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RIEMANNIAN MAPS FROM KAEHLER MANIFOLD WITH GENERIC FIBERS

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Abstract. We study Riemannian maps from almost Hermitian manifolds to Riemannian manifolds for the case when the fibers are generic submanifold of the total space. We obtain the integrability conditions for the distributions while vertical distribution is always integrable. We also study the geometry of the leaves of the distribution which arise from such maps, and obtain the necessary and sufficient conditions for the fibers as well as the total manifold to be generic product manifolds. We, further, obtain the necessary and sufficient condition for such maps to be totally geodesic.

1. Introduction

The theory of smooth maps between Riemannian manifolds was widely used in Riemannian geometry. These map are generally used to compare the geometric structures between two manifolds. From this point of view, such smooth maps are the isometric immersion between Riemannian manifolds which are characterized by the Riemannian metrics and Jacobian matrices. More precisely, Let(M, g_M) and (B, g_B) be two Riemannian manifolds, then a smooth map

$$F : (M, g_M) \longrightarrow (B, g_B),$$

where $dimM = m$ and $dimB = n$, is called an isometric immersion if the Jacobian map which we denote by F_* is injective and satisfies

$$g_B(F_*X, F_*Y) = g_M(X, Y), \tag{1.1}$$

for any vector fields X and Y tangent to M.

Here, one can notice that $m \leq n$. On the other hand, in case when $m \geq n$, the smooth map F was studied in the name Riemannian submersions and was first studied by $B.O'Neill$ [12],[13] and Gray [10].

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The smooth map

$$F : (M, g_M) \longrightarrow (B, g_B),$$

is called a Riemannian submersion if F_* is onto and satisfies (1.1) for vector fields tangent to the horizontal space $(KerF_*)^\perp$.

The idea of a Riemannian map between Riemanian manifolds was first introduced by Fischer, A. E. [9]. The idea of Riemannian maps generalizes and infact unifies the notions of isometric immersion, Riemannian submersion and an isometry. He has shown that every injective Riemannian map is an injective isometric immersion, and that on a connected manifold, every surjective Riemannian map is a surjective Riemaninan submersion and every bijective Riemannian map is an isometry. The notions of an immersion and submersion play a key role in the theory of smooth maps between smooth manifolds(finite or infinite). If we consider the Riemannian manifolds, then the theory of smooth maps between Riemannian manifolds, the two notions of an immersion and a submersion get into the notions of an isometric immersion and Riemannian submersion respectively, and were widely used in differential geometry [12],[23]. But there is no Riemannian analogue which corresponds to the general map between smooth manifolds.

Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a smooth map between Riemannian manifolds (M, g_M) and (B, g_B) such that $0 < rankF < min\{m, n\}$. If we denote the Kernal space of F_* by $KerF_*$ and its orthogonal complement by $(KerF_*)^\perp$ in TM the tangent bundle of M, then TM has the following decomposition

$$TM = (KerF_*) \oplus (KerF_*)^\perp.$$

We call $(KerF_*)$ and respectively $(KerF_*)^\perp$ the vertical and horizontal space of TM.

We denote by $rangF_*$ the range of F_* and consider the orthogonal complementary space to $rangF_*$ in the tangent bundle TB of B and denote it by $(rangF_*)^\perp$. Because of the fact that $rankF < min\{m, n\}$, the $(rangF_*)^\perp$ is non-empty. Hence, the tangent bundle TB of B is decomposed as:

$$TB = (rangF_*) \oplus (rangF_*)^\perp.$$

Now, let $F : (M, g_M) \longrightarrow (B, g_B)$ be a smooth map between Riemannian manifolds (M, g_M) and (B, g_B). Then the smooth map F is said to be a Riemannian map at a point $p \in M$ if the horizontal restriction F^h_{*p} of the derivative map F_* at p, i.e. ,

$$F^h_{*p} : (KerF_{*p})^\perp \longrightarrow (rangF_{*p})$$

is a linear isometry (also known isometric isomorphism) between the inner product spaces $\left((\ker F_{*p})^{\perp}, g_M(p)|_{(\ker F_{*p})^{\perp}}\right)$ and $\left((\mathrm{range} F_{*p}), g_B(q)|_{(\mathrm{range} F_{*p})}\right)$, where $q = F(p)$.

The map F is a Riemannian map if F is a Riemannian map at each point $p \in M$.

Hence, Fischer in the abstract of the paper [9] has remarked that a Riemannian map is a map that is "as isometric as it can be"subject to the limitations imposed upon it as a differential mapping. Also he has remarked that F is a Riemannian map at $p \in M$, $q = F(p) \in B$, if for all $X, Y \in (kerF_*)^{\perp} \subset T_pM$

$$
\begin{aligned}
g_M(p)(X,Y) &= g_B(q)(F^h_{*p}X, F^h_{*p}Y) \\
&= g_B(q)(F_{*p}X, F_{*p}Y)
\end{aligned}
$$

Since $F^h_{*p} = F_{*p}|_{\left(\left(\ker F_{*p}\right)^{\perp}\right)}$ (also restriction on the range).

In view of equation (1.1) it follows that isometric immersion and Riemannian submersion are particular cases of Riemannian maps with $kerF_* = \{0\}$ and $(rangF_*)^{\perp} = \{0\}$ respectively.

Recently, B. Sahin [14] introduced the notion of anti-invariant Riemannian maps which are Riemannian maps from almost Hermitian manifolds to Riemannian manifolds such that the vertical distributions(or, for that matter the fibers) are anti-invariant under the almost complex structure of the total space. Further, as a generalization of anti-invariant Riemannian maps, he introduced the notion of conformal semi-invariant Riemannian maps when the base manifold is a Riemannian manifold and a Kaehler manifold [15],[22]. He has shown that such maps are very much useful to study the geometry of the total space of the Riemannian maps. In the present article, we study the Riemannian maps from almost Hermitian manifolds under the assumption that the integral manifolds of vertical distribution $kerF_*$(or, for that matter the fibers) are generic submanifolds of the total space and call it the Riemannian maps with generic fibers, and it is not hard to say that one can see it as a generalization of semi-invariant Riemannian maps. The paper is organized as follows: In section 2, we give some basic notions of almost Hermitian manifolds, Riemannian submersion and brief introduction of anti-invariant and semi-invariant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. In section 3, we give the definition of Riemannian maps with generic fibers and investigate the geometry of the distributions and their integral manifolds(also known as the leaves of the distributions). In the end of this section we obtain the decomposition theorems. In the last section 4, we find the condition for such Riemannian maps to be totally geodesic.

2. Preliminaries

In this section we recall some basic definitions and notions of almost Hermitian manifolds, Kaehler manifold, CR-submanifolds and give a brief review of basic fact of Riemannian maps. For the notion of Riemannian maps we follow Fischer [9] and B. Sahin [14],[16].

Let M be an almost complex manifold, that is, M admits a tensor field J of type $(1,1)$ with the property that $J^2 = -I$. An almost complex manifold is necessarily orientable and is of even dimension. An almost complex manifold (M, J) endowed with a chosen Riemannian metric g and satisfying the condition

$$g(JX, JY) = g(X, Y), \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$, is called an almost Hermitian manifold.

The Levi-Civita connection ∇ of the almost Hermitian manifold (M, J) can be extended to whole tensor algebra on M, and in this way we obtain tensor fields like $(\nabla_X J)$ and that

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y, \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$.

An almost Hermitian manifold $\overline{M}$ is called Kaehler manifold if

$$(\overline{\nabla}_X J)(Y) = 0, \ \forall X, Y \in \Gamma(T\overline{M}) \tag{2.3}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M be a real submanifold of $\overline{M}$, and let

$$D_p = T_pM \cap JT_pM, \ \forall p \in M$$

such that D_p is the maximal subspace of T_pM.

Definition 2.1 ([4]). A submanifold M is said to be a CR-submanifold of an almost Hermitian manifold $(\overline{M}, g)$ if there exists on M a C^∞-holomorphic distribution D such that its orthogonal complementary distribution $D^\perp$ is totally real, i.e., $JD_p^\perp \subseteq T_p^\perp M$ for all $p \in M$. A CR-submanifold M is said to be proper if neither $D = \{0\}$, *nor* $D^\perp = \{0\}$.

Now, we recall the definition of generic submanifold which is the generalization of CR-submanifolds. These submanifold are defined by relaxing the condition on the complementary distribution to the holomorphic distribution.

Definition 2.2 ([4]). Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and let M be a real submanifold of $\overline{M}$. Then M is said to be a generic submanifold of $\overline{M}$ if the maximal complex subspace D_p has constant dimension at each point $p \in M$ and defines a differentiable distribution on M.

We denote by $D^{\perp}$ be orthogonal complementary distribution to D in $\Gamma(TM)$ and observe that $JD^{\perp} \cap D^{\perp} = \{0\}$. If, in particular, $JD^{\perp} \subset \Gamma(T^{\perp}M)$, we have the concept of CR-submanifold. We call D and $D^{\perp}$ the holomorphic and the purely real distribution on M.

