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On Hypersurface of a Finsler space with the Generalized Matsumoto metric

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In this paper, we study some geometrical properties of a Finsler space with the generalized Matsumoto metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$. Further, we prove the necessary and sufficient condition for Finsler Hypersurface satisfies the condition of hyperplanes of first, second kinds and but not the hyperplane of the third kind with respect to the above metric.

Keywords: Finsler space, Generalized Matsumoto metric, Hyperplanes, Induced Cartan's connection , (α, β)-metric.
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1. Introduction

In 1992,[1] the notion of Finsler spaces with an (α, β)-metric was first proposed by M. Matsumoto named as function of $L(\alpha, \beta)$. After the Matsumoto's accomplishment in the development of Finsler geometry there are lot of contributions were given by several authors they have studied a special forms of (α, β)-metrics like Rander's metric, Kropina metric, generalized Kropina metric, Shen's square metric etc. The systematic theory of the hypersurface of a Finsler space was built by Matsumoto in 1985, along with this he explained the hyperplane of the first kind, second kind and third kind are the classification of hypersurfaces. Further, many researchers were considered these three kinds of hyperplanes in different types of (α, β)-metrics of Finsler spaces and they came with a various conclusions. Recent years, in 2009, H. G. Nagaraja, S. K. Narasimhamurthy, Pradeep Kumar and S. T. Aveesh obtained some results on geometrical properties of Finslerian hypersurfaces with (α, β)-metrics [3,4]. In 2018, K. Vineet and R. K. Gupta worked on some special (α, β)-metric. In 2020, Brijesh kumar Tripathi introduced same aspect with deformed Berwald-infinite series metric.

In this paper, our aim is to express certain geometrical properties of hypersurface of a Finsler spaces and we discussed the different kinds of hyperplanes with generalized Matsumoto metric, $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$. We have derived that the necessary and sufficient condition for hypersurface of generalized Matsumoto metric satisfies the conditions of hyperplane of first, second and but not third kind for above metric.

2. Preliminaries

We consider an n-dimensional Finsler space with smooth manifold M^n assigned with L i.e., $F^n = (M^n, L(\alpha,\beta))$, where α-Riemannian metric and β-differential 1–form. Here

$$L = \frac{\alpha^{m+1}}{(\alpha-\beta)^m} \tag{1}$$

Now differentiate the (1) partially with respect to α and β also with first and second order. We get,

$$L_\alpha = \frac{\alpha^m(\alpha - m\beta - \beta)}{(\alpha-\beta)^{m+1}}, \quad L_\beta = \frac{m\alpha^{m+1}}{(\alpha-\beta)^{m+1}}, \tag{2}$$

$$L_{\alpha\alpha} = -2m(m+1)\left(\frac{\alpha^m}{(\alpha-\beta)^{m+1}} - \frac{\alpha^{m-1}}{2(\alpha-\beta)^m} - \frac{\alpha^{m+1}}{2(\alpha-\beta)^{m+2}}\right), \tag{3}$$

$$L_{\beta\beta} = m(m+1)\frac{\alpha^{m+1}}{(\alpha-\beta)^{m+2}}, \tag{4}$$

$$L_{\alpha\beta} = m(m+1)\left(\frac{\alpha^m}{(\alpha-\beta)^{m+1}} - \frac{\alpha^{m+1}}{(\alpha-\beta)^{m+2}}\right). \tag{5}$$

where, $L_\alpha = \dfrac{\partial L}{\partial \alpha}, L_\beta = \dfrac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \dfrac{\partial L_\alpha}{\partial \alpha}, L_{\beta\beta} = \dfrac{\partial L_\beta}{\partial \beta}, L_{\alpha\beta} = \dfrac{\partial L_\alpha}{\partial \beta}$.

