{n}\}$. We have

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}gy_n=\lim_{n\to\infty}Ty_n=0\in S(X)\cap T(X).$$

Therefore, both pairs (f,S) and (g,T) satisfy the (CLR_{ST})-property. We see that mappings (f,S) and (g,T) commute at 0 which is the coincidence point. Also, $f(x)\subseteq T(X)$ and $g(X)\subseteq S(X)$. We can verify the contraction condition (3.1) by a simple calculation for the case $x,y\in X$ as follows:

If $x,y\in(0,5]$, then

$$\psi(sd(fx,gy))=\frac{7}{5}\times 4\le\frac{9}{10}\times 7=\frac{9}{10}d(Sx,Ty)\le\frac{9}{10}\psi(M_s(x,y)).$$

Thus the contraction condition (3.1) is satisfied for $x,y\in(0,5]$. Similarly, we can verify for other cases. Thus all the conditions of Theorem 3 are satisfied and 0 is a unique common fixed point of mappings f, g, S and T.

Let (X,d) be a b-metric space and f, g, S, $T:X\to X$ be mappings with $f(X)\subseteq T(X)$ and $g(X)\subseteq S(X)$ satisfying the inequality (3.1). If the pairs (f,S) and (g,T) satisfy the common $(E.A)$-property and $S(X)$ or $T(X)$ is closed. Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. If the pairs (f, S) and (g, T) satisfy the common $(E.A)$-property, then there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$. Since $S(X)$ is closed, then there exists a point $p \in X$ such that $Sp = z$. Now, we show that $fp = z$. We put $x = p$ and $y = y_n$ in (3.1), we get

$$\begin{aligned}\psi(d(fp, gy_n)) &\leq \psi(sd(fp, gy_n)) \\ &\leq F(\psi(M_s(p, y_n)), \varphi(M_s(p, y_n))),\end{aligned} \tag{3.6}$$

where

$$M_s(p, y_n) = \max\{d(Sp, Ty_n), d(fp, Sp), d(gy_n, Ty_n), \frac{d(fp, Ty_n) + d(Sp, gy_n)}{2s}\}$$

taking limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} M_s(p, y_n) &= \max\{d(z, z), d(fp, z), d(z, z), \frac{d(fp, z) + d(z, z)}{2s}\} \\ &= d(fp, z).\end{aligned}$$

Taking limit as $n \to \infty$ in (3.6) and definition of ψ and φ, we get

$$\psi(d(fp, z)) \leq F(\psi(d(fp, z)), \varphi(d(fp, z))),$$

which implies $\psi(d(fp, z)) = 0$ or $\varphi(d(fp, z)) = 0$ which gives $fp = z$. Thus p is a coincidence point of the pair (f, S). Since the pair (f, S) is weakly compatible and $fp = Sp$,
therefore $fSp = Sfp$ which implies that $fz = Sz$.

Since $f(X) \subseteq T(X)$, then there exists a point $q \in X$ such that $Tq = z$. Now, we show that $gq = z$. We put $x = p$ and $y = q$ in (3.1), we get

$$\begin{aligned}\psi(d(fp, gq)) &\leq \psi(sd(fp, gq)) \\ &\leq F(\psi(M_s(p, q)), \varphi(M_s(p, q))),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}M_s(p, q) &= \max\{d(Sp, Tq), d(fp, Sp), d(gq, Tq), \frac{d(fp, Tq) + d(Sp, gq)}{2s}\} \\ &= \max\{d(z, z), d(z, z), d(gq, z), \frac{d(z, z) + d(z, gq)}{2s}\} \\ &= d(z, gq).\end{aligned}$$

Thus, from (3.7) and definition of ψ and φ, we get

$$\psi(d(z, gq)) \leq F(\psi(d(z, gq)), \varphi(d(z, gq))),$$

which implies $\psi(d(z, gq)) = 0$ or $\varphi(d(z, gq)) = 0$ which gives $gq = z$. Thus q is a coincidence point of the pair (g, T). Since the pair (g, T) is weakly compatible and $gq = Tq$,
therefore $gTq = Tgq$ which implies that $gz = Tz$.