For $U \in \Gamma(TM)$, we put

$$JU = PU + FU, \tag{2.4}$$

where PU and FU are the tangential and normal part of JU respectively.

For a generic submanifold we have

$$(i) PD = D,\ FD = \{0\}, (ii) PD^{\perp} \subset D^{\perp},\ FD^{\perp} \subset T^{\perp}M. \tag{2.5}$$

For the theory of Riemannian maps we follow A. E, Fischer [9] and B. Sahin [14],[16].

Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian map between the Riemannian manifolds (M, g_M) and (B, g_B), where $dimM = m$ and $dimB = n$ with $0 < rankF < min\{m, n\}$. The letters $\mathcal{H}$ and $\mathcal{V}$ are used to denote the orthogonal projections of $\Gamma(TM)$ on the distributions $\Gamma(kerF_*)^{\perp}$ and $\Gamma(kerF_*)$ respectively. The geometry of Riemannian maps are characterized by the tensor fields T and A of the Riemannian map F defined for arbitrary vector fields E and F on M by

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.6}$$

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.7}$$

where ∇ is the Levi-Civita connection of g_M. Indeed, one can see that these tensor fields are B. O'Neill's fundamental tensor fields defined for the Riemannian submersion. It is easy to see that the Riemannian map $F : M \longrightarrow B$ has totally geodesic fibers if and only if T vanishes identically. For any $E \in \Gamma(TM)$, T_E and A_E are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and vertical spaces. It is also seen that T is vertical, i.e., $T_E = T_{\mathcal{V}E}$ and A is horizontal, i.e., $A_E = A_{\mathcal{H}E}$. We observe that the tensor fields T and A satisfy

$$(i) T_U V = T_V U,\ U, V \in \Gamma(kerF_*), (ii) A_X Y = -A_Y X,\ X, Y \in \Gamma(kerF_*)^{\perp} \tag{2.8}$$

On the other hand, (2.6) and (2.7) give the following lemma.

Lemma 2.1 *([9]). We have*

$$\nabla_U V = T_U V + \hat{\nabla}_U V, \tag{2.9}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + T_U X, \tag{2.10}$$

$$\nabla_X U = A_X U + \mathcal{V}\nabla_X U, \tag{2.11}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y, \tag{2.12}$$

for any $X, Y \in \Gamma(kerF_*)^{\perp}$ and $U, V \in \Gamma(kerF_*)$, where $\hat{\nabla}_U V = \mathcal{V}\nabla_U V$.

Next,we recall the following definition.

Definition 2.3 ([16]). Let$F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from an almost Hermition manifold (M, g_M, J) to a Riemannian manifold (B, g_B). Then F is said to be semi-invariant Riemannian map if there is a distribution $D_1 \in \Gamma(kerF_*)$, such that

$$kerF_* = D_1 \oplus D_2, JD_1 = D_1, JD_2 \subset \Gamma(kerF_*)^{\perp},$$

where D_2 is the orthogonal complement of D_1 in $\Gamma(kerF_*)$.

Finally, we recall the notion of second fundamental form of a map between Riemannian manifolds. Let (M, g_M) and (B, g_B) be Riemannian manifolds, and let $\phi : M \longrightarrow B$ be a smooth map between them. Then the differential ϕ_* of ϕ can be viewed as a section of the bundle $Hom((TM), \phi^{-1}(TB)) \longrightarrow M$, where, $\phi^{-1}(TB) = T_{\phi(p)}B, p \in M$. $Hom(TM, \phi^{-1}(TB))$ has a connection ∇ induced from Levi-Civita connection $\overline{\nabla}$ on M and the pullback connection.

The second fundamental form of ϕ is then given by

$$(\nabla\phi_*)(X, Y) = \nabla^{\phi}_X \phi_*(Y) - \phi_*(\overline{\nabla}_X Y) \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$, where ∇^{ϕ} is the pullback connection. It is known that the second fundamental form is symmetric. We now state the result of B. Sahin [14], which shows that the second fundamental form $(\nabla\phi_*)(X, Y)$,
$\forall X, Y \in (kerF*)^{\perp}$, of a Riemannian map ϕ has no component in $rang\phi_*$.

Lemma 2.2 *([14]). Let ϕ be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (B, g_B). Then*

$$g_B((\nabla\phi_*)(X, Y), \phi_* Z) = 0, \ \forall X, Y, Z \in \Gamma(kerF_*)^{\perp}. \tag{2.14}$$

3. Riemannian Maps With Generic Fibers

B. Sahin [14],[16] defined anti-invariant and semi-invariant Riemannian maps from almost Hermition manifolds to a Riemannian manifold. In these two cases he has defined them for the cases where the vertical distribution are infact anti-invariant and semi-invariant respectively. That means the integral manifolds(or for that matter the fibers) $F^{-1}(q),\ q \in B$ of $(kerF_*)$ are respectively anti-invariant and semi-invariant submanifolds of M. In this section we consider Riemannian maps for the case when the integral manifolds(the fibers) of $kerF_*$ are generic submanifolds of M which inturn generalizes these above mentioned maps. We obtain the integrability conditions for the distributions and investigate the geometry of the distribution $(kerF_*)$ and $(kerF_*)^{\perp}$. Also, we obtain the necessary and sufficient condition for such maps to be totally geodesic. We also obtain product theorem for the total manifold of such Riemannian maps.

Let F be a Riemannian map from an almost Hermition manifold (M, g_M, J) to a Riemannian manifold (B, g_B). We say that the integral manifold(or for that matter the fibers) $F^{-1}(q), q \in B$ of the vertical distribution $kerF_*$ is a generic submanifold of M if the maximal complex space
$D_p = (kerF_{*p}) \cap (JkerF_{*p}),\ p \in M,$ defines on $F^{-1}(q)$ a differential distribution
$D : p \longrightarrow D_p \subset (kerF_{*p})$ such that

$$kerF_* = D_1 \oplus D_2, JD_1 = D_1, \tag{3.1}$$

where D_2 is the orthogonal complement of D in $\Gamma(kerF_*)$, and is called a purely real distribution of the fibers of the Riemannian map F.

Definition 3.1 The Riemannian map $F : (M, g_M, J) \longrightarrow (B, g_B)$ satisfying condition (3.1) is called a Reimannian map with generic fibers.

For any $V \in \Gamma(kerF_*)$ we set

$$JV = \phi V + \omega V, \tag{3.2}$$

where $\phi V \in \Gamma(D_1)$ and $\omega V \in \Gamma(kerF_*)^\perp$. We denote the orthogonal complementary distribution to ωD_2 in $(kerF_*)^\perp$ by μ. Then we can write

$$(kerF_*)^\perp = \omega D_2 \oplus \mu. \tag{3.3}$$

It is easy to see that μ is J-invariant. Thus, for any $X \in \Gamma(kerF_*)^\perp$ we have

$$JX = BX + CX, \tag{3.4}$$

where $BX \in \Gamma(D_2)$ and $CX \in \Gamma(\mu)$.

Using (3.1) through (3.4) we obtain

Lemma 3.1 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then We have*

(i) $\phi D_1 = D_1,\ \omega D_1 = 0$
(ii) $\phi D_2 \subset D_2,\ B(kerF_*)^\perp = D_2$
(iii) $\phi^2 + B\omega = -id,\ \omega\phi + C\omega = 0$
(iv) $BC + \phi B = 0,\ \omega B + C^2 = -id.$

Next, using equations (2.1),(3.2)and Lemma 2.1 we have

Lemma 3.2 *Let (M, g_M, J) be an almost Hermitian manifold and F be a Riemannian map from (M, g_M, J) to a Riemannian manifold (B, g_B)*

with generic fibers. Then

$$(i) \quad g_M(JY, \phi V) = g(BY, JV), \tag{3.5}$$

$$(ii) \quad g_M(CY, \omega V) = 0, \tag{3.6}$$

$$(iii) \quad g_M(\overline{\nabla}_X BY, JV) = -g_M(BY, \mathcal{V}\nabla_X \phi V) - g_M(BY, A_X \omega V), \tag{3.7}$$

$$(iv) \quad g_M(\overline{\nabla}_U BY, JV) = -g_M(BY, \hat{\nabla}_U \phi V) - g_M(BY, T_U \omega V), \tag{3.8}$$

$$(v) \quad g_M(\overline{\nabla}_U BY, CX) = g_M(CX, T_U BY) = -g(BY, T_U CX), \tag{3.9}$$

for any $X, Y \in \Gamma(ker F_*)^\perp$ *and* $U, V \in \Gamma(ker F_*)$.

