Riemannian Maps From Kaehler Manifold
With Generic Fibers

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Abstract. We study Riemannian maps from almost Hermitian manifolds to Riemannian manifolds for the case when the fibers are generic submanifold of the total space. We obtain the integrability conditions for the distributions while vertical distribution is always integrable. We also study the geometry of the leaves of the distribution which arise from such maps, and obtain the necessary and sufficient conditions for the fibers as well as the total manifold to be generic product manifolds. We, further, obtain the necessary and sufficient condition for a Riemannian maps with generic fibers to be totally geodesic map.

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1. Introduction

In Riemannian geometry, the theory of smooth maps between Riemannian manifolds was extensively used. Typically, these maps are used to compare the geometric structures of two manifolds. From this perspective, these smooth maps are the isometric immersion between Riemannian manifolds which are characterized by the Riemanniann metrics and Jacobians matrices. Let \((M,g_M)\) and \((B,g_B)\) be two Riemannian manifolds, then a smooth map

\[ F : (M,g_M) \longrightarrow (B,g_B), \]

where \(dimM = m\), \(dimM_2 = n\) and \(m \leq n\), is called an isometric immersion if the Jacobian map which we denote by \(F_*\) is injective and satisfies

\[ g_B(F_*X, F_*Y) = g_M(X, Y), \quad (1.1) \]

for any vector fields \(X\) and \(Y\) tangent to \(M\).

If \(m \geq n\) then the smooth map \(F\) is called the Riemannian submersions.
This term Riemannian submersion was first studied by B.O’Neill [4],[5] and Gray[3].

Fischer, A.E[2] introduced the idea of a Riemannian maps. The concept of isometric immersion, Riemannian submersion and an isometry are generalised and in fact unifies by the concept of Riemannian maps. Every injective Riemannian map is an injective isometric immersion, and that on a connected manifold, every surjective Riemannian map is a surjective Riemannian submersion and every bijective Riemannian map is an isometry. The theory of the smooth maps between Riemannian manifolds, the two notions of an immersion and submersion get into the notions of an isometric immersion and Riemannian submersion respectively, and were widely used in differential geometry[4],[10]. But there is no Riemannian analogue which corresponds to the general map between smooth manifolds.

Let $F: (M, g_M) \to (B, g_B)$ be a smooth map between Riemannian manifolds $(M, g_M)$ and $(B, g_B)$ such that $0 < \text{rank} F < \min\{m, n\}$. If kernal space of $F_*$ is denoted by $\ker F_*$ and its orthogonal complement is denoted by $(\ker F_*)^\perp$ in tangent bundle $TM$ of $M$, then the tangent bundle of $M$ has the following decomposition

$$TM = (\ker F_*) \oplus (\ker F_*)^\perp.$$ 

We call $(\ker F_*)$ and respectively $(\ker F_*)^\perp$ the vertical and horizontal space of $TM$.

In the tangent bundle $TB$ of $B$, the range of $F_*$ is denoted by $\text{rang} F_*$ and its orthogonal complement is denoted by $(\text{rang} F_*)^\perp$ because $(\text{rang} F_*)^\perp$ is non-empty. Hence, the tangent bundle $TB$ of $B$ is decomposed as:

$$TB = (\text{rang} F_*) \oplus (\text{rang} F_*)^\perp.$$ 

Now, let $F: (M, g_M) \to (B, g_B)$ be a smooth map between Riemannian manifolds. Then the smooth map $F$ is said to be a Riemannina map at a point $p \in M$ if the horizontal restrictions $F^h_{*p}$ of the derivative map $F_*$ at $p$, i.e.,

$$F^h_{*p}: (\ker F_{*p})^\perp \to (\text{rang} F_{*p})$$

is a linear isometry (also known isometric isomorphism) between the inner product spaces $\left( (\ker F_{*p})^\perp, g_M(p) \right)$ and $\left( (\text{rang} F_{*p}), g_B(q) \right)$, where $q = F(p)$.

The map $F$ is a Riemannian map if $F$ is a Riemannian map at each $p \in M$. Hence, Fischer in the abstract of the paper [2] has remarked that a Riemannian map is a map that is “as isometric as it can be” subject to the limitations imposed upon it as a differential mapping. Also he has remarked that $F$ is a Riemannian map at $p \in M$, $q = F(p) \in B$, if for all $X, Y \in (\ker F_*)^\perp \subset T_p M$.

B. Sahin[6], [7],[9] recently introduced the generalization of anti-invariant Riemannian map, he introduced the notion of conformal semi-invariant Riemannian maps when the base manifold is a Riemannian manifold and a
Kaehler manifold. In this article, we study the Riemannian maps from almost Hermitian manifolds under the assumptions that the integral manifolds of vertical distribution are generic submanifolds of the total space and call it the Riemannian maps with generic fibers. The paper is structured as follows: In section 2, we give some basic definitions and brief introduction of anti-invariant and semi-invariant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. In section 3, we defined Riemannian maps with generic fibers and obtain some results. In section 4, we obtain the integrability condition for the distributions and decomposition theorems and in the last section, we find the necessary and sufficient conditions for such Riemannian maps to be totally geodesic.

2. Preliminaries

In this section we recall some basic definitions and notions of almost Hermitian manifolds, Kaehler manifold, CR-submanifolds and give a brief review of basic fact of Riemannian maps. For the notion of Riemannian maps we follow Fischer [2] and B. Sahin [6],[8].

Let $M$ be an almost complex manifold, that is, $M$ admits a tensor field $J$ of type $(1,1)$ with the property that $J^2 = -I$. An almost complex manifold is necessarily orientable and is of even dimension. An almost complex manifold $(M,J)$ endowed with a chosen Riemannian metric $g$ and satisfying the condition

$$g(JX, JY) = g(X, Y),$$

(2.1)

for all $X, Y \in \Gamma(TM)$, is called an almost Hermitian manifold.

The Levi-Civita connection $\nabla$ of the almost Hermitian manifold $(M, J)$ can be extended to whole tensor algebra on $M$, and in this way we obtain tensor fields like $(\nabla_X J)$ and that

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y,$$

(2.2)

for all $X, Y \in \Gamma(TM)$.

An almost Hermitian manifold $\overline{M}$ is called Kaehler manifold if

$$(\nabla_X J)(Y) = 0, \ \forall X, Y \in \Gamma(\overline{TM})$$

(2.3)

where $\nabla$ is the Levi-Civita connection on $\overline{M}$.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and $M$ be a real submanifold of $\overline{M}$, and let

$$D_p = T_p M \cap JT_p M, \ \forall p \in M$$

such that $D_p$ is the maximal subspace of $T_p M$.

**Definition 2.1** ([1]). A submanifold $M$ is said to be a CR-submanifold of an almost Hermitian manifold $(\overline{M}, g)$ if there exists on $M$ a $C^\infty$-holomorphic distribution $D$ such that its orthogonal complementary distribution $D^\perp$ is totally real, i.e., $JD_p^\perp \subseteq T_p^\perp M$ for all $p \in M$. A CR-submanifold $M$ is said to be proper if neither $D = \{0\}$, nor $D^\perp = \{0\}$. 

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Now, we recall the definition of generic submanifold which is the generalization of CR-submanifolds. These submanifolds are defined by relaxing the condition on the complementary distribution to the holomorphic distribution.

**Definition 2.2** ([1]). Let \((\overline{M}, g, J)\) be an almost Hermitian manifold and let \(M\) be a real submanifold of \(\overline{M}\). Then \(M\) is said to be a generic submanifold of \(\overline{M}\) if the maximal complex subspace \(D_p\) has constant dimension at each point \(p \in M\) and defines a differentiable distribution on \(M\).

We denote by \(D_\perp\) be orthogonal complementary distribution to \(D\) in \(\Gamma(TM)\) and observe that \(J D_\perp \cap D_\perp = \{0\}\). If, in particular, \(J D_\perp \subset \Gamma(T_\perp M)\), we have the concept of CR-submanifold. We call \(D\) and \(D_\perp\) the holomorphic and the purely real distribution on \(M\).