The normalized element with supporting element $l_i = \dot{\partial}_i L$ and angular metric tensor $h_{ij} = L^{-1}\dot{\partial}_i\dot{\partial}_j L$ are given by,

$$l_i = \alpha^{-1}L_\alpha Y_i + L_\beta b_i$$

$$h_{ij} = Pa_{ij} + Q_0 b_i b_j + Q_1(b_i Y_j + b_j Y_i) + Q_2 Y_i Y_j \tag{6}$$

where, $Y_i = a_{ij}y^j, \dot{\partial}_i = \dfrac{\partial}{\partial y^i}$ and

$$P = LL_\alpha \alpha^{-1}, \qquad Q_0 = LL_{\beta\beta}, \qquad Q_1 = LL_{\alpha\beta}\alpha^{-1}, \qquad Q_2 = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) \tag{7}$$

The constant quantities of fundamental function $L(\alpha,\beta)$ of (1) are given below by using (7), we get

$$P = \frac{\alpha^{2m}(m+1)}{(\alpha-\beta)^{2m}} - \frac{m\alpha^{2m+1}}{(\alpha-\beta)^{2m+1}},$$

$$P = \frac{\alpha^{2m}(m+1)}{(\alpha-\beta)^{2m}} - \frac{m\alpha^{2m+1}}{(\alpha-\beta)^{2m+1}},$$

$$Q_0 = m(m+1)\frac{\alpha^{2m+2}}{(\alpha-\beta)^{2m+2}},$$

$$Q_1 = m(m+1)\left(\frac{\alpha^{2m}}{(\alpha-\beta)^{2m+1}} - \frac{\alpha^{2m+1}}{(\alpha-\beta)^{2m+2}}\right)$$

$$Q_2 = \frac{m(m+1)\alpha^{2m}}{(\alpha-\beta)^{2m+2}} + \frac{(m^2-1)\alpha^{2m-2}}{(\alpha-\beta)^{2m}} - \frac{2m(m+\frac{1}{2})\alpha^{2m-1}}{(\alpha-\beta)^{2m+1}} \tag{8}$$

Now the fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$is defined as,

$$g_{ij} = Pa_{ij} + P_0 b_i b_j + P_1(b_i Y_j + b_j Y_i) + Q_2 Y_i Y_j. \tag{9}$$

where,

$$P_0 = Q_0 + L_\beta^2, \qquad P_1 = Q_1 + L^{-1}PL_\beta, \qquad P_2 = Q_2 + P^2L^{-2}. \tag{10}$$

In addition, the reciprocal of tensor g_{ij} is g^{ij} given by

$$g^{ij} = P^{-1}a^{ij} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j \tag{11}$$

where,

$$b^i = a^{ij}b_j \quad S_0 = \frac{(PP_0 + (P_0P_2 - P_1^2)\alpha^2)}{\mu P}, \tag{12}$$

$$S_1 = \frac{(PP_1 + (P_0P_2 - P_1^2)\beta)}{\mu P}, \qquad S_2 = \frac{(PP_2 + (P_0P_2 - P_1^2)b^2)}{\mu P}, \tag{13}$$

$$\mu = P(P + P_0 b^2 + P_1\beta) + (P_0P_2 - P_1^2)(\alpha^2 b^2 - \beta^2). \tag{14}$$

The Cartan tensor is given by [5],

$$2PC_{ijk} = P_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k. \tag{15}$$

where,

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \qquad \gamma_1 = P\left(\frac{\partial P_0}{\partial \beta}\right) - 3P_1Q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{16}$$

The associated Riemannian space containing the components of the Christoffel's symbol $\left\{ {i \atop jk} \right\}$ and the covariant derivative ∇_k with respect to x^k corresponding to this Christoffel's symbols.

We have the following components of the symmetric and skew symmetric tensors respectively,

$$E_{ij} = \frac{(b_{ij} + b_{ji})}{2}, \qquad F_{ij} = \frac{(b_{ij} - b_{ji})}{2}. \tag{17}$$

where $b_{ij} = \nabla_j b_i$.

The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ of the special Finsler space F^n is given by

$$\begin{cases} D^i_{jk} = & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + \\ & C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}). \end{cases} \tag{18}$$

where,

$$B_k = P_0 b_k + P_1 Y_k, \;\; B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}, \;\; B_0 = b_i y^i, \tag{19}$$

$$B_{ij} = \frac{\{P_1(a_{ij} - \alpha^{-2} Y_i Y_j) + \dfrac{\partial P_0}{\partial \beta} m_i m_j\}}{2}, \tag{20}$$

$$A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \tag{21}$$

$$B^k_i = g^{kj} B_{ji}, \;\; \lambda^m = B^m E_{00} + 2 B_0 F^m_0. \tag{22}$$

Here and also for the following we denote 0 as contraction with y^i except for the quantities P_0, Q_0 and S_0.