Proof. Proof of (i) and (ii) directly follows from equations (2.1),(3.1),(3.2),(3.3) and (3.4).
(iii), Using the fact that for any $Y \in \Gamma(ker F_*)^\perp$, $BY \in \Gamma(D_2)$ and for any $V \in \Gamma(ker F_*)$, $\phi V \in \Gamma(D_1)$, for any $X \in \Gamma(ker F_*)^\perp$ we have

$$\begin{aligned} g_M(\overline{\nabla}_X BY, JV) &= -g_M(\overline{\nabla}_X BY, \phi V) + g_M(\overline{\nabla}_X BY, \omega V) \\ &= -g_M(BY, \nabla_X \phi V) - g_M(BY, \overline{\nabla}_X \omega V) \\ &= -g_M(BY, \mathcal{V}\nabla_X \phi V) - g_M(BY, \mathcal{V}\nabla_X \omega V) \\ &= -g_M(BY, \mathcal{V}\nabla_X \phi V) - g_M(BY, A_X \omega V). \end{aligned}$$

Similarly, we obtain (iv) and(v). ☐

Proposition 3.1 *Let* $F : (M, g_M, J) \longrightarrow (B, g_B)$ *be a Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (B, g_B) *with generic fibers. Then we have*

$$\begin{aligned} g_M(\overline{\nabla}_X CY, \omega V) &= g_M(BY, A_X \omega V) + g_M(\omega A_X Y, \omega V) \\ &= -g_M(A_X BY, \omega V) + g_M(\omega A_X Y, \omega V), \end{aligned} \tag{3.10}$$

for any $X, Y \in \Gamma(ker F_*)^\perp$ *and* $V \in \Gamma(ker F_*)$.

Proof. From Lemma 3.1(i), Lemma 3.2(ii) and equations (2.2),(2.3) for any $X, Y \in \Gamma(ker F_*)^\perp$ and $V \in \Gamma(ker F_*)$ we have

$$\begin{aligned} g_M(\overline{\nabla}_X CY, \omega V) &= -g_M(CY, \overline{\nabla}_X \omega V) \\ &= -g_M(JY, \overline{\nabla}_X \omega V) + g_M(BY, \overline{\nabla}_X \omega V) \\ &= g_M(\overline{\nabla}_X JY, \omega V) - g_M(\overline{\nabla}_X BY, \omega V) \\ &= g_M(J\overline{\nabla}_X Y, \omega V) - g_M(\overline{\nabla}_X BY, \omega V) \\ &= -g_M(\mathcal{H}\overline{\nabla}_X Y, J\omega V) + g_M(\mathcal{V}\overline{\nabla}_X Y, J\omega V) - g_M(\mathcal{H}\overline{\nabla}_X BY, \omega V) \\ &= -g_M(\mathcal{H}\overline{\nabla}_X Y, J\omega V) - g_M(\mathcal{V}\overline{\nabla}_X Y, J\omega V) - g_M(A_X BY, \omega V) \\ &= -g_M(\mathcal{H}\overline{\nabla}_X Y, J\omega V) - g_M(A_X Y, J\omega V) - g_M(A_X BY, \omega V) \\ &= g_M(J\mathcal{H}\overline{\nabla}_X Y, \omega V) + g_M(JA_X Y, \omega V) - g_M(A_X BY, \omega V) \\ &= g_M(B\mathcal{H}\overline{\nabla}_X Y, \omega V) + g_M(C\mathcal{H}\overline{\nabla}_X Y, \omega V) + g_M(JA_X Y, \omega V) \\ &- g_M(A_X BY, \omega V). \end{aligned}$$

Since for any $X \in \Gamma(kerF_*)^\perp, BX \in \Gamma(D_2)$ and for $V \in \Gamma(kerF_*), \omega V \in \Gamma(kerF_*)^\perp$. Also, using Lemma 3.1 again we have

$$g_M(\overline{\nabla}_X CY, \omega V) = -g_M(A_X BY, \omega V) + g_M(\omega A_X Y, \omega V).$$

which completes the proof. □

Since g_M is a non-degenerate metric on $\overline{M}$, from Proposition 3.1 we have

Corollary 3.1 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then*

$$\overline{\nabla}_X CY = -A_X BY + \omega A_X Y, \tag{3.11}$$

for any $X, Y \in \Gamma(kerF_*)^\perp$.

Further, as a consequences of Proposition 3.1 we have

Corollary 3.2 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannain manifold (B, g_B) with generic fibers. Then $\overline{\nabla}_X CY \in \Gamma(\mu)$ if and only if*

$$A_X BY = \omega A_X Y,$$

for any $X, Y \in \Gamma(kerF_*)^\perp$.

We define the covariant derivative of ϕ and ω as follow:

$$\begin{aligned}(\nabla_V \phi)W &= \hat{\nabla}_V \phi W - \phi \hat{\nabla}_V W\\ (\nabla_V \omega)W &= \mathcal{H}(\nabla_V \omega W) - \omega \hat{\nabla}_V W\end{aligned}$$

Then, using Lemma 2.1 and equations (3.2),(3.4) we obtain

$$\begin{aligned}(\nabla_V \phi)W &= BT_V W - T_V \omega W\\ (\nabla_V \omega)W &= CT_V W - T_V \phi W\end{aligned}$$

for any $V, W \in \Gamma(kerF_*)$.

We now have the following proposition.

Proposition 3.2 *Let F be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Reimannian manifold (B, g_B) with generic fibers. Then*

$$(i) \quad A_X \phi V + \mathcal{H}\overline{\nabla}_X \omega V = CA_X V + \omega(\mathcal{V}\overline{\nabla}_X V)$$

and

$$\mathcal{V}(\overline{\nabla}_X \phi V) + A_X \omega V = BA_X V + \phi(\mathcal{V}\overline{\nabla}_X V),$$

$$(ii) \quad A_X BY + \mathcal{H}(\overline{\nabla}_X CY) = C(\mathcal{H}\overline{\nabla}_X Y) + \omega A_X Y$$

and

$$\mathcal{V}(\overline{\nabla}_X BY) + A_X CY = B(\mathcal{H}\overline{\nabla}_X Y) + \phi A_X Y,$$

$$(iii) \quad T_V\phi W + \mathcal{H}(\overline{\nabla}_V\omega W) = CT_VW + \omega\hat{\nabla}_VW$$

and

$$\hat{\nabla}_V\phi W + T_V\omega W = BT_VW + \phi\hat{\nabla}_VW,$$

for any $X, Y \in \Gamma(kerF_*)^\perp$ *and* $V, W \in \Gamma(kerF_*)$.

Proof. (i) For a Keahler manifold M, we have on using equation (2.3)

$$\overline{\nabla}_XJV = J\overline{\nabla}_XV,$$

for any $X \in \Gamma(kerF_*)^\perp, V \in \Gamma(kerF_*)$.
Further, using Lemma 2.1 and equations (3.1),(3.4) we get

$$\overline{\nabla}_X\phi V + \overline{\nabla}_X\omega V = B(A_XV) + C(A_XV) + \phi(\mathcal{V}\overline{\nabla}_XV) + \omega(\mathcal{V}\overline{\nabla}_XV)$$

or,

$$\begin{aligned}\mathcal{H}\overline{\nabla}_X\phi V + \mathcal{V}\overline{\nabla}_X\phi V + \mathcal{H}\overline{\nabla}_X\omega V + \mathcal{V}\overline{\nabla}_X\omega V &= B(A_XV) + C(A_XV) \\ &+ \phi(\mathcal{V}\overline{\nabla}_XV) + \omega(\mathcal{V}\overline{\nabla}_XV)\end{aligned}$$

or,

$$\begin{aligned}A_X\phi V + \mathcal{V}\overline{\nabla}_X\phi V + \mathcal{H}\overline{\nabla}_X\omega V + A_X\omega V &= B(A_XV) + C(A_XV) \\ &+ \phi(\mathcal{V}\overline{\nabla}_XV) + \omega(\mathcal{V}\overline{\nabla}_XV).\end{aligned}$$

Comparing horizontal and vertical parts, we get

$$A_X\phi V + \mathcal{H}\overline{\nabla}_X\omega V = C(A_XV) + \omega(\mathcal{V}\overline{\nabla}_XV),$$
$$\mathcal{V}\overline{\nabla}_X\phi V + A_X\omega V = B(A_XV) + \phi(\mathcal{V}\overline{\nabla}_XV).$$

On similar lines we get the proof of (ii) and (iii). □

Theorem 3.1 *Let F be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Reimmanian manifold (B, g_B) with generic fibers. Then we have*

$$g_B((\nabla F_*)(V, W), F_*J\xi) = -g_B((\nabla F_*)(V, \phi W), F_*\xi) - g_B((\nabla F_*)(V, \omega W), F_*\xi),$$

for any $V, W \in \Gamma(kerF_*)$ *and* $\xi \in \Gamma(\mu)$.