For \(U \in \Gamma(TM)\), we put
\[
JU = PU + FU,
\]
where \(PU\) and \(FU\) are the tangential and normal part of \(JU\) respectively.

For a generic submanifold we have
\[
(i) PD = D, \quad FD = \{0\}, (ii) PD_\perp \subset D_\perp, \quad FD_\perp \subset T_\perp M.
\]

For the theory of Riemannian maps we follow A. E, Fischer [2] and B. Sahin [6],[8].

Let \(F : (M, g_M) \rightarrow (B, g_B)\) be a Riemannian map between the Riemannian manifolds \((M, g_M)\) and \((B, g_B)\), where \(\text{dim}M = m\) and \(\text{dim}B = n\) with \(0 < \text{rank}F < \text{min}\{m, n\}\). The letters \(H\) and \(V\) are used to denote the orthogonal projections of \(\Gamma(TM)\) on the distributions \(\Gamma(\ker F^*)_\perp\) and \(\Gamma(\ker F^*)\) respectively. The geometry of Riemannian maps are characterized by the tensor fields \(T\) and \(A\) of the Riemannian map \(F\) defined for arbitrary vector fields \(E\) and \(F\) on \(M\) by
\[
A_E F = H \nabla_{HE} VF + V \nabla_{HE} HF, \tag{2.6}
\]
\[
T_E F = H \nabla_{VE} VF + V \nabla_{VE} HF, \tag{2.7}
\]
where \(\nabla\) is the Levi-Civita connection of \(g_M\). Indeed, one can see that these tensor fields are B. O’Neill’s fundamental tensor fields defined for the Riemannian submersion. It is easy to see that the Riemannian map \(F : M \rightarrow B\) has totally geodesic fibers if and only if \(T\) vanishes identically. For any \(E \in \Gamma(TM), T_E\) and \(A_E\) are skew-symmetric operators on \((\Gamma(TM), g)\) reversing the horizontal and vertical spaces. It is also seen that \(T\) is vertical, i.e., \(T_E = T_{VE}\) and \(A\) is horizontal, i.e., \(A_E = A_{HE}\). We observe that the tensor fields \(T\) and \(A\) satisfy
\[
(i) T_U V = T_V U, \quad U, V \in \Gamma(\ker F^*), (ii) A_X Y = -A_Y X, \quad X, Y \in \Gamma(\ker F^*)_\perp. \tag{2.8}
\]

On the other hand, (2.6) and (2.7) give the following lemma.

**Lemma 2.1** ([2]). We have
\[
\nabla_U V = T_U V + \hat{\nabla}_U V, \tag{2.9}
\]
\[
\nabla_U X = H \nabla_U X + T_U X, \tag{2.10}
\]
\[ \nabla_X U = A_X U + \nabla_X U, \quad (2.11) \]
\[ \nabla_X Y = \mathcal{H} \nabla_X Y + A_X Y, \quad (2.12) \]

for any \( X, Y \in \Gamma(\ker F^*)^\perp \) and \( U, V \in \Gamma(\ker F^*_*) \), where \( \tilde{\nabla} U = \nabla \nabla U \).

Next, we recall the following definition.

**Definition 2.3** ([8]). Let \( F : (M, g_M, J) \to (B, g_B) \) be a Riemannian map from an almost Hermitian manifold \((M, g_M, J)\) to a Riemannian manifold \((B, g_B)\). Then \( F \) is said to be semi-invariant Riemannian map if there is a distribution \( D_1 \in \Gamma(\ker F^*) \), such that

\[ \ker F^* = D_1 \oplus D_2, \quad J D_1 = D_1, \quad J D_2 \subset \Gamma(\ker F^*)^\perp, \]

where \( D_2 \) is the orthogonal complement of \( D_1 \) in \( \Gamma(\ker F^*) \).

Finally, we recall the notion of second fundamental form of a map between Riemannian manifolds. Let \((M, g_M)\) and \((B, g_B)\) be Riemannian manifolds, and let \( \phi : M \to B \) be a smooth map between them. Then the differential \( \phi^* \) of \( \phi \) can be viewed as a section of the bundle \( \text{Hom}(\mathcal{T}M, \phi^{-1}(\mathcal{T}B)) \to M \), where, \( \phi^{-1}(\mathcal{T}B) = T_{\phi(p)}B, \quad p \in M, \quad \text{Hom}(TM, \phi^{-1}(TB)) \) has a connection \( \nabla \) induced from Levi-Civita connection \( \nabla \) on \( M \) and the pullback connection.

The second fundamental form of \( \phi \) is then given by

\[ (\nabla_{\phi^*}(X, Y)) = \nabla^\phi_{\phi^*}(Y) - \phi^*(\tilde{\nabla}_XY) \quad (2.13) \]

for any \( X, Y \in \Gamma(TM) \), where \( \nabla^\phi \) is the pullback connection. It is known that the second fundamental form is symmetric. We now state the result of B. Sahin [6], which shows that the second fundamental form \( (\nabla_{\phi^*}(X, Y)) \), \( \forall X, Y \in (\ker F^*)^\perp \), of a Riemannian map \( \phi \) has no component in \( \text{rang} \phi^* \).

**Lemma 2.2** ([6]). Let \( \phi \) be a Riemannian map from a Riemannian manifold \((M, g_M)\) to a Riemannian manifold \((B, g_B)\). Then

\[ g_B((\nabla_{\phi^*}(X, Y), \phi^*Z) = 0, \quad \forall X, Y, Z \in \Gamma(\ker F^*)^\perp. \quad (2.14) \]

### 3. Riemannian Maps With Generic Fibers

B. Sahin [6],[8] defined anti-invariant and semi-invariant Riemannian maps from almost Hermitian manifolds to a Riemannian manifold. In these two cases he has defined them for the cases where the vertical distribution are infact anti-invariant and semi-invariant respectively. That means the integral manifolds(or for that matter the fibers) \( F^{-1}(q), \ q \in B \) of \( \ker F_* \) are respectively anti-invariant and semi-invariant submanifolds of \( M \). In this section we consider Riemannian maps for the case when the integral manifolds(the fibers) of \( \ker F_* \) are generic submanifolds of \( M \) which inturn generalizes these above mentioned maps. We obtain the integrability conditions for the distributions and investigate the geometry of the distribution \( (\ker F_*)^\perp \) and \( (\ker F_*)^\perp \). Also, we obtain the necessary and sufficient condition for such maps to be totally geodesic. We also obtain product theorem for the total manifold of such Riemannian maps.
Let $F$ be a Riemannian map from an almost Hermitian manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$. We say that the integral manifold (or for that matter the fibers) $F^{-1}(q), q \in B$ of the vertical distribution $\ker F_*$ is a generic submanifold of $M$ if the maximal complex space $D_p = (\ker F_* p) \cap (J \ker F_* p), p \in M$, defines on $F^{-1}(q)$ a differential distribution $D: p \mapsto D_p \subset (\ker F_* p)$ such that

$$\ker F_* = D_1 \oplus D_2, J D_1 = D_1,$$

where $D_2$ is the orthogonal complement of $D$ in $\Gamma(\ker F_*)$, and is called a purely real distribution of the fibers of the Riemannian map $F$.

**Definition 3.1** The Riemannian map $F: (M, g_M, J) \rightarrow (B, g_B)$ satisfying condition (3.1) is called a Reimannian map with generic fibers.

For any $V \in \Gamma(\ker F_*)$ we set

$$JV = \phi V + \omega V,$$

where $\phi V \in \Gamma(D_1)$ and $\omega V \in \Gamma(\ker F_* ^\perp)$. We denote the orthogonal complementary distribution to $\omega D_2$ in $(\ker F_*)^\perp$ by $\mu$. Then we can write

$$(\ker F_*)^\perp = \omega D_2 \oplus \mu.$$  

It is easy to see that $\mu$ is $J$-invariant. Thus, for any $X \in \Gamma(\ker F_*^\perp)$ we have

$$JX = BX + CX,$$

where $BX \in \Gamma(D_2)$ and $CX \in \Gamma(\mu)$.

