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On a slant submanifold of a Kaehler-Norden manifold

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The present paper contains the study of a slant submanifold of a Kaehler-Norden manifold. Also, some results related to totally geodesic and umbilical submanifold have been derived.

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1. Introduction

The idea of slant submanifold was introduced by B. Y. Chen in 1990. Chen [3] generalised the concept of holomorphic and totally real submanifold in complex geometry. In 1996, A. Lotto [5] extended this concept in contact manifold. Recently Siraj Uddin and Cenap Ozeln[8] have studied a classification theorem on totally umbilical submanifolds in a cosymplectic manifold. T. Adati [1] have studied submanifold of an almost product Riemannian manifold and defined invariant, anti-invariant and non-invariant submanifold of locally product manifold. Mehmet Atceken [2] studied slant submanifold of a Riemannian product manifold in 2010. Totally umbilical proper-slant submanifold of a nearly Kaehler manifold was studied by K. Singh , S. Uddin and M. A. Khan [7], R. Prasad, S. S. Shukla, A. Haseeb and S. Kumar have studied quasi hemi-slant submanifolds of Kaehler manifolds [9] and Hemi-slant submanifolds in metallic Riemannian manifolds was studied by Cristina E. Hretcanu and Adara M. Blaga [10]. In the cosequences of these V. A. Khan and M. A. Khan [4] studied semi-slant submanifold of a nearly Kaehler manifold. These studies inspired us for the study of slant submanifold of a Kaehler-Norden manifold. Throughout the paper, we have considered non-degenerate submanifolds of Kaehler-Norden manifold. This paper contains six sections, first section is the introductory. In section 2 and 3 we have defined the terms which are required for the studies. In section 4 the formation and proof of theorems are given. In section 5 and 6 discussion and conflict of interest are given.

2. Kaehler-Norden manifold

An even n-dimensional differentiable manifold M is said to be an almost complex manifold with almost complex structure F if

$$F^2 + I = 0. \tag{1}$$

A semi Riemannian metric g is said to be an anti-Hermitian (Norden) if the metric g satisfies

$$g(FX, Y) = g(X, FY), \tag{2}$$

for any $X, Y \in TM$. An almost complex manifold M with an anti-Hermitian (Norden) metric define by (2) is called an almost anti-Hermitian (Norden) manifold.
An anti-Hermitian (Norden) manifold is said to be an anti-Kaehler or Kaehler-Norden manifold if [6]

$$(\overline{\nabla}_X F)Y = 0, \tag{3}$$

where $\overline{\nabla}$ is a Levi-civita connection.

3. Submanifold

Through out this paper TM and $T^{\perp}M$ denote the Lie algebra of vector fields in M and the set of all vector fields normal to M respectively. Now if we take two connections ∇ and $\overline{\nabla}$ on M and $\overline{M}$ then the Gauss-Weigarten farmula are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{4}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{5}$$

for any $X, Y \in TM$ and any $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle, σ is the second fundamental form and A_V is the Weigarten endomorphism associated with V. The second fundamental form and the shap operator A are associated by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{6}$$

A manifold M is called totally geodesic if its second fundamental form σ vanishes that is $\sigma = 0$, from (6) which gives $A_V = 0$. A manifold is said to be totally umbilical submanifold in $\overline{M}$ if for all $X, Y \in$ TM, we have

$$\sigma(X, Y) = g(X, Y)H, \tag{7}$$

where H is the mean curvature vector field M in $\overline{M}$. If $H = 0$, then it is called minimal submanifold.
For any $X \in TM$, we can write

$$FX = TX + NX, \tag{8}$$

where TX and NX are tangetial and normal part of FX respectively.
Similarly, for any $V \in T^{\perp}M$, we have

$$FV = tV + nV, \tag{9}$$

where tV and nV are tangential and normal part of FV respectively.
If $NX = 0$ i.e. $FX = TX \in TM$, the submanifold is said to be an invariant for any $X \in TM$. However if $TX = 0$ i.e. $FX = NX \in T^{\perp}M$, the submanifold is called anti-invariant $X \in TM$.
Replacing X by FX in equation (8), we get

$$F^2X = FTX + FNX. \tag{10}$$

With the help of equations (1), (8), (9) and (10), we can write

$$-X = T^2X + TNX + NtX + NnX, \tag{11}$$

compairing tangential and normal part of equation (11) , we have

$$T^2 + Nt = -I, \tag{12}$$

and

$$TN + Nn = 0. \tag{13}$$

Simiarly for any $V \in T^{\perp}M$, we get

$$n^2 + Nt = -I, \tag{14}$$

and

$$tT + tn = 0. \tag{15}$$

From equation (3), we have

$$\overline{\nabla}_X FY = F\overline{\nabla}_X Y, \tag{16}$$

using equation (8) and (9) in (16), we get

$$\begin{aligned} \nabla_X TY + &\sigma(X,TY) - A_{NY}X + \nabla^{\perp}_X NY \\ &= T\nabla_X Y + N\nabla_X Y + t\sigma(X,Y) + n\sigma(X,Y), \end{aligned} \tag{17}$$