Proof. Since M is a kaehler manifold, for any $V, W \in \Gamma(kerF_*)$ we have,

$$\overline{\nabla}_VJW = J\overline{\nabla}_VW.$$

Using Lemma 2.1 and equations (3.2),(3.4), we get

$$\begin{aligned}&\mathcal{H}(\overline{\nabla}_V\phi W) + \mathcal{V}(\overline{\nabla}_V\phi W) + \mathcal{H}(\overline{\nabla}_V\omega W) + \mathcal{V}(\overline{\nabla}_V\omega W) \\ &= B\mathcal{H}\overline{\nabla}_VW + C\mathcal{H}\overline{\nabla}_VW + \phi\mathcal{V}\overline{\nabla}_VW + \omega\mathcal{V}\overline{\nabla}_VW \\ &= B\mathcal{H}\overline{\nabla}_VW + C\mathcal{H}\overline{\nabla}_VW + \phi\hat{\nabla}_VW + \omega\hat{\nabla}_VW. \qquad (3.12)\end{aligned}$$

Equating horizontal parts in (3.12), we have

$$\mathcal{H}\nabla_V\phi W + \mathcal{H}\overline{\nabla}_V\omega W = C\mathcal{H}\overline{\nabla}_VW + \omega\hat{\nabla}_VW. \qquad (3.13)$$

Taking Riemannian inner product in (3.13) with a vector $\xi \in \Gamma(\mu)$, we obtain

$$\begin{aligned} g_M(\mathcal{H})\overline{\nabla}_V \phi W, \xi) + g_M(\mathcal{H}\overline{\nabla}_V \omega W, \xi) &= g_M(C\mathcal{H}\overline{\nabla}_V W, \xi) \\ g_M(\overline{\nabla}_V \phi W, \xi) + g_M(\overline{\nabla}_V \omega W, \xi) &= g_M(J\mathcal{H}\overline{\nabla}_V W, \xi) \\ &= -g_M(\overline{\nabla}_V W, J\xi). \end{aligned}$$

Since F is a Riemannian map, we have

$$g_B(F_*(\overline{\nabla}_V \phi W), F_*\xi) + g_B(F_*(\overline{\nabla}_V \omega W), F_*\xi) = -g_B(F_*(\overline{\nabla}_V W), F_* J\xi).$$

Using equation (2.13) we get the result. □

From Theorem 3.1 we have

Corollary 3.3 *Let F be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannain manifold (B, g_B) with generic fibers. Then we have*

$$g_B((\nabla F_*)(V, W), F_* J\xi) = -g_B((\nabla F_*)(V, JW), F_*\xi),$$

for any $V \in \Gamma(ker F_*), W \in \Gamma(D_1)$ and $\xi \in \Gamma(\mu)$.

Lemma 3.3 *Let F be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B). Then*

$$g(JT_V W, \xi) = g(T_V JW, \xi),$$

for any $V \in \Gamma(ker F_*), W \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$

Proof. Since M is a Kaehler manifold, then for any $V \in \Gamma(ker F_*), W \in \Gamma(D_1)$, using equation (2.3) we have

$$J\overline{\nabla}_V W = \overline{\nabla}_V JW.$$

On using Lemma 2.1 we get

$$J(T_V W + \hat{\nabla}_V W) = T_V JW + \hat{\nabla}_V JW.$$

Taking inner product with a vector field $\xi \in \Gamma(\mu)$, we get

$$g(JT_V W, \xi) + g(J\hat{\nabla}_V W, \xi) = g(T_V JW, \xi) + g(\hat{\nabla}_V JW, \xi)$$

$$g(JT_V W, \xi) - g(\hat{\nabla}_V W, J\xi) = g(T_V JW, \xi) + g(\hat{\nabla}_V JW, \xi). \quad (3.14)$$

Since μ is invariant under J, the result then follows from (3.14). □

Next, we have

Theorem 3.2 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then we have*

$$\begin{aligned} g_B((\nabla F_*)(X, V), F_* J\xi) &= g_B((\nabla F_*)(X, \phi V) + (\nabla F_*)(X, \omega V), F_*\xi) \\ &- g_B(\nabla^F_X F_*(\omega V), F_*\xi), \end{aligned}$$

for any vector field $X \in \Gamma(ker F_*)^\perp, V \in \Gamma(ker F_*)$ and $\xi \in \Gamma(\mu)$.

Proof. For $X \in \Gamma(kerF_*)^\perp, V \in \Gamma(kerF_*)$,
using equation (2.3) for a Kaehler manifold we have

$$\overline{\nabla}_X JV = J\overline{\nabla}_X V.$$

Using Lemma 2.1 and equations (3.2),(3.4) we get

$$\mathcal{H}\overline{\nabla}_X\phi V+\mathcal{V}\overline{\nabla}_X\phi V+\mathcal{H}\overline{\nabla}_X\omega V+\mathcal{V}\overline{\nabla}_X\omega V = B\mathcal{H}\overline{\nabla}_X V+C\mathcal{H}\overline{\nabla}_X V+\phi\mathcal{V}\overline{\nabla}_X V+\omega\mathcal{V}\overline{\nabla}_X V. \tag{3.15}$$

Equating horizontal component in (3.15) we get

$$\mathcal{H}\overline{\nabla}_X\phi V + \mathcal{H}\overline{\nabla}_X\omega V = C\mathcal{H}\overline{\nabla}_X V + \omega\mathcal{V}\overline{\nabla}_X V. \tag{3.16}$$

Taking Riemannian product in (3.16) with a vector $\xi \in \Gamma(\mu)$ we obtain

$$\begin{aligned} g_M(\mathcal{H}\overline{\nabla}_X\phi V, \xi) + g(\mathcal{H}\overline{\nabla}_X\omega V, \xi) &= g(C\mathcal{H}\overline{\nabla}_X V, \xi) + g_M(\omega\mathcal{V}\overline{\nabla}_X V, \xi) \\ g_M(\overline{\nabla}_X\phi V, \xi) + g(\overline{\nabla}_X\omega V, \xi) &= g_M(J\mathcal{H}\overline{\nabla}_X V, \xi) - g_M(B\mathcal{H}\overline{\nabla}_X V, \xi) \\ &+ g_M(J\mathcal{V}\overline{\nabla}_X V, \xi) - g_M(\phi\mathcal{V}\overline{\nabla}_X V\xi). \end{aligned}$$

But, for generic fibers $B(kerF_*)^\perp = D_2$ and $\phi\mathcal{V}\overline{\nabla}_X V \in (kerF_*)$, (Lemma 3.1) we then have

$$\begin{aligned} g_M(\overline{\nabla}_X\phi V, \xi) + g_M(\overline{\nabla}_X\omega VC, \xi) &= -g_M(\mathcal{H}\overline{\nabla}_X V, \xi) \\ &= -g_M(\overline{\nabla}_X V, \xi). \end{aligned} \tag{3.17}$$

Since F is a Riemannian map, from (3.17) we have

$$g_B(F_*(\overline{\nabla}_X\phi V), F_*\xi) + g_B(F_*(\overline{\nabla}_X\omega V), F_*\xi) = g_B(F_*(\overline{\nabla}_X V), F_*J\xi),$$

which on using (2.13) yields

$$\begin{aligned} -g_B((\nabla F_*)(X, \phi V), F_*\xi) - g_B((\nabla F_*)(X, \omega V), F_*\xi) + g_B(\nabla^F_X F_*(\omega V), F_*\xi) \\ = -g_B((\nabla F_*)(X, V), F_*J\xi) \end{aligned}$$

$$\begin{aligned} g_B((\nabla F_*)(X, V), F_*J\xi) = g_B((\nabla F_*)(X, \phi V) + (\nabla F_*)(X, \omega V), F_*\xi) \\ -g_B(\nabla^F_X F_*(\omega V), F_*\xi). \end{aligned}$$

Which completes the proof. □

As a consequences of above result we have

Corollary 3.4 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then*

$$g_B((\nabla F_*)(X, V), F_*J\xi) = g_B((\nabla F_*)(X, JV), F_*\xi),$$

for any $X \in \Gamma(kerF_*)^\perp, V \in \Gamma(D_1), \xi \in \Gamma(\mu)$.

4. Integrability of Distributions

In this section we obtain the integrability conditions for the distribution D_1 and D_2. Since we have seen that the fibers of the Riemannian map F under consideration are the generic submanifolds of the manifold M and T works as the second fundamental form of the fibers and ∇F_* is the second fundamental form of the Riemannian map F, we have following theorem;

Theorem 4.1 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then the distribution D_1 is integrable if and only if*

$$g_B((\nabla F_*)(V, JW) - (\nabla F_*)(W, JV), F_*\omega U) = 0. \tag{4.1}$$

or, $(\nabla F_*)(V, JW) - (\nabla F_*)(W, JV)$ has no component in $\Gamma(F_*(\omega D_2))$, for any $V, W \in \Gamma(D_1)$ and $U \in \Gamma(D_2)$

Proof. Let $V, W \in \Gamma(D_1), U \in \Gamma(D_2)$. Then on using equations (2.3),(3.1) and (3.2), we have

$$\begin{aligned}
\omega[V, W] &= J[V, W] - \phi[V, W] \\
&= J\nabla_V W - J\nabla_W V - \phi[V, W] \\
&= \nabla_V JW - J\nabla_W JV - \phi[V, W] \\
&= \mathcal{H}\nabla_V JW - \mathcal{H}\nabla_W JV + \mathcal{V}\nabla_V JW - \mathcal{V}\nabla_W JV - \phi[V, W]
\end{aligned}$$

$$\omega[V, W] - \mathcal{H}\nabla_V JW - \mathcal{H}\nabla_W JV = \mathcal{V}\nabla_V JW - \mathcal{V}\nabla_W JV - \phi[V, W] \tag{4.2}$$