Using (3.1) through (3.4) we obtain

**Lemma 3.1** Let $F: (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from an almost Hermitian manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then We have

(i) $\phi D_1 = D_1, \omega D_1 = 0$
(ii) $\phi D_2 \subset D_2, B(\ker F_* ^\perp) = D_2$
(iii) $\phi^2 + B \omega = -id, \phi \omega + C \omega = 0$
(iv) $BC + \phi B = 0, \omega B + C^2 = -id$.

Next, using equations (2.1),(3.2)and Lemma 2.1 we have

**Lemma 3.2** Let $(M, g_M, J)$ be an almost Hermitian manifold and $F$ be a Riemannian map from $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then

(i) $g_M(JY, \phi V) = g(BY, JV),$
(ii) $g_M(CY, \omega V) = 0,$
(iii) $g_M(\nabla_X BY, JV) = -g_M(BY, \nabla_X \phi V) - g_M(BY, A_X \omega V),$
(iv) $g_M(\nabla_U BY, JV) = -g_M(BY, \hat{\nabla}_U \phi V) - g_M(BY, T_U \omega V),$
(v) $g_M(\nabla_U BY, CX) = g_M(CX, T_U BY) = -g(BY, T_U CX),$

for any $X, Y \in \Gamma(\ker F_*^\perp)$ and $U, V \in \Gamma(\ker F_*)$.  

Proof. Proof of (i) and (ii) directly follows from equations (2.1), (3.1), (3.2), (3.3) and (3.4). (iii), Using the fact that for any $Y \in \Gamma(kerF_*)$, $BY \in \Gamma(D_2)$ and for any $V \in \Gamma(kerF_*)$, $\phi V \in \Gamma(D_1)$, for any $X \in \Gamma(kerF_*)$ we have
\[
g_M(\nabla_X BY, JV) = -g_M(\nabla_X BY, \phi V) + g_M(\nabla_X BY, \omega V)
\]
\[
= -g_M(BY, \nabla_X \phi V) - g_M(BY, \nabla_X \omega V)
\]
\[
= -g_M(BY, \nabla_X \phi V) - g_M(BY, J\nabla_X \omega V)
\]
\[
= -g_M(BY, J\nabla_X \phi V) - g_M(BY, A_X \omega V).
\]
Similarly, we obtain (iv) and (v). □

**Proposition 3.1** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then we have
\[
g_M(\nabla_X CY, \omega V) = g_M(BY, A_X \omega V) + g_M(\omega A_X Y, \omega V)
\]
\[
= -g_M(A_X BY, \omega V) + g_M(\omega A_X Y, \omega V),
\]
for any $X, Y \in \Gamma(kerF_*)$ and $V \in \Gamma(kerF_*)$.

Proof. From Lemma 3.1(i), Lemma 3.2(ii) and equations (2.2), (2.3) for any $X, Y \in \Gamma(kerF_*)$ and $V \in \Gamma(kerF_*)$ we have
\[
g_M(\nabla_X CY, \omega V) = -g_M(CY, \nabla_X \omega V)
\]
\[
= -g_M(JY, \nabla_X \omega V) + g_M(BY, \nabla_X \omega V)
\]
\[
= g_M(\nabla_X JY, \omega V) - g_M(\nabla_X BY, \omega V)
\]
\[
= g_M(J\nabla_X Y, \omega V) - g_M(\nabla_X BY, \omega V)
\]
\[
= -g_M(H\nabla_X Y, J\omega V) + g_M(J\nabla_X Y, J\omega V) - g_M(H\nabla_X BY, \omega V)
\]
\[
= -g_M(H\nabla_X Y, J\omega V) - g_M(\nabla_X BY, J\omega V) - g_M(A_X BY, \omega V)
\]
\[
= -g_M(H\nabla_X Y, J\omega V) - g_M(A_X BY, J\omega V) - g_M(A_X BY, \omega V)
\]
\[
= g_M(BH\nabla_X Y, \omega V) + g_M(CH\nabla_X Y, \omega V) + g_M(JAX Y, \omega V)
\]
\[
- g_M(A_X BY, \omega V).
\]
Since for any $X \in \Gamma(kerF_*)$, $BX \in \Gamma(D_2)$ and for $V \in \Gamma(kerF_*)$, $\omega V \in \Gamma(kerF_*)$. Also, using Lemma 3.1 again we have
\[
g_M(\nabla_X CY, \omega V) = -g_M(A_X BY, \omega V) + g_M(\omega A_X Y, \omega V).
\]
which completes the proof. □

Since $g_M$ is a non-degenerate metric on $M$, from Proposition 3.1 we have

**Corollary 3.1** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then
\[
\nabla_X CY = -A_X BY + \omega A_X Y,
\]
(3.11)
for any $X, Y \in \Gamma(\ker F^*)^\perp$.

Further, as a consequence of Proposition 3.1 we have

**Corollary 3.2** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then $\nabla_X CY \in \Gamma(\mu)$ if and only if

$$A_X BY = \omega A_X Y,$$

for any $X, Y \in \Gamma(\ker F^*)^\perp$.

We define the covariant derivative of $\phi$ and $\omega$ as follow:

$$(\nabla_V \phi)W = \hat{\nabla}_V \phi W - \phi \hat{\nabla}_V W$$

$$(\nabla_V \omega)W = \mathcal{H}(\nabla_V \omega W) - \omega \hat{\nabla}_V W$$

Then, using Lemma 2.1 and equations (3.2), (3.4) we obtain

$$(\nabla_V \phi)W = BT_V W - T_V \omega W$$

$$(\nabla_V \omega)W = CT_V W - T_V \phi W$$

for any $V, W \in \Gamma(\ker F^*)$.

We now have the following proposition.

**Proposition 3.2** Let $F$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then

(i) $A_X \phi V + \mathcal{H} \nabla_X \omega V = CA_X V + \omega (V \nabla_X V)$

and

$$\nabla(\nabla_X \phi V) + A_X \phi V = B A_X V + \phi (V \nabla_X V),$$

(ii) $A_X BY + \mathcal{H} (\nabla_X CY) = C (\mathcal{H} \nabla_X Y) + \omega A_X Y$

and

$$\nabla(\nabla_X BY) + A_X CY = B (\mathcal{H} \nabla_X Y) + \phi A_X Y,$$

(iii) $T_V \phi W + \mathcal{H}(\nabla_V \omega W) = CT_V W + \omega \hat{\nabla}_V W$

and

$$\hat{\nabla}_V \phi W + T_V \omega W = BT_V W + \phi \hat{\nabla}_V W,$$

for any $X, Y \in \Gamma(\ker F^*)^\perp$ and $V, W \in \Gamma(\ker F^*)$.

**Proof.** (i) For a Kaehler manifold $M$, we have on using equation (2.3)

$$\nabla_X JV = J \nabla_X V,$$

for any $X \in \Gamma(\ker F^*)^\perp, V \in \Gamma(\ker F^*)$.

Further, using Lemma 2.1 and equations (3.1), (3.4) we get

$$\nabla_X \phi V + \nabla_X \omega V = B(A_X V) + C(A_X V) + \phi (V \nabla_X V) + \omega (V \nabla_X V)$$

or,

$$\mathcal{H} \nabla_X \phi V + V \nabla_X \phi V + \mathcal{H} \nabla_X \omega V + V \nabla_X \omega V = B(A_X V) + C(A_X V) + \phi (V \nabla_X V)$$

$$+ \omega (V \nabla_X V)$$

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or,
\[ A_X\phi V + \nabla_X \phi V + \mathcal{H}\nabla_X \omega V + A_X \omega V = B(A_X V) + C(A_X V) + \phi(\nabla_X V) + \omega(\nabla_X V). \]

Comparing horizontal and vertical parts, we get
\[ A_X\phi V + \mathcal{H}\nabla_X \omega V = C(A_X V) + \omega(\nabla_X V), \]
\[ \nabla_X \phi V + A_X \omega V = B(A_X V) + \phi(\nabla_X V). \]

On similar lines we get the proof of (ii) and (iii). □

**Theorem 3.1** Let \( F \) be a Riemannian map from a Kaehler manifold \((M, g_M, J)\) to a Reimannain manifold \((B, g_B)\) with generic fibers. Then we have
\[ g_B((\nabla F_*)(V, W), F_*(J\xi)) = -g_B((\nabla F_*)(V, \phi W), F_*(\xi)) - g_B((\nabla F_*)(V, \omega W), F_*(\xi)), \]
for any \( V, W \in \Gamma(\ker F^*) \) and \( \xi \in \Gamma(\mu) \).