compairing tangential and normal part of equation (17), we get

$$(\nabla_X T)Y = A_{NY}X + t\sigma(X,Y), \tag{18}$$

and

$$(\nabla_X N)Y = n\sigma(X,Y) - \sigma(X,TY), \tag{19}$$

where the covariant derivative of T and N are defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{20}$$

and

$$(\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{21}$$

for any $X, Y \in TM$.

Definition 1. A manifold M is said to be slant if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in M$ and $X \in TM$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant. Let M be a slant submanifold of an anti-Hermitian metric $\overline{M}$. The FT_xM is subspace of $T^{\perp}M$. Thus for any $x \in M$ we decompose the normal space as

$$T^{\perp}M = FTM \oplus \mu. \tag{22}$$

4. Main results

Theorem 2. *Let M be a submanifold of a Kaehler-Norden manifold $\overline{M}$ then M is slant if and only if there exists a constant $\lambda \in [-1\,, 0]$ such that $T^2 = \lambda I$, where $\lambda = -cos^2\theta$.*

Proof: Suppose that M is slant manifold then for any $X \in TM$, we have

$$Cos\theta(X) = \frac{\|TX\|}{\|FX\|}, \tag{23}$$

where $\theta(X)$ is slant angle.
With the help of equation(2) and (23), we have

$$\begin{aligned} g(T^2X,\, X) &= g(TX,\, TX) \\ &= Cos^2\theta(X) g(FX,\, FX) \\ &= -Cos^2\theta(X) g(X,\, X). \end{aligned} \tag{24}$$

Equation (24), can be written as

$$T^2X = -Cos^2\theta(X)X. \tag{25}$$

Let $\lambda = -Cos^2\theta(X)$, then equation (25) becomes

$$T^2 = \lambda\, I, \tag{26}$$

where $\lambda \in [-1,\, 0]$.
Conversaly suppose that $T^2 = \lambda\, I$, where $\lambda \in [-1,\, 0]$, then from equation (1), (24) and (25), we get

$$\begin{aligned} Cos\theta(X) &= \frac{g(FX,\, TX)}{\|FX\|\,\|TX\|} = \frac{g(X\,, T^2X)}{\|FX\|\,\|TX\|} \\ &= -\frac{Cos^2\theta(X) g(X,\, X)}{\|FX\|\,\|TX\|} = \frac{Cos^2\theta(X) g(FX,\, FX)}{\|FX\|\,\|TX\|} \\ &= -\frac{\lambda g(FX,\, FX)}{\|FX\|\,\|TX\|} = -\frac{\lambda\,\|FX\|}{\|TX\|}. \end{aligned} \tag{27}$$

Using equation (23) in (27), we get

$$Cos^2\theta(X) = -\lambda, \tag{28}$$

which implies that $\theta(X)$ is constant so M is slant. □

Using equation (8) in (2), we have

$$g(TX + NX, TY + NY) = -g(X, Y). \tag{29}$$

Equation (29) implies that

$$g(TX, TY) = -Cos^2\theta \;\; g(X, Y), \tag{30}$$

and

$$g(NX, NY) = -Sine^2\theta \;\; g(X, Y). \tag{31}$$

Thus we conclude:

Lemma 3. *If M be a slant submanifold of a Kaehler-Norden manifold $\overline{M}$ with slant angle θ then for any $X, Y \in TM$, we get*

$$(i)\, g(TX, TX) = -Cos^2\theta \;\; g(X, Y),$$
$$(ii)\, g(NY, NY) = -Sine^2\theta \;\; g(X, Y).$$

Now we propose:

Lemma 4. *If M be a slant submanifold of a Kaehler-Norden manifold $\overline{M}$ then N is parallel if and only if*

$$A_{n\,V}\, Z = A_V\, TZ,$$

for all $Z \in TM$ and $V \in T^{\perp}M$.