Since $\omega[V, W] \in \Gamma(kerF_*)^\perp$. In equation (4.2) the right hand side is vertical where as the left hand side is horizontal. Comparing horizontal and vertical parts, we get

$$\omega[V, W] = \mathcal{H}\nabla_V JW - \mathcal{H}\nabla_W JV. \tag{4.3}$$

$$\phi[V, W] = \mathcal{V}\nabla_V JW - \mathcal{V}\nabla_W JV. \tag{4.4}$$

Now, in view of decomposition (3.3) and equation (3.4) for each vector field $Z \in \Gamma(\omega D_2) \subset \Gamma(kerF_*)^\perp$ there exist a vector $U \in \Gamma(D_2)$ such that $\omega U = Z$. Taking Riemannian inner product in (4.3) with $\omega U \in \Gamma(\omega D_2)$ we get

$$\begin{aligned}
g_M(\omega[V, W], \omega U) &= g_M(\mathcal{H}\nabla_V JW, \omega U) - g_M(\mathcal{H}\nabla_W JV, \omega U) \\
&= g_M(\overline{\nabla}_V JW, \omega U) - g_M(\overline{\nabla}_W JV, \omega U)
\end{aligned}$$

Since F is a Riemannian map,

$$g_M(\omega[V, W], \omega U) = g_B(F_*(\overline{\nabla}_V JW), F_*\omega U) - g_B(F_*(\overline{\nabla}_W JV), F_*\omega U).$$

On using equation (2.13), we get

$$
\begin{aligned}
g_M(\omega[V,W],\omega U) &= g_B((\nabla F_*)(V,JW),F_*\omega U) - g_B((\nabla F_*)(W,JV),F_*\omega U) \\
&= g_B((\nabla F_*)(V,JW) - (\nabla F_*)(W,JV),F_*\omega U). \qquad (4.5)
\end{aligned}
$$

Hence the distribution D_1 is integrable if and only if $\omega[V,W] = 0$. That is D_1 is inetegrable if and only if

$$g_B((\nabla F_*)(V,JW) - (\nabla F_*)(W,JV),F_*\omega U) = 0. \qquad (4.6)$$

or, $(\nabla F_*)(V,JW) - (\nabla F_*)(W,JV)$ has no component in $\Gamma(F_*(\omega U))$. Which completes the proof. □

Since $F_*(kerF_*)^{\perp} = rangeF_*$ and F is a Riemannian map, using Lemma 3.2(ii) it follows that $g_B(F_*\omega V,F_*X) = 0$, for any $X \in \Gamma(kerF_*)^{\perp}$ and $V \in \Gamma(kerF_*)$, which then implies that

$$TB = F_*(\omega D_2) \oplus F_*(\mu) \oplus (rangeF_*)^{\perp}.$$

From equation (4.6) we also have

Theorem 4.2 *Let $F : (M,g_M,J) \longrightarrow (B,g_B)$ be a Riemannian map from a Kaehler manifold (M,g_M,J) to a Riemannian manifold (B,g_B) with generic fibers, then the distribution D_1 is integrable if and only if*

$$(\nabla F_*)(V,JW) - (\nabla F_*)(W,JV) \in \Gamma(F_*(\mu)),$$

for any $V,W \in \Gamma(D_1)$.

Lemma 4.1 *Let $F : (M,g_M,J) \longrightarrow (B,g_B)$ be a Riemannian map from an almost Hermitian manifold (M,g_M,J) to a Riemannian manifold (B,g_B) with generic fibers, then the distribution D_2 is integrable if and only if*

$$\phi[V,W] \in \Gamma(D_2),$$

for any $V,W \in \Gamma(D_2)$.

Proof. Since M is an almost Hermitian manifold and the distribution $(kerF_*)$ is integrable, for any $V,W \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we have

$$
\begin{aligned}
g_M([V,W],Z) &= g_M(J[V,W],JZ) \\
&= g_M(\phi[V,W],JZ) + g_M(\omega[V,W],JZ) \\
&= g_M(\phi[V,W],JZ).
\end{aligned}
$$

The result then follow immediately. □

Next we have,

Theorem 4.3 *Let $F : (M,g_M,J) \longrightarrow (B,g_B)$ be a Riemannian map from a Kaehler manifold (M,g_M,J) to a Riemannian manifold (B,g_B) with generic fibers, then the distribution D_2 is integrable if and only if*

$$T_V\omega W - T_W\omega V + \hat{\nabla}_V\phi W - \hat{\nabla}_W\phi V \in \Gamma(D_2),$$

for any $V,W \in \Gamma(D_2)$.

Proof. Since M is a Kaehler manifold, for any $V, W \in \Gamma(D_2)$ using Lemma 2.1 and equations (2.3),(3.2),(3.4) we obtain

$$\begin{aligned}[V, W] &= \nabla_V W - \nabla_W V \\ &= -J(\nabla_V JW - \nabla_W JV)\end{aligned}$$

$$\begin{aligned}&= B(\mathcal{H}\nabla_W JV - \mathcal{H}\nabla_V JW) + C(\mathcal{H}\nabla_W JV - \mathcal{H}\nabla_V JW) \\ &\quad +\phi(\mathcal{V}\nabla_W JV - \mathcal{V}\nabla_V JW) + \omega(\mathcal{V}\nabla_W JV - \mathcal{V}\nabla_V JW). \qquad (4.7)\end{aligned}$$

Since $(kerF_*)$ is always integrable, therefore $[V, W] \in \Gamma(kerF_*)$. Comparing the vertical part in (4.7), we get

$$[V, W] = B(\mathcal{H}\nabla_W JV - \mathcal{H}\nabla_V JW) + \phi(\mathcal{V}\nabla_W JV - \mathcal{V}\nabla_V JW). \quad (4.8)$$

Taking Riemannian inner product in (4.8) with a vector $Z \in \Gamma(D_1)$ and further using equation (3.2) and Lemma 3.1, we get

$$\begin{aligned}g_M([V, W], Z) &= g_M(\phi(\mathcal{V}\nabla_W JV), Z) - g_M(\phi\mathcal{V}\nabla_V JW, Z) \\ &= g_M(J(\mathcal{V}\nabla_W JV), Z) - g_M(J(\mathcal{V}\nabla_V JW), Z) \\ &= -g_M(\mathcal{V}\nabla_W \phi V, JZ) - g_M(\mathcal{V}\nabla_W \omega V, JZ) + g_M(\mathcal{V}\nabla_V \phi W, JZ) \\ &+ g_M(\mathcal{V}\nabla_V \omega W, JZ) \\ &= -g_M\left(\hat{\nabla}_W \phi V, JZ\right) - g_M(T_V \omega V, JZ) + g_M\left(\hat{\nabla}_V \phi W, JZ\right) \\ &+ g_M(T_W \omega V, JZ).\end{aligned}$$

Finally, we get

$$g_M([V, W], Z) = g_M\left(T_V \omega W - T_W \omega V + \hat{\nabla}_V \phi W - \hat{\nabla}_W \phi V, JZ\right). \quad (4.9)$$

Since for, $JZ \in \Gamma(D_1)$. From equation (4.9) it follows that the distribution D_2 is integrable if and only if

$$T_V \omega W - T_W \omega V + \hat{\nabla}_V \phi W - \hat{\nabla}_W \phi V \in \Gamma(D_2),$$

for any $V, W \in \Gamma(D_2)$. Which completes the proof. □

We now discuss the geometry of the leaves of the distributions D_1 and D_2, and relate it with the geometry of the base manifold B using the second fundamental form of the Riemannian map F, and we have the following propositions.

Proposition 4.1 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then the distribution D_1 defines a totally geodesic foliation if and only if*

$$g_M\left(\hat{\nabla}_U JV, \phi W_2\right) = g_B((\nabla F_*)(U, JV), F_* \omega W_2)$$

and

$$g_M\left(\hat{\nabla}_{U_1} JV_1, BX\right) = g_B((\nabla F_*)(U_1, JV_1), F_* CX),$$

for any vector fields $U, V \in \Gamma(D_1), W_2 \in \Gamma(D_2)$ and $X \in \Gamma(kerF_*)^\perp$.