**Proof.** Since \( M \) is a kaehler manifold, for any \( V, W \in \Gamma(\ker F^*) \) we have,
\[ \nabla_V JW = J\nabla_V W. \]

Using Lemma 2.1 and equations (3.2),(3.4), we get
\[ H(\nabla_V \phi W) + \mathcal{V}(\nabla_V \phi W) + H(\nabla_V \omega W) + \mathcal{V}(\nabla_V \omega W) = B(\nabla_V W) + C(\nabla_V W) + \phi(\nabla_V W) + \omega(\nabla_V W). \]

Equating horizontal parts in (3.12), we have
\[ H(\nabla_V \phi W) + H(\nabla_V \omega W) = C(\nabla_V W) + \omega(\nabla_V W). \]

Taking Riemannian inner product in (3.13) with a vector \( \xi \in \Gamma(\mu) \), we obtain
\[ g_M(H(\nabla_V \phi W), \xi) + g_M(H(\nabla_V \omega W), \xi) = g_M(C(\nabla_V W), \xi) \]
\[ g_M(\nabla_V \phi W, \xi) + g_M(\nabla_V \omega W, \xi) = g_M(J(\nabla_V W), \xi) \]
\[ = -g_M(\nabla_V W, J\xi). \]

Since \( F \) is a Riemannian map, we have
\[ g_B(F_*(\nabla_V \phi W), F_*(\nabla_V \omega W), F_*(\xi)) = -g_B(F_*(\nabla_V W), F_*(J\xi)). \]

Using equation (2.13) we get the result. □

From Theorem 3.1 we have

**Corollary 3.3** Let \( F \) be a Riemannian map from a Kaehler manifold \((M, g_M, J)\) to a Riemannain manifold \((B, g_B)\) with generic fibers. Then we have
\[ g_B((\nabla F_*)(V, W), F_*(J\xi)) = -g_B((\nabla F_*)(V, JW), F_*(\xi)), \]
for any \( V \in \Gamma(\ker F_*), W \in \Gamma(D_1) \) and \( \xi \in \Gamma(\mu) \).
Lemma 3.3 Let $F$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$. Then

$$g(JT_V W, \xi) = g(T_V J W, \xi),$$

for any $V \in \Gamma(\ker F^*), W \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$

Proof. Since $M$ is a Kaehler manifold, then for any $V \in \Gamma(\ker F^*), W \in \Gamma(D_1)$, using equation (2.3) we have

$$J \nabla_V W = \nabla_V J W.$$

On using Lemma 2.1 we get

$$J(\nabla_V W + \hat{\nabla}_V W) = \nabla_V J W + \hat{\nabla}_V J W.$$

Taking inner product with a vector field $\xi \in \Gamma(\mu)$, we get

$$g(JT_V W, \xi) + g(J\hat{\nabla}_V W, \xi) = g(T_V J W, \xi) + g(\hat{\nabla}_V J W, \xi).$$

(3.14)

Since $\mu$ is invariant under $J$, the result then follows from (3.14). □

Next, we have

Theorem 3.2 Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then we have

$$g_B((\nabla F_*)(X, V), F_* \xi) = g_B((\nabla F_*)(X, \phi V) + (\nabla F_*)(X, \omega V), F_* \xi) - g_B(\hat{\nabla}_X F_* (\omega V), F_* \xi),$$

for any vector field $X \in \Gamma(\ker F_*), V \in \Gamma(\ker F_*)$ and $\xi \in \Gamma(\mu)$.

Proof. For $X \in \Gamma(\ker F_*), V \in \Gamma(\ker F_*)$, using equation (2.3) for a Kaehler manifold we have

$$\nabla_X J V = J \nabla_X V.$$

Using Lemma 2.1 and equations (3.2),(3.4) we get

$$\mathcal{H} \nabla_X \phi V + \nu \nabla_X \phi V + \mathcal{H} \nabla_X \omega V + \nu \nabla_X \omega V = B \mathcal{H} \nabla_X V + C \mathcal{H} \nabla_X V + \phi \nu \nabla_X V + \omega \nu \nabla_X V.$$  

(3.15)

Equating horizontal component in (3.15) we get

$$\mathcal{H} \nabla_X \phi V + \mathcal{H} \nabla_X \omega V = C \mathcal{H} \nabla_X V + \omega \nu \nabla_X V.  

(3.16)$$

Taking Riemannian product in (3.16) with a vector $\xi \in \Gamma(\mu)$ we obtain

$$g_M(\mathcal{H} \nabla_X \phi V, \xi) + g(\mathcal{H} \nabla_X \omega V, \xi) = g(M \mathcal{H} \nabla_X V, \xi) + g_M(\omega \nu \nabla_X V, \xi)$$

$$g_M(\nabla_X \phi V, \xi) + g(\nabla_X \omega V, \xi) = g_M(\mathcal{H} \nabla_X V, \xi) - g_M(B \mathcal{H} \nabla_X V, \xi) + g_M(\mathcal{H} \nabla_X V, \xi) - g_M(\phi \nu \nabla_X V, \xi).$$
But, for generic fibers $B(\ker F_*)^\perp = D_2$ and $\phi V \nabla_X V \in (\ker F_*)$, (Lemma 3.1) we then have

$$g_M(\nabla_X \phi V, \xi) + g_M(\nabla_X \omega V C, \xi) = -g_M(\mathcal{H}\nabla_X V, \xi)$$

$$= -g_M(\bar{\nabla}_X V, \xi).$$ (3.17)

Since $F$ is a Riemannian map, from (3.17) we have

$$g_B(F_*(\nabla_X \phi V), F_*\xi) + g_B(F_*(\nabla_X \omega V), F_*\xi) = g_B(F_*(\nabla_X V), F_*J\xi),$$

which on using (2.13) yields

$$-g_B((\nabla F_*)(X, \phi V), F_*\xi) - g_B((\nabla F_*)(X, \omega V), F_*\xi) + g_B(\nabla^F_X F_*(\omega V), F_*\xi)$$

$$= -g_B((\nabla F_*)(X, V), F_*J\xi)$$

$$g_B((\nabla F_*)(X, V), F_*J\xi) = g_B((\nabla F_*)(X, \phi V) + (\nabla F_*)(X, \omega V), F_*\xi)$$

$$- g_B(\nabla^F_X F_*(\omega V), F_*\xi).$$

Which completes the proof. \hfill \Box

As a consequences of above result we have

**Corollary 3.4** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then

$$g_B((\nabla F_*)(X, V), F_*\xi) = g_B((\nabla F_*)(X, J V), F_*\xi),$$

for any $X \in \Gamma(\ker F_*)^\perp$, $V \in \Gamma(D_1), \xi \in \Gamma(\mu)$.