3. Induced Cartan Connection

Let F^{n-1} be a Hypersurface which contains a equation $x^i = x^i(u^\alpha), \alpha = 1, 2, ...(n-1)$, where u_α be a Gaussian coordinates on the hypersurface F^{n-1}. Let us consider that the matrix of the projection factor $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ is of rank $(n-1)$, then

$$y^i = B^i_\alpha(u) v^\alpha \tag{23}$$

Here y^i is the supporting element of F^n is tangential to F^{n-1} and thus the $v = v^\alpha$ is the element of support of F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and Cartan tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma. \tag{24}$$

A unit normal vector $N^i(u, v)$ is at each point u_α of F^{n-1} is expressed by,

$$g_{ij}(x(u,v), y(u,v)) B^i_\alpha N^i = 0, \qquad g_{ij}(x(u,v), y(u,v)) N^i N^j = 1. \tag{25}$$

In terms of angular metric tensor h_{ij}, thus

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad h_{ij} B^i_\alpha N^j = 0, \qquad h_{ij} N^i N^j = 1. \tag{26}$$

If B^α_i is the inverse of B^i_α , then

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N^i = 0, \tag{27}$$

$$N_i = g_{ij} N^j, \qquad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j. \tag{28}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ of F^{n-1}. From (25) and (27), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N_i = 0, \qquad N^i B^\alpha_i = 0, \qquad N^i N_i = 1. \tag{29}$$

And also we have,

$$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j. \tag{30}$$

The induced Cartan's connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\ \gamma}, G^\alpha_\beta, C^\alpha_{\beta\ \gamma})$ of F^{n-1} generated from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{j\ k}, \Gamma^{*i}_{0\ k}, C^i_{j\,k})$ is given by [6],

$$\begin{aligned}
\Gamma^{*\alpha}_{\beta\ \gamma} =& B^\alpha_i(B^i_{\beta_\gamma} + \Gamma^{*i}_{j\ k} B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma\\
G^\alpha_\beta =& B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0\ j} B^j_\beta),\\
C^\alpha_{\beta\ \gamma} =& B^\alpha_i C^i_{j\,k} B^j_\beta B^k_\gamma.
\end{aligned}$$

where,

$$M_{\beta\gamma} = N_i C^i_{jk} B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma} M_{\beta\gamma}, \quad H_\beta = \dot{N}_i(B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_\beta)$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha$$

Here the second fundamental v - tensor and normal curvature vector are $M_{\beta\gamma}$ and H_β respectively, the second fundamental h - tensor $H_{\beta\gamma}$ is stated as [6]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M_\beta H_\gamma. \tag{31}$$

where,

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k. \tag{32}$$

The second fundamental h-tensor and the normal curvature vector are $H_{\alpha\beta}$ and H_α of F^{n-1} respectively, are given by

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta \tag{33}$$

and

$$H_\alpha = N_i(B^i_{0\alpha} + G^i_j B^j_\alpha) \tag{34}$$

where $M_\alpha = C_{ijk} B^i_\alpha N^j N^k, B^i_{\alpha\beta} = \dfrac{\partial x^i}{\partial u^\alpha \partial u^\beta}$ and $B^i_{0\alpha} = B^i_{\beta\alpha} v^\beta$. It is clearly expresses that $H_{\alpha\beta}$ is not symmetric and

$$H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha. \tag{35}$$

The equations (33) and (34) gives

$$H_{0\alpha} = H_{\beta\alpha} v^\beta = H_\alpha, \quad H_{\alpha 0} = H_{\alpha\beta} v^\beta = H_\alpha + M_\alpha H_0. \tag{36}$$