Proof. For $U_1, V_1 \in \Gamma(D_1)$ and $X \in \Gamma(kerF_*)^\perp$ using equation (3.4) we have

$$\begin{aligned} g_M\left(\nabla_{U_1}V_1, X\right) &= g_M\left(J\nabla_{U_1}V_1, JX\right) \\ &= g_M\left(\nabla_{U_1}JV_1, BX\right) + g_M\left(\nabla_{U_1}JV_1, CX\right) \\ &= g_M\left(\hat{\nabla}_{U_1}JV_1, BX\right) + g_M\left(\overline{\nabla}_{U_1}JV_1, CX\right) \\ &= g_M\left(\hat{\nabla}_{U_1}JV_1, BX\right) + g_B\left(F_*(\overline{\nabla}_{U_1}JV_1), F_*CX\right), \end{aligned}$$

where we have used the fact that F be a Riemannian map. Using now equation (2.13) we have

$$g_M\left(\nabla_{U_1}V_1, X\right) = g_M\left(\hat{\nabla}_{U_1}JV_1, BX\right) - g_B\left((\nabla F_*)(U_1, JV_1), F_*CX\right).$$

Hence $\nabla_{U_1}V_1 \in \Gamma(kerF_*)$ if and only if

$$g_M\left(\hat{\nabla}_{U_1}JV_1, BX\right) = g_B\left((\nabla F_*)(U_1, JV_1), F_*CX\right). \tag{4.10}$$

Further, for $U_1, V_1 \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$, using equations (2.1),(2.3) and (3.1) we have

$$\begin{aligned} g_M\left(\nabla_{U_1}V_1, W_2\right) &= g_M\left(J\nabla_{U_1}V_1, JW_2\right) \\ &= g_M\left(J\nabla_{U_1}JV_1, JW_2\right) \\ &= g_M\left(\mathcal{H}\nabla_{U_1}JV_1, \phi W_2\right) + g_M\left(\mathcal{V}\nabla_{U_1}JV_1, \phi W_2\right) + g\left(\mathcal{H}\nabla_{U_1}JV_1, \omega W_2\right) \\ &+ g_M\left(\mathcal{V}\nabla_{U_1}JV_1, \omega W_2\right) \\ &= g_M\left(\hat{\nabla}_{U_1}JV_1, \phi W_2\right) + g_M\left(\overline{\nabla}_{U_1}JV_1, \omega W_2\right). \end{aligned}$$

Since F is a Riemannian map, using (2.13) we have

$$\begin{aligned} g_M\left(\nabla_{U_1}V_1, W_2\right) &= g_M\left(\hat{\nabla}_{U_1}JV_1, \phi W_2\right) + g_B\left(F_*(\overline{\nabla}_{U_1}JV_1), F_*\omega W_2\right) \\ &= g_M\left(\hat{\nabla}_{U_1}JV_1, \phi W_2\right) - g_B\left((\nabla F_*)(U_1, JV_1), F_*\omega W_2\right). \end{aligned}$$

Hence $\nabla_{U_1}V_1 \in \Gamma(D_1)$ if and only if

$$g_M\left(\hat{\nabla}_{U_1}JV_1, \phi W_2\right) = g_B\left((\nabla F_*)(U_1, JV_1), F_*\omega W_2\right). \tag{4.11}$$

The result then follows from (4.10) and (4.11) and which completes the proof. □

Proposition 4.2 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then the distribution D_2 defines a totally geodesic foliation if and only if*

$$\begin{aligned} g_M\left(\hat{\nabla}_{V_2}\phi W_2, BX\right) + g_M\left(T_{V_2}\omega W_2, BX\right) &= g_B((\nabla F_*)(V_2, \phi W_2) \\ &+ (\nabla F_*)(V_2, \omega W_2), F_*CX) \end{aligned}$$

and

$$\hat{\nabla}_{V_2}\phi W_2 + T_{V_2}\omega W_2 \in \Gamma(D_2),$$

for any $V_2, W_2 \in \Gamma(D_2)$ *and* $X \in \Gamma(kerF_*)^{\perp}$.

Proof. For any $V_2, W_2 \in \Gamma(D_2), X \in \Gamma(kerF_*)^{\perp}$.
Using equations (2.1),(2.3),(2.13),(3.2),(3.4),Lemma 2.1 and the fact that F is a Riemannian map we have

$$\begin{aligned} g_M\left(\nabla_{V_2}W_2, X\right) &= g_M\left(\nabla_{V_2}JW_2, JX\right) \\ &= g_M\left(\hat{\nabla}_{V_2}\phi W_2, BX\right) + g_M\left(v\nabla_{V_2}\omega W_2, BX\right) \\ &+ g_M\left(\overline{\nabla}_{V_2}\phi W_2, CX\right) + g_M\left(\overline{\nabla}_{V_2}\omega W_2, CX\right) \\ &= g_M\left(\hat{\nabla}_{V_2}\phi W_2, BX\right) + g_M\left(T_{V_2}\omega W_2, BX\right) \\ &+ g_M\left(F_*(\overline{\nabla}_{V_2}\phi W_2), F_*CX\right) + g_M\left(F_*(\overline{\nabla}_{V_2}\omega W_2), F_*CX\right) \\ &= g_M\left(\hat{\nabla}_{V_2}\phi W_2, BX\right) + g_M\left(T_{V_2}\omega W_2, BX\right) \\ &- g_B\left((\nabla F_*)(V_2, \phi W_2), F_*CX\right) - g_B\left((\nabla F_*)(V_2, \omega W_2), F_*CX\right). \end{aligned} \tag{4.12}$$

Equation (4.12) yields that $\nabla_{V_2}W_2 \in \Gamma(KerF_*)$ if and only if

$$\begin{aligned} g_M\left(\hat{\nabla}_{V_2}\phi W_2, BX\right) + g_M\left(T_{V_2}\omega W_2, BX\right) &= g_B\left((\nabla F_*)(V_2, \phi W_2), F_*CX\right) \\ &+ g_B\left((\nabla F_*)(V_2, \omega W_2), F_*CX\right) \end{aligned} \tag{4.13}$$

On the other hand, for $U_1 \in \Gamma(D_1)$ and $V_2, W_2 \in \Gamma(D_2)$ we have

$$\begin{aligned} g_M\left(\nabla_{V_2}W_2, U_1\right) &= g_M\left(\nabla_{V_2}JW_2, JU_1\right) \\ &= g_M\left(\mathcal{V}\nabla_{V_2}JW_2, JU_1\right) \\ &= g_M\left(\mathcal{V}\nabla_{V_2}\phi W_2, U_1\right) + g_M\left(\mathcal{V}\nabla_{V_2}\omega W_2, U_1\right) \\ &= g_M\left(\hat{\nabla}_{V_2}\phi W_2 + T_{V_2}\omega W_2, JU_1\right). \end{aligned}$$

Since for $U_1 \in \Gamma(D_1), JU_1 \in \Gamma(D_1)$, the above relation implies that $\nabla_{V_2}W_2 \in \Gamma(D_2)$ if and only if

$$\hat{\nabla}_{V_2}\phi W_2 + T_{V_2}\omega W_2 \in \Gamma(D_2). \tag{4.14}$$

The result then follows from (4.13) and (4.14) □

We now recall the following definition.

Definition 4.1 ([3]). Let g be a metric tensor on the manifold $M = B \times F$ and assume that the canonical distribution D_B and D_F intersect perpendicularly everywhere, then g is the metric tensor of a usual product of Riemannian manifold if and only if D_B and D_F are totally geodesic foliation.

From Proposition 4.1 and Proposition 4.2 we have the following theorem.

Theorem 4.4 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $F : (M, g_M, J)$ to a Riemannian manifold (B, g_B) with generic fibers. Then the integral manifold of the distributions $(kerF_*)$ is a locally Riemannian product of the leaves of the distribution D_1 and D_2 if and only if*

$$\begin{aligned} g_M\left(\hat{\nabla}_{U_1} JV_1, \phi W_2\right) &= g_B\left((\nabla F_*)(U_1, JV_1), F_*\omega W_2\right), \\ g_M\left(\hat{\nabla}_{U_1} JV_1, BX\right) &= g_B\left((\nabla F_*)(U_1, JV_1), F_*CX\right) \\ \text{and } g_M\left(\hat{\nabla}_{V_1}\phi W_2, BX\right) + g_M\left(T_{V_1}\omega W_2, BX\right) &= g_B\left((\nabla F_*)(V_1, \phi W_2), F_*CX\right) \\ &+ g_B\left((\nabla F_*)(V_1, \omega W_2), F_*CX\right); \end{aligned}$$

$$\hat{\nabla}_{V_1}\phi W_2 + T_V\omega W_2 \in \Gamma(D_2),$$

for any vector fields $U_1, V_1 \in \Gamma(D_1), V_2, W_2 \in \Gamma(D_2)$ and $X \in \Gamma(kerF_*)^\perp$.

Since the integral manifolds of the distribution $(kerF_*)$ are infact the fibers of the Riemannian map F, the above theorem can be re-stated as:

Theorem 4.5 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $F : (M, g_M, J)$ to a Riemannian manifold (B, g_B) with generic fibers. Then the fibers of F are the locally Riemannian product of the leaves of D_1 and D_2 if and only if*

$$\begin{aligned} g_M\left(\hat{\nabla}_{U_1} JV_1, \phi W_2\right) &= g_B\left((\nabla F_*)(U_1, JV_1), F_*\omega W_2\right), \\ g_M\left(\hat{\nabla}_{U_1} JV_1, BX\right) &= g_B\left((\nabla F_*)(U_1, JV_1), F_*CX\right) \\ \text{and } g_M\left(\hat{\nabla}_{V_1}\phi W_2, BX\right) + g_M\left(T_{V_1}\omega W_2, BX\right) &= g_B\left((\nabla F_*)(V_1, \phi W_2), F_*CX\right) \\ &+ g_B\left((\nabla F_*)(V_1, \omega W_2), F_*CX\right); \end{aligned}$$

$$\hat{\nabla}_{V_1}\phi W_2 + T_V\omega W_2 \in \Gamma(D_2),$$

for any vector fields $U_1, V_1 \in \Gamma(D_1), V_2, W_2 \in \Gamma(D_1)$ and $X \in \Gamma(kerF_*)^\perp$.