4. Integrability of Distributions

In this section we obtain the integrability conditions for the distribution $D_1$ and $D_2$. Since we have seen that the fibers of the Riemannian map $F$ under consideration are the generic submanifolds of the manifold $M$ and $T$ works as the second fundamental form of the fibers and $\nabla F_*$ is the second fundamental form of the Riemannian map $F$, we have following theorem;

**Theorem 4.1** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the distribution $D_1$ is integrable if and only if

$$g_B((\nabla F_*)(V, J W) - (\nabla F_*)(W, J V), F_*\omega U) = 0.$$ (4.1)

or, $(\nabla F_*)(V, J W) - (\nabla F_*)(W, J V)$ has no component in $\Gamma(F_*(\omega D_2))$, for any $V, W \in \Gamma(D_1)$ and $U \in \Gamma(D_2)$

**Proof.** Let $V, W \in \Gamma(D_1), U \in \Gamma(D_2)$. Then on using equations (2.3),(3.1) and (3.2), we have

\[ \omega[V, W] - \mathcal{H} \nabla_V JW - \mathcal{H} \nabla_W JV = \mathcal{V} \nabla_V JW - \mathcal{V} \nabla_W JV - \phi[V, W]. \]  
(4.2)

Since \( \omega[V, W] \in \Gamma(\ker F_\ast) \perp \). In equation (4.2) the right hand side is vertical where as the left hand side is horizontal. Comparing horizontal and vertical parts, we get

\[ \omega[V, W] = \mathcal{H} \nabla_V JW - \mathcal{H} \nabla_W JV. \]  
(4.3)

\[ \phi[V, W] = \mathcal{V} \nabla_V JW - \mathcal{V} \nabla_W JV. \]  
(4.4)

Now, in view of decomposition (3.3) and equation (3.4) for each vector field \( Z \in \Gamma(\omega D_2) \subset \Gamma(\ker F_\ast) \perp \) there exist a vector \( U \in \Gamma(D_2) \) such that \( \omega U = Z \). Taking Riemannian inner product in (4.3) with \( \omega \in \Gamma(\omega D_2) \) we get

\[ g_M(\omega[V, W], \omega U) = g_M(\mathcal{H} \nabla_V JW, \omega U) - g_M(\mathcal{H} \nabla_W JV, \omega U) \]

\[ = g_M(\mathcal{V} \nabla_V JW, \omega U) - g_M(\mathcal{V} \nabla_W JV, \omega U) \]

Since \( F \) is a Riemannian map,

\[ g_M(\omega[V, W], \omega U) = g_B(F_\ast(\mathcal{V} \nabla_V JW), F_\ast \omega U) - g_B(F_\ast(\mathcal{V} \nabla_W JV), F_\ast \omega U). \]

On using equation (2.13), we get

\[ g_M(\omega[V, W], \omega U) = g_B((\nabla F_\ast)(V, JW), F_\ast \omega U) - g_B((\nabla F_\ast)(W, JV), F_\ast \omega U) \]

\[ = g_B((\nabla F_\ast)(V, JW) - (\nabla F_\ast)(W, JV), F_\ast \omega U). \]  
(4.5)

Hence the distribution \( D_1 \) is integrable if and only if \( \omega[V, W] = 0 \). That is \( D_1 \) is integrable if and only if

\[ g_B((\nabla F_\ast)(V, JW) - (\nabla F_\ast)(W, JV), F_\ast \omega U) = 0. \]  
(4.6)

or, \( (\nabla F_\ast)(V, JW) - (\nabla F_\ast)(W, JV) \) has no component in \( \Gamma(F_\ast(\omega U)) \).

Which completes the proof. \( \square \)

Since \( F_\ast(\ker F_\ast) \perp = \text{range} F_\ast \) and \( F \) is a Riemannian map, using Lemma 3.2(ii) it follows that \( g_B(F_\ast \omega V, F_\ast X) = 0 \), for any \( X \in \Gamma(\ker F_\ast) \perp \) and \( V \in \Gamma(\ker F_\ast) \), which then implies that

\[ TB = F_\ast(\omega D_2) \oplus F_\ast(\mu) \oplus (\text{range} F_\ast) \perp. \]

From equation (4.6) we also have

**Theorem 4.2** Let \( F : (M, g_M, J) \rightarrow (B, g_B) \) be a Riemannian map from a Kaehler manifold \( (M, g_M, J) \) to a Riemannian manifold \( (B, g_B) \) with generic fibers, then the distribution \( D_1 \) is integrable if and only if

\[ (\nabla F_\ast)(V, JW) - (\nabla F_\ast)(W, JV) \in \Gamma(F_\ast(\mu)), \]

for any \( V, W \in \Gamma(D_1) \).

**Lemma 4.1** Let \( F : (M, g_M, J) \rightarrow (B, g_B) \) be a Riemannian map from an almost Hermitian manifold \( (M, g_M, J) \) to a Riemannian manifold \( (B, g_B) \) with generic fibers, then the distribution \( D_2 \) is integrable if and only if

\[ \phi[V, W] \in \Gamma(D_2), \]

for any \( V, W \in \Gamma(D_2) \).
Proof. Since \( M \) is an almost Hermitian manifold and the distribution \((\ker F_*)\) is integrable, for any \( V, W \in \Gamma(D_2) \) and \( Z \in \Gamma(D_1) \), we have
\[
g_M([V, W], Z) = g_M(J[V, W], JZ) = g_M(\phi[V, W], JZ) + g_M(\omega[V, W], JZ) = g_M(\phi[V, W], JZ).
\]
The result then follow immediately. \( \square \)

Next we have,

**Theorem 4.3** Let \( F : (M, g_M, J) \longrightarrow (B, g_B) \) be a Riemannian map from a Kaehler manifold \((M, g_M, J)\) to a Riemannian manifold \((B, g_B)\) with generic fibers, then the distribution \( D_2 \) is integrable if and only if
\[
T_V\omega W - T_W\omega V + \hat{\nabla}_V\phi W - \hat{\nabla}_W\phi V \in \Gamma(D_2),
\]
for any \( V, W \in \Gamma(D_2) \).

**Proof.** Since \( M \) is a Kaehler manifold, for any \( V, W \in \Gamma(D_2) \) using Lemma 2.1 and equations (2.3),(3.2),(3.4) we obtain
\[
[V, W] = \nabla_V W - \nabla_W V = -J(\nabla_V JW - \nabla_W J V)
\]
\[
= B(\mathcal{H}\nabla_V JW - \mathcal{H}\nabla_W J V) + C(\mathcal{H}\nabla_V JW - \mathcal{H}\nabla_W J V)
+ \phi(\mathcal{V}\nabla_V JW - \mathcal{V}\nabla_W J V) + \omega(\mathcal{V}\nabla_V JW - \mathcal{V}\nabla_W J V).
\]
(4.7)
Since \((\ker F_*)\) is always integrable, therefore \([V, W] \in \Gamma(\ker F_*)\). Comparing the vertical part in (4.7), we get
\[
[V, W] = B(\mathcal{H}\nabla_V JW - \mathcal{H}\nabla_W J V) + \phi(\mathcal{V}\nabla_V JW - \mathcal{V}\nabla_W J V).
\]
(4.8)
Taking Riemannian inner product in (4.8) with a vector \( Z \in \Gamma(D_1) \) and further using equation (3.2) and Lemma 3.1, we get
\[
g_M ([V, W], Z) = g_M (\mathcal{D}_V \phi W, JZ) - g_M (\mathcal{D}_W \omega V, JZ) + g_M (T_V\omega W, J Z)
+ g_M (T_W\omega V, J Z).
\]
Finally, we get
\[
g_M([V, W], Z) = g_M \left(T_V\omega W - T_W\omega V + \hat{\nabla}_V\phi W - \hat{\nabla}_W\phi V, J Z\right).
\]
(4.9)
Since for, \( JZ \in \Gamma(D_1) \). From equation (4.9) it follows that the distribution \( D_2 \) is integrable if and only if
\[
T_V\omega W - T_W\omega V + \hat{\nabla}_V\phi W - \hat{\nabla}_W\phi V \in \Gamma(D_2),
\]
for any \( V, W \in \Gamma(D_2) \). Which completes the proof. \( \square \)
We now discuss the geometry of the leaves of the distributions $D_1$ and $D_2$, and relate it with the geometry of the base manifold $B$ using the second fundamental form of the Riemannian map $F$, and we have the following propositions.

**Proposition 4.1** Let $F: (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the distribution $D_1$ defines a totally geodesic foliation if and only if

$$g_M \left( \nabla_U J V, \phi W_2 \right) = g_B \left( (\nabla F_*)(U, JV), F_* \phi W_2 \right)$$

and

$$g_M \left( \nabla_U J V, B X \right) = g_B \left( (\nabla F_*)(U_1, JV_1), F_* C X \right),$$

for any vector fields $U, V \in \Gamma(D_1), W_2 \in \Gamma(D_2)$ and $X \in \Gamma(ker F_*)^\perp$.