Now we show that z is a common fixed point of the pair (f, S). Putting $x = z$, $y = q$ in (3.1), we get

$$\begin{aligned}\psi(d(fz, gq)) &\leq \psi(sd(fz, gq))\\ &\leq F(\psi(M_s(z, q)), \varphi(M_s(z, q))),\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}M_s(z, q) &= \max\{d(Sz, Tq), d(fz, Sz), d(gq, Tq), \frac{d(fz, Tq) + d(Sz, gq)}{2s}\}\\ &= \max\{d(fz, z), d(fz, Sz), d(z, z), \frac{d(fz, z) + d(fz, z)}{2s}\}\\ &= d(fz, z).\end{aligned}$$

Thus, from (3.8) and definition of ψ and φ, we get

$$\psi(d(fz, z)) \leq F(\psi(d(fz, z)), \varphi(d(fz, z))),$$

which implies $\psi(d(fz, z)) = 0$ or $\varphi(d(fz, z)) = 0$ which gives $fz = z$. Hence $fz = z = Sz$. Thus z is a common fixed point of the pair (f, S). Similarly, we can show that $gz = z = Tz$. Hence z is a common fixed point of f, g, S and T.

Now, we show that z is a unique common fixed point. Let t be another common fixed point of f, g, S and T. Putting $x = z$, $y = t$ in (3.1), we get

$$\begin{aligned}\psi(d(fz, gt)) &\leq \psi(sd(fz, gt))\\ &\leq F(\psi(M_s(z, t)), \varphi(M_s(z, t))),\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}M_s(z, t) &= \max\{d(Sz, Tt), d(fz, Sz), d(gt, Tt), \frac{d(fz, Tt) + d(Sz, gt)}{2s}\}\\ &= \max\{d(z, t), d(z, z), d(t, t), \frac{d(z, t) + d(z, t)}{2s}\}\\ &= d(z, t).\end{aligned}$$

Thus, from (3.9) and definition of ψ and φ, we get

$$\psi(d(z, t)) \leq F(\psi(d(z, t)), \varphi(d(z, t))),$$

which implies $\psi(d(z, t)) = 0$ or $\varphi(d(z, t)) = 0$ which gives $z = t$. Hence z is a unique common fixed point of mappings f, g, S and T. □

Remark 3.4. If we put $\psi(t) = t$ in Theorem 3, we get the following result.

Let (X, d) be a b-metric space and f, g, S, $T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that

$$sd(fx, gy) \leq F(M_s(x, y), \varphi(M_s(x, y))), \ \textit{for all } x, y \in X,$$

where

$$M_s(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(fx, Ty) + d(Sx, gy)}{2s}\}.$$

Suppose that the pairs (f, S) and (g, T) satisfy the common $(E.A)$-property and $S(X)$ or $T(X)$ is closed. Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Remark 3.5. If we put $g = f$ and $S = T$ in Theorem 3, we get the following result.

Let (X, d) be a b-metric space and f, $T : X \to X$ be mappings with $f(X) \subseteq T(X)$ such that

$$\psi(sd(fx, fy)) \leq F(\psi(M_s(x, y)), \varphi(M_s(x, y))), \text{ for all } x, y \in X,$$

$$\text{where } M_s(x, y) = \max\{d(Tx, Ty), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(Tx, fy)}{2s}\}.$$

Suppose that the pair (f, T) satisfy the common $(E.A)$-property and one of subspaces $f(X)$ and $T(X)$ is closed. Then the pair (f, T) has a point of coincidence in X. Moreover, if the pair (f, T) is weakly compatible, then f and T have a unique common fixed point.

Now, we illustrate an example to validate our Theorem 3.

Example 3.6. Consider $X = [0, 1]$, $d(x, y) = \max\{x, y\}$, $\psi(t) = t$, $\varphi(t) = \frac{t}{10}$ and $F(s, t) = \frac{9}{10}s$. Then (X, d) is a b-metric space with constant $s = \frac{49}{25}$. Define mappings f, g, S and T on X such that

$$f(x) = \begin{cases} 0, & \text{if } x \in \{0\} \cup (\frac{1}{2}, 1]; \\ \frac{1}{4}, & \text{if } x \in (0, \frac{1}{2}]; \end{cases} \quad g(x) = \begin{cases} 0, & \text{if } x \in \{0\} \cup (\frac{1}{2}, 1]; \\ \frac{1}{3}, & \text{if } x \in (0, \frac{1}{2}]; \end{cases}$$

$$S(x) = \begin{cases} 0, & \text{if } x = 0; \\ \frac{4}{5}, & \text{if } x \in (0, \frac{1}{2}]; \\ \frac{x+3}{12}, & \text{if } x \in (\frac{1}{2}, 1]; \end{cases} \quad T(x) = \begin{cases} 0, & \text{if } x = 0; \\ \frac{3}{5}, & \text{if } x \in (0, \frac{1}{2}]; \\ x - \frac{1}{2}, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

We take sequences $x_n = \{0\}$ and $y_n = \{\frac{1}{2} + \frac{1}{n}\}$. We have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = 0 \in X.$$