Here,

$$M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k. \tag{37}$$

where $M_{\alpha\beta}$ is the second fundamental v-tensor. The projection factors B_i^α and N^i contains the h and v-covariant derivatives evolves with respect to induced Cartan connection $IC\Gamma$ respectively are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i = M_{\alpha\beta}N^i, \quad N^i_{|\beta} = -H_{\alpha\beta}B^\alpha_j g^{ij} = -M_{\alpha\beta}B^\alpha_j g^{ij} \tag{38}$$

and in terms of h and v-covariant derivatives of vector field X^i are as follows:

$$X_{i|\beta} = X_{i|j}B^j_\beta + X_{i|j}N^j H_\beta, \quad X_{i|\beta} = X_{i|j}B^j_\beta. \tag{39}$$

The following lemmas states the different kinds of hypersurfaces and their characteristics conditions which are defined by M. Matsumoto [6].

Lemma 1. *A hypersurface of a Finsler space F^{n-1} is a hyperplane of the 1^{st} kind if and only if $H_\alpha = 0$.*

Lemma 2. *A hypersurface of a Finsler space F^{n-1} is a hyperplane of the 2^{nd} kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 3. *A hypersurface of a Finsler space F^{n-1} is a hyperplane of the 3^{rd} kind if and only if $H_\alpha = 0$ and $M_{\alpha\beta} = H_{\alpha\beta} = 0$.*

4. Hypersurface F^{n-1} of Finsler space

Let us consider a Finsler space F^n with the (α,β)- metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$, where, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is 1-form metric and $b_i(x) = \dfrac{\partial b}{\partial x^i}$ is a gradient of scalar function $b(x)$. In this part we prove the necessary and sufficient condition of hypersurface of a Finsler space to be hyperplane which satisfies the 1^{st} kind, 2^{nd} kind and but not 3^{rd} kind. We have,

$$b_iB^i_\alpha = 0, \quad b_iy^i = \beta = 0. \tag{40}$$

Accordingly, the induced metric $L(u,v)$ of hypersurface of (40) which is Riemannian metric given by

$$L(u,v) = \sqrt{a_{\alpha\beta}(u)v^\alpha v^\beta}, \quad a_{\alpha\beta} = a_{ij}B^i_\alpha B^j_\beta. \tag{41}$$

Henceforth $F^{n-1}(c)$, now from the equations (8), (10) and (12), we have

$$P = 1 \qquad Q_0 = m(m+1) \qquad Q_1 = 0 \qquad Q_2 = -\frac{1}{\alpha^2} \qquad P_0 = m(2m+1)$$

$$P_1 = \frac{m}{\alpha}, \quad P_2 = 0, \quad \mu = 1 + (m^2+m)b^2, \quad S_0 = \frac{m^2+m}{1+(m^2+m)b^2}, \quad S_1 = \frac{m}{\alpha(1+(m^2+m)b^2)}$$

$$S_2 = -\frac{m^2b^2}{\alpha^2(1+(m^2+m)b^2)}. \tag{42}$$

wherefore (11) yields,

$$g^{ij} = a^{ij} - \frac{(m^2+m)}{1+(m^2+m)b^2} b^i b^j - \frac{m}{\alpha(1+(m^2+m)b^2)}(b^i y^j + b^j y^i) + \frac{m^2 b^2}{\alpha^2(1+(m^2+m)b^2)} y^i y^j. \tag{43}$$

Hence with F^{n-1}, operating the $b_i b_j$ using (40) and (43) we get

$$g^{ij} b_i b_j = \frac{b^2}{1+(m^2+m)b^2} \tag{44}$$

Therefore, we obtain

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+(m^2+m)b^2}} N_i, \qquad b^2 = a^{ij} b_i b_j. \tag{45}$$

where b is the length of vector b^i. From (43) and (45), we get

$$b^i = a^{ij} b_j = \sqrt{b^2\left((m^2+m)b^2+1\right)} N^i + \frac{mb^2}{\alpha} y^i. \tag{46}$$

Theorem 1. *Let F^{n-1} be a hypersurface of Finsler space F^n with metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ with scalar function $b_i(x) = \partial_i b(x)$ given by (44) and (45) and a hypersurface F^{n-1} of F^n. Then the induced Riemannian metric is lead by (40).*