**Proof.** For $U_1, V_1 \in \Gamma(D_1)$ and $X \in \Gamma(ker F_*)^\perp$ using equation (3.4) we have

$$g_M \left( \nabla_U V_1, X \right) = g_M \left( J \nabla_U V_1, J X \right) = g_M \left( \nabla_U J V_1, B X \right) + g_M \left( \nabla_U J V_1, C X \right) = g_M \left( \nabla_U J V_1, B X \right) + g_M \left( \nabla_U J V_1, C X \right) = g_M \left( \nabla_U J V_1, B X \right) + g_B \left( F_*(\nabla_U J V_1), F_* C X \right),$$

where we have used the fact that $F$ be a Riemannian map. Using now equation (2.13) we have

$$g_M \left( \nabla_U V_1, X \right) = g_M \left( \nabla_U J V_1, B X \right) - g_B \left( (\nabla F_*)(U_1, JV_1), F_* C X \right).$$

Hence $\nabla_U V_1 \in \Gamma(ker F_*)$ if and only if

$$g_M \left( \nabla_U J V_1, B X \right) = g_B \left( (\nabla F_*)(U_1, JV_1), F_* C X \right). \quad (4.10)$$

Further, for $U_1, V_1 \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$, using equations (2.1),(2.3) and (3.1) we have

$$g_M \left( \nabla_U V_1, W_2 \right) = g_M \left( J \nabla_U V_1, J W_2 \right) = g_M \left( H \nabla_U J V_1, \phi W_2 \right) + g_M \left( V \nabla_U J V_1, \phi W_2 \right) + g_M \left( \nabla_U J V_1, \omega W_2 \right) + g_M \left( \nabla_U J V_1, \omega W_2 \right) = g_M \left( \nabla_U J V_1, \phi W_2 \right) + g_M \left( \nabla_U J V_1, \omega W_2 \right).$$

Since $F$ is a Riemannian map, using (2.13) we have

$$g_M \left( \nabla_U V_1, W_2 \right) = g_M \left( \nabla_U J V_1, \phi W_2 \right) + g_B \left( F_*(\nabla_U J V_1), F_* \omega W_2 \right) = g_M \left( \nabla_U J V_1, \phi W_2 \right) - g_B \left( (\nabla F_*)(U_1, JV_1), F_* \omega W_2 \right).$$
Hence $\nabla_{U_1} V_1 \in \Gamma(D_1)$ if and only if
\[
g_M \left( \nabla_{U_1} J V_1, \phi W_2 \right) = g_B \left( (\nabla F_*)(U_1, J V_1), F_* \omega W_2 \right).  \tag{4.11}
\]
The result then follows from (4.10) and (4.11) and which completes the proof. \hfill \Box

**Proposition 4.2** Let $F : (M, g_M, J) \to (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the distribution $D_2$ defines a totally geodesic foliation if and only if
\[
g_M \left( \nabla_{V_2} \phi W_2, B X \right) + g_M \left( T_{V_2} \omega W_2, B X \right) = g_B \left( (\nabla F_*)(V_2, \phi W_2) \right)
+ \left( \nabla F_*(V_2, \omega W_2), F_* C X \right)
\]
and
\[
\nabla_{V_2} \phi W_2 + T_{V_2} \omega W_2 \in \Gamma(D_2),
\]
for any $V_2, W_2 \in \Gamma(D_2)$ and $X \in \Gamma(\ker F_*)^\perp$.

**Proof.** For any $V_2, W_2 \in \Gamma(D_2), X \in \Gamma(\ker F_*)^\perp$.

Using equations (2.1), (2.3), (2.13), (3.2), (3.4), Lemma 2.1 and the fact that $F$ is a Riemannian map we have
\[
g_M \left( \nabla_{V_2} W_2, X \right) = g_M \left( \nabla_{V_2} J W_2, J X \right)
= g_M \left( \nabla_{V_2} \phi W_2, B X \right) + g_M \left( V \nabla_{V_2} \omega W_2, B X \right) + g_M \left( \nabla_{V_2} \phi W_2, C X \right)
+ g_M \left( \nabla_{V_2} \omega W_2, C X \right)
= g_M \left( \nabla_{V_2} \phi W_2, B X \right) + g_M \left( T_{V_2} \omega W_2, B X \right)
+ g_M \left( F_* (\nabla_{V_2} \phi W_2), F_* C X \right) + g_M \left( F_* (\nabla_{V_2} \omega W_2), F_* C X \right)
= g_M \left( \nabla_{V_2} \phi W_2, B X \right) + g_M \left( T_{V_2} \omega W_2, B X \right)
- g_B \left( (\nabla F_*)(V_2, \phi W_2), F_* C X \right) - g_B \left( (\nabla F_*)(V_2, \omega W_2), F_* C X \right) \tag{4.12}
\]
Equation (4.12) yields that $\nabla_{V_2} W_2 \in \Gamma(\ker F_*)$ if and only if
\[
g_M \left( \nabla_{V_2} \phi W_2, B X \right) + g_M \left( T_{V_2} \omega W_2, B X \right) = g_B \left( (\nabla F_*)(V_2, \phi W_2), F_* C X \right)
+ g_B \left( (\nabla F_*)(V_2, \omega W_2), F_* C X \right). \tag{4.13}
\]
On the other hand, for $U_1 \in \Gamma(D_1)$ and $V_2, W_2 \in \Gamma(D_2)$ we have
\[
g_M \left( \nabla_{V_2} W_2, U_1 \right) = g_M \left( \nabla_{V_2} J W_2, J U_1 \right)
= g_M \left( V \nabla_{V_2} J W_2, J U_1 \right)
= g_M \left( V \nabla_{V_2} \phi W_2, U_1 \right) + g_M \left( V \nabla_{V_2} \omega W_2, U_1 \right)
= g_M \left( \nabla_{V_2} \phi W_2 + T_{V_2} \omega W_2, J U_1 \right).
\]
Since for \( U_1 \in \Gamma(D_1), JU_1 \in \Gamma(D_1) \), the above relation implies that \( \nabla_{V_2} W_2 \in \Gamma(D_2) \) if and only if
\[
\hat{\nabla}_{V_2} \phi W_2 + T_{V_2} \omega W_2 \in \Gamma(D_2).
\]
(4.14)

The result then follows from (4.13) and (4.14) \( \square \)

We now recall the following definition.

**Definition 4.1** ([?]). Let \( g \) be a metric tensor on the manifold \( M = B \times F \) and assume that the canonical distribution \( D_B \) and \( D_F \) intersect perpendicularly everywhere, then \( g \) is the metric tensor of a usual product of Riemannian manifold if and only if \( D_B \) and \( D_F \) are totally geodesic foliation.

From Proposition 4.1 and Proposition 4.2 we have the following theorem.

**Theorem 4.4** Let \( F : (M, g_M, J) \longrightarrow (B, g_B) \) be a Riemannian map from a Kaehler manifold \( F : (M, g_M, J) \) to a Riemannian manifold \( (B, g_B) \) with generic fibers. Then the integral manifold of the distributions \( (\ker F_\ast) \) is a locally Riemannian product of the leaves of the distribution \( D_1 \) and \( D_2 \) if and only if
\[
g_M \left( \hat{\nabla}_{U_1, J V_1, \phi W_2} \right) = g_B \left( (\nabla F_\ast)(U_1, J V_1), F_\ast \omega W_2 \right),
g_M \left( \hat{\nabla}_{U_1, J V_1, B X} \right) = g_B \left( (\nabla F_\ast)(U_1, J V_1), F_\ast CX \right)
\]
and \( g_M \left( \hat{\nabla}_{V_1, \phi W_2, B X} \right) + g_M \left( T_{V_1} \omega W_2, B X \right) = g_B \left( (\nabla F_\ast)(V_1, \phi W_2), F_\ast CX \right) + g_B \left( (\nabla F_\ast)(V_1, \omega W_2), F_\ast CX \right);
\[
\hat{\nabla}_{V_1} \phi W_2 + T_V \omega W_2 \in \Gamma(D_2),
\]
for any vector fields \( U_1, V_1 \in \Gamma(D_1), V_2, W_2 \in \Gamma(D_2) \) and \( X \in \Gamma(\ker F_\ast)^\perp \).