Proof: From equation (2) and (19), we get

$$\begin{aligned} g((\nabla_X\, N)\, Z, V) &= g(n\sigma(X\,, Z) - \sigma(X, \, TZ), V) \\ &= g((\sigma(X\,, Z), nV) - g(\sigma(X\,, TZ))\,, V), \end{aligned} \tag{32}$$

using (6), in (32), we have

$$g((\nabla_X\, N)\, Z, V) = g(A_{n\,V}\, Z - A_V\, TZ, \; X). \tag{33}$$

If we take

$$(\nabla_X\, N) = 0, \tag{34}$$

then from (33) and (34), we have

$$A_{n\,V}\, Z = A_V\, TZ. \tag{35}$$

Conversaly if

$$A_{n\,V}\, Z = A_V\, TZ,$$

then from equation (33), we get

$$(\nabla_X N) = 0. \qquad \square$$

Since in a Norden manifold the metric tensor g satisfies

$$g(Z, (\overline{\nabla}_Y F) X) = g((\overline{\nabla}_Y F) Z, X). \tag{36}$$

Equation (36) can be written as

$$g(Z, (\overline{\nabla}_Y FX - F\overline{\nabla}_Y X)) = g((\overline{\nabla}_Y FZ - F\overline{\nabla}_Y Z), X). \tag{37}$$

Using equation (8) in (37), we have

$$g(Z, (\overline{\nabla}_Y (TX + NX) - F\overline{\nabla}_Y X)) = g((\overline{\nabla}_Y (TZ + NZ) - F\overline{\nabla}_Y Z), X). \tag{38}$$

From (4), (5) and (38), we have

$$\begin{aligned} g(Z, (\nabla_Y TX + &\sigma(Y, TX) - A_{NX} Y + \nabla^{\perp}_Y NX)) - g(Z, F\overline{\nabla}_Y X) \\ &= g((\nabla_Y TZ + \sigma(Y, TZ) - A_{NZ} Y \\ &+ \nabla^{\perp}_Y NZ), X) - g(F\overline{\nabla}_Y Z, X). \end{aligned} \tag{39}$$

Using (8), (20) and (21) in (39), we get

$$\begin{aligned} &g(Z, (\nabla_Y T)X) + g(Z, \sigma(Y, TX)) - g(Z, A_{NX} Y) + g(Z, (\nabla_Y N)X) \\ &= g((\nabla_Y T)Z, X) + g(\sigma(Y, TZ), X) - g(A_{NZ} Y, X) \\ &+ g((\nabla_Y N)Z, X) + g(Z, F\sigma(Y, X)) - g(F\sigma(Y, Z), X), \end{aligned} \tag{40}$$

from (36) and (40), we have

$$\begin{aligned} g(Z, \sigma(Y, TX) - F\sigma(Y, X)) &- g(\sigma(Y, TZ) - F\sigma(Y, Z), X) \\ &= g(Z, A_{NX} Y) - g(A_{NZ} Y, X). \end{aligned} \tag{41}$$

Suppose that

$$\sigma(Y, TX) = \sigma(Y, X) = 0,$$

then from equation (41), we have

$$g(Z, A_{NX} Y) = g(A_{NZ} Y, X). \tag{42}$$

Thus we conclude:

Theorem 5. *If M be a slant submanifold of a Kaehler-Norden manifold $\overline{M}$ then the manifold M is totally geodesic if*

$$g(Z, A_{NX} Y) = g(A_{NZ} Y, X).$$

Now we propose:

Theorem 6. *If M be a slant submanifold of a Kaehler-Norden manifold $\overline{M}$ then the manifold M is totally geodesic if*

$$\nabla_X Y = 0.$$

Proof: From equation (1), (2) and (3), we have

$$g(\overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X FY, FZ). \tag{43}$$

Now using (8) in (43), we have

$$g(\overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X FY, TZ) - g(\overline{\nabla}_X FY, NZ), \tag{44}$$

from (3) and (44), we have

$$g(\overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X Y, FTZ) - g(\overline{\nabla}_X Y, FNZ), \tag{45}$$

using (8) in (45), we get

$$g(\overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X Y, T^2 Z + TNZ) - g(\overline{\nabla}_X Y, TNZ + N^2 Z), \tag{46}$$

equation (46), can be written as

$$g(\overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X Y, T^2 Z) - 2g(\overline{\nabla}_X Y, TNZ) - g(\overline{\nabla}_X Y, N^2 Z), \tag{47}$$

using lemma (23) in (47), we get

$$g(\overline{\nabla}_X Y, TNZ) = 0. \tag{48}$$

From (4) and (48), we have

$$\nabla_X Y = -\sigma(X, Y). \tag{49}$$

If M be a totally geodesic i.e. $\sigma(X, Y) = 0$, then from (49), we get

$$\nabla_X Y = 0. \tag{50}$$ □

Now we propose:

Theorem 7. *If M is totally umbilical slant submanifold of a Kaeher-Norden manifold $\overline{M}$ then the manifold M is minimal if and only if*

$$(\nabla_{TX} N) X = 0.$$

Proof: Let M is totally umbilical submanifold then from equation (7), we have

$$\sigma(TX, TY) = g(TX, TY) H. \tag{51}$$

Now using equation (4) in (51) , we get

$$\begin{aligned} \overline{\nabla}_{TX} TY - \nabla_{TX} TY &= g(TX, TY) H \\ &= -Cos^2\theta g(X, Y) H. \end{aligned} \tag{52}$$