Now, we study the geometry of the leaves of the distribution $(kerF_*)$ and $(kerF_*)^\perp$.

Proposition 4.3 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then the distribution $(kerF_*)$ defines a totally geodesic foliation if and only if*

$$\begin{aligned} g_B\left((\nabla F_*)(V, \phi W), F_*CX\right) &+ g_B\left((\nabla F_*)(V, \omega W), F_*CX\right) \\ &= g_M\left(\hat{\nabla}_V\phi W, BX\right) + g_M\left(T_V\omega W, BX\right), \end{aligned}$$

for any $V, W \in \Gamma(kerF_*)$ *and* $X \in \Gamma(kerF_*)^{\perp}$.

Proof. For any $V, W \in \Gamma(kerF_*)$ and $X \in \Gamma(kerF_*)^{\perp}$, using equations (2.3),(2.13),(3.2),(3.4),Lemma 2.1 and the fact that F be a Reimannian map we have

$$\begin{aligned}
g_M(\nabla_V W, X) &= g_M(\nabla_V JW, JX) \\
&= g_M(\nabla_V(\phi W + \omega W), BX + CX) \\
&= g_M(\mathcal{V}\nabla_V \phi W, BX) + g_M(\mathcal{V}\nabla_V \omega W, BX) \\
&+ g_M(\mathcal{H}\nabla_V \phi W, CX) + g_M(\mathcal{H}\nabla_V \omega W, CX) \\
&= g_M\left(\hat{\nabla}_V \phi W, BX\right) + g_M(T_V \omega W, BX) \\
&+ g_M\left(\overline{\nabla}_V \phi W, CX\right) + g_M\left(\overline{\nabla}_V \omega W, CX\right) \\
&= g_M\left(\hat{\nabla}_V \phi W, BX\right) + g_M(T_V \omega W, BX) \\
&+ g_M\left(F_*(\overline{\nabla}_V \phi W), F_*CX\right) + g_M\left(F_*(\overline{\nabla}_V \omega W), F_*CX\right) \\
&= g_M\left(\hat{\nabla}_V \phi W, BX\right) + g_M(T_V \omega W, BX) \\
&- g_M((\nabla F_*)(V, \phi W), F_*CX) - g_M((\nabla F_*)(V, \omega W), F_*CX).
\end{aligned} \tag{4.15}$$

From equation (4.15) it follows that $kerF_*$ defines a totally geodesic foliation if and only if

$$\begin{aligned}
&g_B((\nabla F_*)(V, \phi W), F_*CX) + g_B((\nabla F_*)(V, \omega W), F_*CX) \\
&\qquad = g_M\left(\hat{\nabla}_V \phi W, BX\right) + g_M(T_V \omega W, BX).
\end{aligned}$$

Which completes the proof. □

Next, we have

Proposition 4.4 *Let* $F : (M, g_M, J) \longrightarrow (B, g_B)$ *be a Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (B, g_B) *with generic fibers. Then the distribution* $(kerF_*)^{\perp}$ *defines a totally geodesic foliation if and only if*

$$g_M(\omega A_X Y, \omega V) = -g_M(\mathcal{V}\nabla_X BY, \phi V) - g_M(A_X CY, \phi V),$$

for any $X, Y \in \Gamma(kerF_*)^{\perp}$ *and* $V \in \Gamma(kerF_*)$.

Proof. For any $X, Y \in \Gamma(kerF_*)^{\perp}$ and $V \in \Gamma(kerF_*)$, using equations (2.1),(2.3), we have

$$g_M(\nabla_X Y, V) = g_M(\nabla_X JY, JV).$$

Further, on using equations (3.2),(3.4) and Lemma 2.1 we obtain

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\nabla_X BY, \phi V) + g_M(\nabla_X BY, \omega V) \\ &+ g_M(\nabla_X CY, \phi V) + g_M(\nabla_X CY, \omega V) \\ &= g_M(\mathcal{V}\nabla_X BY, \phi V) + g_M(\mathcal{V}\nabla_X CY, \phi V) \\ &+ g_M(\mathcal{V}\nabla_X BY, \omega V) + g_M\left(\overline{\nabla}_X CY, \omega V\right) \\ &= g_M(\mathcal{V}\nabla_X BY, \phi V) + g_M(A_X CY, \omega V) \\ &+ g_M(A_X BY, \omega V) + g_M\left(\overline{\nabla}_X CY, \omega V\right). \end{aligned}$$

Using Proposition 3.1 we get

$$g_M(\nabla_X Y, V) = g_M(\mathcal{V}\nabla_X BY, \phi V) + g_M(A_X CY, \phi V) + g_M(\omega A_X Y, \omega V). \tag{4.16}$$

Hence, from (4.16) it follows that $(kerF_*)^\perp$ defines a totally geodesic foliation if and only if

$$g_M(\omega A_X Y, \omega V) = -g_M(\mathcal{V}\nabla_X BY, \phi V) - g_M(A_X CY, \phi V).$$

Which completes the proof. □

From equation (4.16) and Lemma 3.1 we also have

Proposition 4.5 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then the distribution $(kerF_*)^\perp$ defines a totally geodesic foliation if and only if*

$$\mathcal{V}\nabla_X BY + A_X CY = 0 \text{ and } A_X Y \in \Gamma(D_1)$$

for any $X, Y \in \Gamma(kerF_*)^\perp$.

Theorem 4.4 and Proposition 4.4 yields the following decomposition theorem.

Theorem 4.6 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then the total manifold M is a Riemannian product manifold of the leaves D_1,D_2 and $(kerF_*)^\perp$ i,e.,*
$M = M_{D_1} \times M_{D_2} \times M_{(kerF_*)^\perp}$, if and only if

$$\begin{aligned} g_M\left(\hat{\nabla}_{U_1} JV_1, \phi W_2\right) &= g_M((\nabla F_*)(U_1, JV_1), F_*\omega W); \\ g_M\left(\hat{\nabla}_{U_1} JV_1, BX\right) &= g_B((\nabla F_*)(U_1, JV_1), F_*CX) \\ g_B((\nabla F_*)(V_2, \phi W_2), F_*CX) &+ g_B((\nabla F_*)(V_2, \omega W_2), F_*CX) \\ &= g_M\left(\hat{\nabla}_{V_2}\phi W_2, BX\right) + g_M(T_{V_2}\omega W_2, BX); \end{aligned}$$

$$\hat{\nabla}_{V_2}\phi W_2 + T_{V_2}\omega W_2 \in \Gamma(D_2).$$

and

$$g_M\left(\omega A_X Y, \omega V\right) = -g_M\left(v\nabla_X BY, \phi V\right) - g_M\left(A_X CY, \phi V\right),$$

for any $U_1, V_1 \in \Gamma(D_1), V_2, W_2 \in \Gamma(D_2), V \in \Gamma(kerF_*)$ *and* $X, Y \in \Gamma(kerF_*)^{\perp}$, *where* M_{D_1}, M_{D_2} *and* $M_{(kerF*)^{\perp}}$ *are respectively the leaves of* D_1, D_2 *and* $(kerF_*)^{\perp}$.

From Proposition 4.3 and Proposition 4.4 we have following theorem

Theorem 4.7 *Let* $F : (M, g_M, J) \longrightarrow (B, g_B)$ *be a Riemannian map from Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (B, g_B) *with generic fibers. Then the total space is a generic product manifold i.e,*
$M = M_{(kerF_*)} \times M_{(kerF_*)^{\perp}}$, *if and only if*

$$\begin{aligned} g_M\left((\nabla F_*)(V, \phi W), F_*CX\right) &+ g_M\left((\nabla F_*)(V, \omega W)\right) \\ &= g_M\left(\hat{\nabla}_V \phi W, BX\right) + g_M\left(T_V \omega W, BX\right) \end{aligned}$$

and

$$g_M(\omega A_X Y, \omega V) = -g_M(v\nabla_X BY, \phi V) - g_M(A_X CY, \phi V),$$

for any $V, W \in \Gamma(kerF_*)$ *and* $X, Y \in \Gamma(kerF_*)^{\perp}$.