Since the integral manifolds of the distribution \( (\ker F_\ast) \) are infact the fibers of the Riemannian map \( F \), the above theorem can be re-stated as:

**Theorem 4.5** Let \( F : (M, g_M, J) \longrightarrow (B, g_B) \) be a Riemannian map from a Kaehler manifold \( F : (M, g_M, J) \) to a Riemannian manifold \( (B, g_B) \) with generic fibers. Then the fibers of \( F \) are the locally Riemannian product of the leaves of \( D_1 \) and \( D_2 \) if and only if
\[
g_M \left( \hat{\nabla}_{U_1, J V_1, \phi W_2} \right) = g_B \left( (\nabla F_\ast)(U_1, J V_1), F_\ast \omega W_2 \right),
g_M \left( \hat{\nabla}_{U_1, J V_1, B X} \right) = g_B \left( (\nabla F_\ast)(U_1, J V_1), F_\ast CX \right)
\]
and \( g_M \left( \hat{\nabla}_{V_1, \phi W_2, B X} \right) + g_M \left( T_{V_1} \omega W_2, B X \right) = g_B \left( (\nabla F_\ast)(V_1, \phi W_2), F_\ast CX \right) + g_B \left( (\nabla F_\ast)(V_1, \omega W_2), F_\ast CX \right);
\[
\hat{\nabla}_{V_1} \phi W_2 + T_V \omega W_2 \in \Gamma(D_2),
\]
for any vector fields \( U_1, V_1 \in \Gamma(D_1), V_2, W_2 \in \Gamma(D_1) \) and \( X \in \Gamma(\ker F_\ast)^\perp \).

Now, we study the geometry of the leaves of the distribution \( (\ker F_\ast) \) and \( (\ker F_\ast)^\perp \).
**Proposition 4.3** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the distribution $(\ker F_*)$ defines a totally geodesic foliation if and only if

$$ g_B((\nabla_{F_*})(V, \phi W), F_* CX) + g_B((\nabla_{F_*})(V, \omega W), F_* CX) = g_M(\hat{\nabla}_V \phi W, BX) + g_M(T_V \omega W, BX), $$

for any $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma(\ker F_*)^\perp$.

**Proof.** For any $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma(\ker F_*)^\perp$, using equations (2.3), (2.13), (3.2), (3.4), Lemma 2.1 and the fact that $F$ be a Riemannian map we have

$$ g_M(\nabla_V W, X) = g_M(\nabla_V JW, JX) = g_M(\nabla_V (\phi W + \omega W), BX + CX) = g_M(\nabla \nabla V \phi W, BX) + g_M(\nabla \nabla V \omega W, BX) + g_M(\mathcal{H} \nabla V \phi W, CX) + g_M(\nabla \nabla V \omega W, CX) = g_M(\hat{\nabla}_V \phi W, BX) + g_M(T_V \omega W, BX) = g_M(F_*(\nabla V \phi W), F_* CX) + g_M(F_*(\nabla V \omega W), F_* CX) = g_M(\hat{\nabla}_V \phi W, BX) + g_M(T_V \omega W, BX) - g_M((\nabla_{F_*})(V, \phi W), F_* CX) - g_M((\nabla_{F_*})(V, \omega W), F_* CX). $$

From equation (4.15) it follows that $\ker F_*$ defines a totally geodesic foliation if and only if

$$ g_B((\nabla_{F_*})(V, \phi W), F_* CX) + g_B((\nabla_{F_*})(V, \omega W), F_* CX) = g_M(\hat{\nabla}_V \phi W, BX) + g_M(T_V \omega W, BX). $$

Which completes the proof. □

Next, we have

**Proposition 4.4** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kähler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if

$$ g_M(\omega A_X Y, \omega V) = -g_M(\nabla X BY, \phi V) - g_M(A_X CY, \phi V), $$

for any $X, Y \in \Gamma(\ker F_*)^\perp$ and $V \in \Gamma(\ker F_*)$. 

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Proof. For any $X, Y \in \Gamma(\ker F_*)$ and $V \in \Gamma(\ker F_*)$, using equations (2.1), (2.3), we have
\[
g_M(\nabla X Y, V) = g_M(\nabla X JY, JV).
\]
Further, on using equations (3.2), (3.4) and Lemma 2.1 we obtain
\[
g_M(\nabla X Y, V) = g_M(\nabla X BY, \phi V) + g_M(A_X CY, \omega V) + g_M(A_X BY, \omega V) + g_M(A_X Y, \omega V).
\]
Using Proposition 3.1 we get
\[
g_M(\nabla X Y, V) = g_M(\nabla X BY, \phi V) + g_M(A_X CY, \omega V).
\]
Hence, from (4.16) it follows that $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if
\[
g_M(\omega A_X Y, \omega V) = g_M(\nabla X BY, \phi V) - g_M(A_X CY, \omega V).
\]
Which completes the proof.

From equation (4.16) and Lemma 3.1 we also have

**Proposition 4.5** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if
\[
\nabla X BY + A_X CY = 0 \quad \text{and} \quad A_X Y \in \Gamma(D_1)
\]
for any $X, Y \in \Gamma(\ker F_*)$.

Theorem 4.4 and Proposition 4.4 yields the following decomposition theorem.

**Theorem 4.6** Let $F : (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then the total manifold $M$ is a Riemannian product manifold of the leaves $D_1, D_2$ and $(\ker F_*)^\perp$ i.e.,
\[
M = M_{(D_1, D_2)} \times M_{(\ker F_*)^\perp}, \text{if and only if}
\]
\[
g_M(\nabla U_1 J V_1, \phi W_2) = g_M((\nabla F_*)(U_1, J V_1), F_* \omega W);
\]
\[
g_M(\nabla_U J V_1, B X) = g_B((\nabla F_*)(U_1, J V_1), F_* C X);
\]
\[
g_B((\nabla F_*)(V_2, \phi W_2), F_* C X) + g_B((\nabla F_*)(V_2, \omega W_2), F_* C X) = g_M(\nabla V_2 \phi W_2, B X) + g_M(T_{V_2} \omega W_2, B X);
\]
\[
\hat{\nabla}_V \phi W + T_V \omega W \in \Gamma(D_2).
\]

and
\[
g_M(\omega A_X Y, \omega V) = -g_M(v \nabla_X BY, \phi V) - g_M(A_X CY, \phi V),
\]

for any \(U_1, V_1 \in \Gamma(D_1), \) \(V_2, W_2 \in \Gamma(D_2), \) \(V \in \Gamma(\ker F) \) and \(X, Y \in \Gamma(\ker F) \perp , \)

where \(M_{D_1}, M_{D_2} \) and \(M_{(\ker F) \perp } \) are respectively the leaves of \(D_1, D_2 \) and \((\ker F) \perp \).

From Proposition 4.3 and Proposition 4.4 we have following theorem

**Theorem 4.7** Let \(F : (M, g_M, J) \rightarrow (B, g_B) \) be a Riemannian map from Kaehler manifold \((M, g_M, J)\) to a Riemannian manifold \((B, g_B)\) with generic fibers. Then the total space is a generic product manifold i.e,
\[ M = M_{(\ker F) \perp} \times M_{(\ker F) \perp} \] if and only if
\[
g_M((\nabla F)(V, \phi W), F_* CX) + g_M((\nabla F)(V, \omega W)) = g_M(\hat{\nabla}_V \phi W, BX) + g_M(T_V \omega W, BX)
\]

and
\[
g_M(\omega A_X Y, \omega V) = -g_M(v \nabla_X BY, \phi V) - g_M(A_X CY, \phi V),
\]

for any \(V, W \in \Gamma(\ker F) \) and \(X, Y \in \Gamma(\ker F) \perp \).