Theorem 2. *The second fundamental v-tensor of hypersurface of a finsler space with the metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ vanishes and the second fundamental h-tensor is symmetric.*

Proof: Further, h_{ij} and g_{ij} are identified by (6) and (9) into (42),

$$h_{ij} = a_{ij} + m(m+1) b_i b_j - \frac{1}{\alpha^2} Y_i Y_j \tag{47}$$

$$g_{ij} = a_{ij} + m(2m+1) b_i b_j + \frac{m}{\alpha}(b_i y_j + b_j y_i) \tag{48}$$

where, h_{ij} is an angular metric tensor and g_{ij} is a metric tensor.

Along hypersurface $F^{n-1}(c)$ with $h^{(a)}_{\alpha\beta}$ which represents the angular metric tensor of Riemannian $a_{ij}(x)$, then $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Accordingly from (10), as well as $F^{n-1}(c)$, $\dfrac{\partial P_0}{\partial \beta} = 2m(m+1)(2m+1)\dfrac{\alpha^{2m+2}}{(\alpha-\beta)^{2m+3}}$.

Therefore, (16) yields $\gamma_1 = \dfrac{m(m+1)(m+2)}{\alpha}$, $m_i = b_i$.

Now from (15), we get

$$C_{ijk} = \frac{m}{2\alpha}(h_{ij} b_k + h_{jk} b_i + h_{ki} b_j) + \left(\frac{m(m+1)(m+2)}{2\alpha}\right) b_i b_j b_k. \tag{49}$$

Using (37) and (26) and (49), we obtain

$$M_{\alpha\beta} = \frac{m}{2\alpha}\sqrt{\frac{b^2}{1+(m^2+m)b^2}}h_{\alpha\beta}, \quad M_\alpha = 0 \tag{50}$$

Thus from (50) and (35) thereby $H_{\alpha\beta} = H_{\beta\alpha}$.
we have the following result:

The necessary and sufficient condition for a hypersurface $F^{n-1}(c)$ of a Finsler space with the generalized Matsumoto metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ to be a hyperplane of first kind $b_{ij} = \dfrac{b_i c_j + b_j c_i}{2}$ holds.

Proof: Subsequently, from (40) we get

$$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0. \tag{51}$$

Therefore from (38) and we are using (39), we get

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta}N^i = 0. \tag{52}$$

Along $b_{i|j} = -b_s C^s_{ij}$, becomes $b_{i|j}B^i_\alpha N^j = -\sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}M_\alpha = 0$. Including $b_{i|j}$ is symmetric, from (52) we obtain

$$\sqrt{\frac{b^2}{1+(m^2+m)b^2}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \tag{53}$$

Now, contracting the (53) with v^β and again that with v^α, yields

$$\sqrt{\frac{b^2}{1+(m^2+m)b^2}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0, \tag{54}$$

$$\sqrt{\frac{b^2}{1+(m^2+m)b^2}}H_0 + b_{i|j}y^i y^j = 0. \tag{55}$$

In the context of lemma (1), the hypersurface of Finsler space $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $H_0 = 0$ and also on the other hand $b_{i|j}y^i y^j = 0$. The covariant derivative $b_{i|j}$ depends on y^i.

Since, from (17) which become $E_{ij} = b_{ij},\ F_{ij} = 0$ and $F^i_j = 0$.

Hence 18 reduces to

$$D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_s g^{is} + \lambda^s\left(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}\right) \tag{56}$$

In the context of (42) and (43), the relations in (19) consists of

$$B_i = m(2m+1)b_i + \frac{m}{\alpha}Y_i \qquad B^i = \frac{m^2+m}{1+(m^2+m)b^2}b^i + \frac{m}{\alpha(1+(m^2+m)b^2)}y^i$$

$$B_{ij} = \frac{m}{2\alpha}a_{ij} - \frac{m}{2\alpha^3}Y_iY_j + \frac{m(m+1)(2m+1)}{\alpha}b_ib_j,$$