Therefore, both pairs (f, S) and (g, T) satisfy the common $(E.A)$-property. We see that mappings (f, S) and (g, T) commute at 0 which is coincidence point. Also, $f(x) \subseteq T(X)$ and $g(X) \subseteq S(X)$. We can verify the contraction condition (3.1) by a simple calculation for the case $x, y \in X$ as follows:

If $x, y \in (\frac{1}{2}, 1]$, then

$$\psi(sd(fx, gy)) = \frac{49}{25} \times 0 \leq \frac{9}{10} \times \frac{1}{2} = \frac{9}{10} d(Sx, Ty) \leq \frac{9}{10} \psi(M_s(x, y)).$$

Thus the contraction condition (3.1) is satisfied for $x, y \in (\frac{1}{2}, 1]$. Similarly, we can verify for other cases. Thus all the conditions of Theorem 3 are satisfied and 0 is a unique common fixed point of mappings f, g, S and T.

4. Application to system of integral equations

Let $X = C[a, b]$ be the set of all real continuous functions on $[a, b]$. Let the function $d : X \times X \to [0, \infty)$ be defined by $d(u, v) = \max_{a \leq r \leq b} |u(r) - v(r)|^2$, for

all $u, v \in X$. Obviously, (X, d) is a b-metric space with parameter $s = 2$.
Now, consider the integral equations:

$$u(r) = p(r) + \int_a^b G(r,s)K(r,s,S(u(t)))ds \tag{4.1}$$

$$v(r) = p(r) + \int_a^b G(r,s)J(r,s,T(v(t)))ds \tag{4.2}$$

for all $r \in [a,b]$, where $G : [a,b] \times [a,b] \to \mathbb{R}$ and $K, J : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Suppose that

(1) For all $r, s \in [a,b]$ and $u, v \in X$, we have

$$|K(r,s,S) - J(r,s,T)| \leq \sqrt{\frac{F(|S(u(r)) - T(v(r))|^2, \varphi(|S(u(r)) - T(v(r))|^2))}{2}}.$$

(2) $\max_{a \leq r \leq b} \int_a^b G(r,s)ds \leq 1$.

Then the integral equations (4.1) and (4.2) have a unique common solution.

Proof. Let $f, g : X \to X$ be mappings defined by

$$f(u(r)) = p(r) + \int_a^b G(r,s)K(r,s,S(u(t)))ds$$

$$g(v(r)) = p(r) + \int_a^b G(r,s)J(r,s,T(v(t)))ds$$

From (4.1) and (4.2), we have

$$\begin{aligned}
d(f(u(r)), g(v(r))) &= \max_{a \leq r \leq b} |f(u(r)) - g(v(r))|^2 \\
&= \max_{a \leq r \leq b} \big\{ | \int_a^b G(r,s)K(r,s,S(u(t)))ds \\
&\quad - \int_a^b G(r,s)J(r,s,T(v(t)))ds|^2 \big\} \\
&= \max_{a \leq r \leq b} \big\{ (\int_a^b G(r,s)ds)^2 (\int_a^b |K(r,s,S(u(t))) \\
&\quad - J(r,s,T(v(t)))|^2 ds) \big\}
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{a \leq r \leq b} \big\{ (\frac{F(|S(u(r)) - T(v(r))|^2, \varphi(|S(u(r)) - T(v(r))|^2))}{2}) \\
&(\int_a^b G(r,s)ds)^2 \big\} \\
&\leq \frac{F(d(S(u(r)), T(v(r))), \varphi(d(S(u(r)), T(v(r)))))}{2} \\
&\leq \frac{F(M_s(u,v), \varphi(M_s(u,v)))}{2}
\end{aligned}$$

$$2d(f(u(r)), g(v(r))) \leq F(M_s(u,v), \varphi(M_s(u,v))),$$

where

$$M_s(u,v) = \max\{d(Su,Tv), d(fu,Su), d(gv,Tv), \frac{d(fu,Tv)+d(Su,gv)}{2s}\}.$$

for all $u, v \in X$

$$sd(fu,gv) \leq F(M_s(u,v), \varphi(M_s(u,v))).$$

Hence all hypotheses of Corollary 3 are satisfied. Thus, the system of integral equations (4.1) and (4.2) have a unique common solution. □

5. Conclusion

It concludes that we have established common fixed point theorems for self mappings satisfying the common limit range property with respect to mappings S and T and common $(E.A)$-property via the concept of C-class functions in b-metric spaces. Further we furnish examples and an application to validate our findings.

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