If we consider Proposition 4.5 along with Proposition 4.3, then we have the following theorem

**Theorem 4.8** Let \(F : (M, g_M, J) \rightarrow (B, g_B) \) be a Riemannian map from a Kaehler manifold \((M, g_M, J)\) to a Riemannian manifold \((B, g_B)\) with generic fibers. Then \(M \) is a generic product manifold, i.e., \(M = M_{(\ker F) \perp} \times M_{(\ker F) \perp} \) if and only if
\[
g_M((\nabla F)(V, \phi W), F_* CX) + g_M((\nabla F)(V, \omega W)) = g_M(\hat{\nabla}_V \phi W, BX) + g_M(T_V \omega W, BX)
\]

and
\[
\nabla_X BX + A_X CY = 0; \ A_X Y \in \Gamma(D_1),
\]

for any \(X, Y \in \Gamma(\ker F) \perp \) and \(V, W \in \Gamma(\ker F) \).

5. Totally Geodesic Maps

In this section we obtain the necessary and sufficient condition for the Riemannian map \(F \) to be totally geodesic map. First we recall the following definition.
Definition 5.1 Let $F: (M, g_M) \rightarrow (B, g_B)$ be a Riemannian map between the Riemannian manifolds $(M, g_M)$ and $(B, g_B)$. Then $F$ is said to be totally geodesic map if

$$\left(\nabla F\right)(X, Y) = 0,$$

for all vector fields $X, Y \in \Gamma(TM)$.

We have the following theorem.

Theorem 5.1 Let $F: (M, g_M, J) \rightarrow (B, g_B)$ be a Riemannian map from a Kaehler manifold $(M, g_M, J)$ to a Riemannian manifold $(B, g_B)$ with generic fibers. Then $F$ is a totally geodesic map if and only if

$$\hat{\nabla} V \phi W + T V \omega W \in \Gamma(D_1),$$

$$T V \phi W + H \hat{\nabla} V \omega W \in \Gamma(\omega D_2),$$

$$\hat{\nabla} V BX + T V CX \in \Gamma(D_1),$$

$$T V BX + H \hat{\nabla} V CX \in \Gamma(\omega D_2),$$

for any $V, W \in \Gamma(kerF_*)$ and $X, Y \in \Gamma(kerF_*)^\perp$.

Proof. Since $F$ is a Riemannian map, then from Lemma 2.2 we have

$$\left(\nabla F\right)(X, Y) = 0,$$

for any $X, Y \in \Gamma(kerF_*)^\perp$

For any $V, W \in \Gamma(kerF_*)$ and $X, Y \in \Gamma(kerF_*)^\perp$, using equation (2.13) we have

$$\left(\nabla F\right)(V, W) = -F_*(\hat{\nabla} V W),$$

Now, using equations (2.3),(3.2),(3.4) and Lemma 2.1 we have

$$-\nabla V W = J(\nabla V JW)$$

$$= J\nabla V \phi W + J\nabla V \omega W$$

$$= J \left( H\nabla V \phi W + V\nabla V \phi W \right) + J \left( H\nabla V \omega W + V\nabla V \omega W \right)$$

$$= (B H \nabla V \phi W + B H \nabla V \omega W) + (C H \nabla V \phi W + C H \nabla V \omega W)$$

$$+ (\phi \nabla V \phi W + \phi \nabla V \omega W) + (\omega \nabla V \phi W + \omega \nabla V \omega W)$$

Applying $F_*$ to above equation we have

$$-F_*(\nabla V W) = F_*(C(T V \phi W + H \nabla V \omega W)) + F_*(\omega (\hat{\nabla} V \phi W + T V \omega W)).$$

From (5.3) and (5.4), we have

$$(\nabla F_*)(V, W) = F_*(C(T V \phi W + H \nabla V \omega W)) + F_*(\omega (\hat{\nabla} V \phi W + T V \omega W)).$$
Hence, \((\nabla F^*)(V, W) = 0\) if and only if
\[
\hat{\nabla}_V \phi W + T_V \omega W \in \Gamma(D_1), \ T_V \phi W + \mathcal{H} \nabla_V \phi W \in \Gamma(\omega D_2). \tag{5.5}
\]
On the other hand using (2.13) for any \(V \in \Gamma(ker F_*)\) and \(X \in \Gamma(ker F_*)^\perp\), we have
\[
(\nabla F_*)(V, X) = -F_*(\hat{\nabla}_V X). \tag{5.6}
\]
But
\[
-\nabla_V X = J\nabla_V J X
= J(\nabla_V BX) + J(\nabla_V CX)
= J(\mathcal{H} \nabla_V BX) + J(\mathcal{H} \nabla_V CX) + J(\nabla_V BX) + J(\nabla_V CX)
= B(T_V BX) + C(T_V BX) + B(\nabla_V BX) + C(\nabla_V CX)
+ \phi \hat{\nabla}_V BX + \omega \hat{\nabla}_V CX + \phi \nabla_V CX + \omega \nabla_V CX
= B(T_V BX + \mathcal{H} \nabla_V CX) + \phi \left( \hat{\nabla}_V BX + \nabla_V CX \right)
+ C(T_V BX + \mathcal{H} \nabla_V CX) + \omega \left( \hat{\nabla}_V BX + T_V CX \right).
\]
Therefore,
\[
-F_*(\nabla_V X) = F_*(\omega(\hat{\nabla}_V BX + T_V CX)) + F_*(C(T_V BX + \nabla_V CX)).
\]
Which then yields \((\nabla F_*)(V, X) = 0\) if and only if
\[
\omega \left( \hat{\nabla}_V BX + T_V CX \right) + C \left( T_V BX + \mathcal{H} \nabla_V CX \right) = 0
\]
that is, \((\nabla F_*)(V, X) = 0\) if and only if
\[
\hat{\nabla}_V BX + T_V CX \in \Gamma(D_1); \ T_V BX + \mathcal{H} \nabla_V CX \in \Gamma(\omega D_2). \tag{5.7}
\]
which completes the proof. \(\square\)

Now we recall the following definition:

**Definition 5.2** Let \(F : (M, g_M) \rightarrow (B, g_B)\) be a Riemannian map between the Riemannian manifold \((M, g_M)\) and \((B, g_B)\). Then we say that the fibers of the map \(F\) are totally umbilical if and only if
\[
h(V, W) = g(V, W)\lambda, \tag{5.8}
\]
for any \(X, Y \in \Gamma(TM)\) where \(h\) is the second fundamental form of the fibers when considered as the immersed submanifolds of the total space \(M\) and coincide with the B. O’Neill’s fundamental tensor \(T\) for the vector fields \(V, W \in \Gamma(ker F_*)\). \(\lambda\) is called the mean curvature vector of the fibers and is a horizontal vector field.

Infact, we prove the following.

**Theorem 5.2** Let \(F : (M, g_M, J) \rightarrow (B, g_B)\) be a Riemannian map from a kahler manifold \((M, g_M, J)\) to a Riemannian manifold \((B, g_B)\) with generic and \(D_1\)—totally umbilical fibers. Then \(\lambda \in \Gamma(\omega D_2)\).
Proof. Since $M$ is a Kaehler manifold, for any $V, W \in \Gamma(D_1)$ using (2.3) we have

$$\nabla_V JW = J\nabla_V W.$$  

Now using (3.2) and (3.4) we have

$$h(V, JW) + \hat{\nabla}_V JW = Bh(V, W) + Ch(V, W) + \phi \nabla_V W + \omega \nabla_V W.$$  

Taking Riemannian product in above equation with a vector field $X \in \Gamma(\mu)$ and then using (5.8) we obtain

\begin{align*}
g(h(V, JW), X) &= g(Ch(V, W), X), \\
g(V, JW)g(\lambda, X) &= -g(h(V, W), JX) \\
&= -g(V, W)g(\lambda, JX).
\end{align*}

(5.9)

Interchanging $V$ and $W$ in (5.9), we have

$$g(W, JV)g(\lambda, X) = -g(W, V)g(\lambda, JX).$$

(5.10)

From equation (5.9) and (5.10) on combining them, we get

$$g(\lambda, JX) = 0.$$  

Which gives the result.  

□

References

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