If we consider Proposition 4.5 along with Proposition 4.3, then we have the following theorem

Theorem 4.8 *Let* $F : (M, g_M, J) \longrightarrow (B, g_B)$ *be a Riemannian map from a Keahler manifold* (M, g_M, J) *to a Riemannian manifold* (B, g_B) *with generic fibers. Then* M *is a generic product manifold, i.e.,*
$M = M_{(kerF_*)} \times M_{(kerF_*)^{\perp}}$, *if and only if*

$$\begin{aligned} g_M\left((\nabla F_*)(V, \phi W), F_*CX\right) &+ g_M\left((\nabla F_*)(V, \omega W)\right) \\ &= g_M\left(\hat{\nabla}_V \phi W, BX\right) + g_M\left(T_V \omega W, BX\right) \end{aligned}$$

and

$$\mathcal{V}\nabla_X BX + A_X CY = 0; \;\; A_X Y \in \Gamma(D_1),$$

for any $X, Y \in \Gamma(kerF_*)^{\perp}$ *and* $V, W \in \Gamma(kerF_*)$.

5. Totally Geodesic Maps

In this section we obtain the necessary and sufficient condition for the Riemannian map F to be totally geodesic map. First we recall the following definition.

Definition 5.1 Let $F : (M, g_M) \longrightarrow (B, g_B)$ be a Riemannian map between the Riemannian manifolds (M, g_M) and (B, g_B). Then F is said to be totally geodesic map if

$$(\nabla F_*)(X, Y) = 0, \tag{5.1}$$

for all vector fields $X, Y \in \Gamma(TM)$.

We have the following theorem.

Theorem 5.1 *Let $F : (M, g_M, J) \longrightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (B, g_B) with generic fibers. Then F is a totally geodesic map if and only if*

$$\hat{\nabla}_V \phi W + T_V \omega W \in \Gamma(D_1),$$
$$T_V \phi W + \mathcal{H}\overline{\nabla}_V \omega W \in \Gamma(\omega D_2),$$
$$\hat{\nabla}_V BX + T_V CX \in \Gamma(D_1),$$
$$T_V BX + \mathcal{H}\overline{\nabla}_V CX \in \Gamma(\omega D_2),$$

for any $V, W \in \Gamma(kerF_*)$ and $X, Y \in \Gamma(kerF_*)^\perp$.

Proof. Since F is a Riemannian map, then from Lemma 2.2 we have

$$(\nabla F_*)(X, Y) = 0, \tag{5.2}$$

for any $X, Y \in \Gamma(kerF_*)^\perp$

For any $V, W \in \Gamma(kerF_*)$ and $X, Y \in \Gamma(kerF_*)^\perp$, using equation (2.13) we have

$$(\nabla F_*)(V, W) = -F_*(\overline{\nabla}_V W), \tag{5.3}$$

Now, using equations (2.3),(3.2),(3.4) and Lemma 2.1 we have

$$\begin{aligned}
-(\overline{\nabla}_V W) &= J(\overline{\nabla}_V JW) \\
&= J\overline{\nabla}_V \phi W + J\overline{\nabla}_V \omega W \\
&= J\left(\mathcal{H}\overline{\nabla}_V \phi W + \mathcal{V}\overline{\nabla}_V \phi W\right) + J\left(\mathcal{H}\overline{\nabla}_V \omega W + \mathcal{V}\overline{\nabla}_V \omega W\right) \\
&= \left(B\mathcal{H}\overline{\nabla}_V \phi W + B\mathcal{H}\overline{\nabla}_V \omega W\right) + \left(C\mathcal{H}\overline{\nabla}_V \phi W + C\mathcal{H}\overline{\nabla}_V \omega W\right) \\
&+ \left(\phi\hat{\nabla}_V \phi W + \phi\mathcal{V}\overline{\nabla}_V \omega W\right) + \left(\omega\hat{\nabla}_V \phi W + \omega\mathcal{V}\overline{\nabla}_V \omega W\right) \\
&= B\left(T_V \phi W + \overline{\nabla}_V \omega W\right) + C\left(T_V \phi W + \overline{\nabla}_V \omega W\right) \\
&+ \phi\left(\hat{\nabla}_V \phi W + T_V \omega W\right) + \omega\left(\hat{\nabla}_V \phi W + T_V \omega W\right).
\end{aligned}$$

Applying F_* to above equation we have

$$-F_*(\overline{\nabla}_V W) = F_*\left(C(T_V \phi W + H\overline{\nabla}_V \omega W)\right) + F_*\left(\omega(\hat{\nabla}_V \phi W + T_V \omega W)\right). \tag{5.4}$$

From (5.3) and (5.4), we have

$$(\nabla F_*)(V, W) = F_*\left(C(T_V \phi W + \mathcal{H}\overline{\nabla}_V \omega W)\right) + F_*\left(\omega(\hat{\nabla}_V \phi W + T_V \omega W)\right).$$

Hence, $(\nabla F_*)(V,W)=0$ if and only if

$$\hat{\nabla}_V \phi W + T_V \omega W \in \Gamma(D_1),\ T_V \phi W + \mathcal{H}\overline{\nabla}_V \phi W \in \Gamma(\omega D_2). \tag{5.5}$$

On the other hand using (2.13) for any $V \in \Gamma(kerF_*)$ and $X \in \Gamma(kerF_*)^\perp$, we have

$$(\nabla F_*)(V,X) = -F_*(\overline{\nabla}_V X). \tag{5.6}$$

But

$$\begin{aligned}
-\overline{\nabla}_V X &= J\overline{\nabla}_V JX \\
&= J(\overline{\nabla}_V BX) + J(\overline{\nabla}_V CX) \\
&= J(\mathcal{H}\overline{\nabla}_V BX) + J(\mathcal{H}\overline{\nabla}_V CX) + J(\mathcal{V}\overline{\nabla}_V BX) + J(\mathcal{V}\overline{\nabla}_V CX) \\
&= B(T_V BX) + C(T_V BX) + B(\mathcal{H}\overline{\nabla}_V CX) + C(\mathcal{H}\overline{\nabla}_V CX) \\
&+ \phi\hat{\nabla}_V BX + \omega\hat{\nabla}_V BX + \phi\mathcal{V}\overline{\nabla}_V CX + \omega\mathcal{V}\overline{\nabla}_V CX \\
&= B(T_V BX + \mathcal{H}\overline{\nabla}_V CX) + \phi\left(\hat{\nabla}_V BX + \mathcal{V}\overline{\nabla}_V CX\right) \\
&+ C(T_V BX + H\overline{\nabla}_V CX) + \omega\left(\hat{\nabla}_V BX + T_V CX\right).
\end{aligned}$$

Therefore,

$$-F_*(\overline{\nabla}_V X) = F_*\left(\omega(\hat{\nabla}_V BX + T_V CX)\right) + F_*\left(C(T_V BX + \overline{\nabla}_V CX)\right).$$

Which then yields $(\nabla F_*)(V,X)=0$ if and only if

$$\omega\left(\hat{\nabla}_V BX + T_V CX\right) + C\left(T_V BX + \mathcal{H}\overline{\nabla}_V CX\right) = 0$$

that is, $(\nabla F_*)(V,X)=0$ if and only if

$$\hat{\nabla}_V BX + T_V CX \in \Gamma(D_1);\ T_V BX + \mathcal{H}\overline{\nabla}_V CX \in \Gamma(\omega D_2). \tag{5.7}$$

which completes the proof. □

Now we recall the following definition:

Definition 5.2 Let $F:(M,g_M)\longrightarrow(B,g_B)$ be a Riemannian map between the Riemannian manifold (M,g_M) and (B,g_B). Then we say that the fibers of the map F are totally umbilical if and only if

$$h(V,W) = g(V,W)\lambda, \tag{5.8}$$

for any $X,Y \in \Gamma(TM)$ where h is the second fundamental form of the fibers when considered as the immersed submanifolds of the tatal space M and concide with the B. O'Neill's fundamental tensor T for the vector fields $V,W \in \Gamma(kerF_*)$. λ is called the mean curvature vector of the fibers and is a horizontal vector field.

Infact, we prove the following.

Theorem 5.2 *Let $F:(M,g_M,J)\longrightarrow(B,g_B)$ be a Riemannian map from a kaehler manifold (M,g_M,J) to a Riemannian manifold (B,g_B) with generic and D_1-totally umbilical fibers. Then $\lambda \in \Gamma(\omega D_2)$.*

Proof. Since M is a Kaehler manifold, for any $V, W \in \Gamma(D_1)$ using (2.3) we have

$$\nabla_V JW = J\nabla_V W.$$

Now using (3.2) and (3.4) we have

$$h(V, JW) + \hat{\nabla}_V JW = Bh(V, W) + Ch(V, W) + \phi\nabla_V W + \omega\nabla_V W.$$

Taking Riemannian product in above equation with a vector field $X \in \Gamma(\mu)$ and then using (5.8) we obtain

$$\begin{aligned} g(h(V, JW), X) &= g(Ch(V, W), X), \\ g(V, JW)g(\lambda, X) &= -g(h(V, W), JX) \\ &= -g(V, W)g(\lambda, JX). \end{aligned} \tag{5.9}$$

Interchanging V and W in (5.9), we have

$$g(W, JV)g(\lambda, X) = -g(W, V)g(\lambda, JX). \tag{5.10}$$

From equation (5.9) and (5.10) on combining them, we get

$$g(\lambda, JX) = 0.$$

Which gives the result. □

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