$$B^i_j = g^{ik}B_{kj} = \frac{m}{2\alpha}(\delta^i_j - \alpha^{-2}y^iY_j) + \frac{m(m+1)(3m+2)}{2\alpha(1+(m^2+m)b^2)}b^ib_j - \frac{m^2+2m^2(m+1)(2m+1)b^2}{2\alpha^2(1+(m^2+m)b^2)}y^ib_j$$

$$A^m_k = B^m_k b_{00} + B^m b_{k0} \qquad \lambda^m = B^m b_{00}. \tag{57}$$

In the context of (42) and (43), the relations in (19) consists of

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}$$

$$D^i_{00} = B^i \qquad = b_{00} = \left(\frac{m^2+m}{1+(m^2+m)b^2}b^i + \frac{m}{\alpha(1+(m^2+m)b^2)}y^i\right)b_{00}. \tag{58}$$

By the relation (40), we obtain

$$b_iD^i_{j0} = \frac{(m^2+m)b^2}{1+(m^2+m)b^2}b_{j0} + \frac{m+3m(m+1)^2b^2}{2\alpha(1+(m^2+m)b^2)}b_jb_{00} - \frac{(m^2+m)b^m}{1+(m^2+m)b^2}b_iC^i_{jm}b_{00}. \tag{59}$$

$$b_iD^i_{00} = \frac{(m^2+m)b^2}{1+(m^2+m)b^2}b_{00}. \tag{60}$$

Hence $b_{i|j} = b_{ij} - b_sD^s_{ij}$ with (59) and (60) yields

$$b_{i|j}y^iy^j = \frac{1}{1+(m^2+m)b^2}b_{00}. \tag{61}$$

Subsequently (54) and (55) are expressed as

$$\frac{b}{\sqrt{1+(m^2+m)b^2}}H_\alpha + b_{i0}B^i_\alpha = 0. \tag{62}$$

$$\frac{b}{\sqrt{1+(m^2+m)b^2}}H_0 + \frac{1}{1+(m^2+m)b^2}b_{00} = 0. \tag{63}$$

As consequence of that the condition $H_0 = 0$ for an induced metric which is equivalent to $b_{00} = 0$, using (40) which can be written as $b_{ij}y^iy^j = (b_iy^i)(c_jy^j)$ for some $c_j(x)$, therefore

$$b_{ij} = \frac{b_ic_j + b_jc_i}{2}. \tag{64}$$

And also satisfies the hypersurface of the second kind, $b_{00} = 0$, $b_{ij}B^i_\alpha B^j_\beta = 0$, $b_{ij}B^i_\alpha y^j = 0$. Here (63) gives $H_\alpha = 0$. And from (57) and (64) we obtain $b_{i0}b^i = \frac{c_0b^2}{2}$, $\lambda^m = 0$, $A^i_jB^j_\beta = 0$ and $B_{ij}B^i_\alpha B^j_\beta = 0$.

With the help of (37), (43), (46), (50) and (56), we get $b_s D^s_{ij} B^i_\alpha B^j_\beta = -\dfrac{c_0 b^2 m}{4\alpha(1+(m^2+m)b^2)} h_{\alpha\beta}$, henceforth, (53) reduces to

$$\frac{b}{\sqrt{1+(m^2+m)b^2}} H_{\alpha\beta} + \frac{c_0 b^2 m}{4\alpha(1+(m^2+m)b^2)} h_{\alpha\beta} = 0. \qquad (65)$$

hence the result From (36),(49), (57) and (55) and the Theorem2.

Theorem 3. *The necessary and sufficient condition for a hypersurface $F^{n-1}(c)$ of a Finsler space with (α,β)-metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ to be a hyperplane of second kind* (65) *holds. i.e., $H_{\alpha\beta} = 0$.*

Theorem 4. *The necessary and sufficient condition for a hypersurface $F^{n-1}(c)$ of a Finsler space with (α,β)-metric $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ to be a hyperplane of third kind does not holds. we distinguished that the hypersurface $F^{n-1}(c)$ is not a hyperplane of the third kind.*

Here we concluded that the necessary and sufficient conditions for the hypersurface $F^{n-1}(c)$ of a Finsler space with the $L = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$, in this paper we determined the conditions for hypersrfaces to be a hyperplanes of first, second kind and but not a third kind.

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