Replacing Y with X, we have

$$\begin{aligned} \overline{\nabla}_{TX} TX - \nabla_{TX} TX &= -Cos^2\theta g(X, X) H \\ &= -Cos^2\theta \|X\|^2 H. \end{aligned} \tag{53}$$

Using (8) in (53), we have

$$\overline{\nabla}_{TX} FX - \overline{\nabla}_{TX} NX - \nabla_{TX} TX = -Cos^2\theta \|X\|^2 H. \tag{54}$$

From (4), (5) and (54), we have

$$\begin{aligned} -(\nabla_{TX} T)X + (\overline{\nabla}_{TX} F)X + N\nabla_{TX} X + g(TX, X)FH \\ - \nabla^{\perp}_{TX} NX + A_{NX} TX \\ = -Cos^2\theta \, \|X\|^2 H. \end{aligned} \tag{55}$$

Taking normal part of Equation (55), we have

$$N\nabla_{TX} X - \nabla^{\perp}_{TX} NX = -Cos^2\theta \, \|X\|^2 H. \tag{56}$$

Now taking inner product in equation (56) with NX, we get

$$g(N\nabla_{TX} X, NX) - g(\nabla^{\perp}_{TX} NX, NX) = -Cos^2\theta \, \|X\|^2 g(H, NX), \tag{57}$$

using (21) in (57), we get

$$g((\nabla_{TX} N)X \;, NX) = Cos^2\theta \, \|X\|^2 g(H, NX). \tag{58}$$

If we take

$$(\nabla_{TX} N)X = 0, \tag{59}$$

then from equation (58) and (59), we get

$$H = 0. \tag{60}$$

Conversaly if manifold be minimal i.e. $H = 0$, then from equation (58) and (60), we get

$$(\nabla_{TX} N)X = 0.$$

□

Now from equation(3), we get

$$\overline{\nabla}_X FY = F\overline{\nabla}_X Y, \tag{61}$$

using (5), (6) and (8) in (61), we have

$$\nabla_X TY + g(X, TY)H - A_{NY} X + \nabla^{\perp}_X NY = T\nabla_X Y + N\nabla_X Y + F\sigma(X, Y). \tag{62}$$

From (7) and (62), we get

$$\nabla_X TY + g(X, TY)H - A_{NY} X + \nabla^{\perp}_X NY = T\nabla_X Y + N\nabla_X Y + Fg(X, Y)H. \tag{63}$$

Now taking inner product of (63) by FH, we have

$$\begin{aligned} & g(\nabla_X TY, FH) + g(X, TY)\, g(H, FH) \\ & - g(NY, H)\, g(X, FH) + g(\nabla^{\perp}_X NY, FH) \\ & = g(T\nabla_X Y, FH) + g(N\nabla_X Y, FH) \\ & + g(X, Y)\, g(FH, FH). \end{aligned} \tag{64}$$

equation (64) implies

$$\begin{aligned} & g(\nabla_X TY, FH) + g(X, TY)\, g(H, FH) \\ & - g(NY, H)\, g(X, FH) + g(\nabla^{\perp}_X NY, FH) \\ & = g(T\nabla_X Y, FH) + g(N\nabla_X Y, FH) - g(X, Y)\, \|H\|^2 . \end{aligned} \tag{65}$$

From equation (21) and (65), we get

$$g(((\nabla_X T) + (\nabla_X N))Y, FH) + g(X, TY)\, g(H, FH) - g(NY, H)\, g(X, FH) = -g(X, Y)\, \|H\|^2 . \tag{66}$$

Replacing H by FH in (66), we get

$$g(((\nabla_X T) + (\nabla_X N))Y, H) + g(X, TY)\, g(FH, H) - g(NY, FH)\, g(X, H) = -g(X,Y)\, \|H\|^2 . \tag{67}$$

Equation (67) can be written as

$$H = -\frac{((\nabla_X T) + (\nabla_X N))Y + g(X, TY)\, FH - g(NY, FH)}{g(X,Y)}. \tag{68}$$

Now if M be minimal i.e. $H = 0$, then from (68), we have

$$(\nabla_X T)\, Y + (\nabla_X N)\, Y = 0. \tag{69}$$

Thus we conclude:

Theorem 8. *If M be a totally umbilical slant submanifold of a Kaeher-Norden manifold $\overline{M}$ then the slant submanifold M is minimal if*

$$\nabla_X T = -\nabla_X N.$$

5. Discussion

In the present we have studied slant submanifold of a Kaehler-Norden manifold. In this paper, we have proved some interesting results in Kaehler-Nordan manifold.

6. Conflict of interest

The authors declare that they have no conflict of interest.

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