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# Conformal Vector Fields on Finsler - Kropina Metric

Natesh N.[1], S.K. Narasimhamurthy[2] and Roopa.M.K[3]
*Department of Mathematics, Govt.Science College, Chitradurga - 5770501*
*Department of Mathematics, Kuvempu University,*
*Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA*
*E-mail[1]: nateshmaths@gmail.com*
*E-mail[2]: nmurthysk@gmail.com*
*E-mail[2]: roopamk1002@gmail.com*

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In this paper, we study the conformal vector fields on a class of Finsler metrics. In particular Kropina metric $F = \frac{\alpha^2}{\beta}$ is defined in Riemannian metric $\alpha$ and 1-form $\beta$ and its norm. Then we characterize the PDE's of conformal vector fields on Kropina metric.

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## 1. Introduction

Finsler geometry has been developing rapidly since last few decades, after its emergence in 1917. Finsler geometry has been influenced by group theory. The celebrated Erlangen program of F. Klein, posed in 1872, greatly influenced the development of geometry. Klein proposed to categorize the geometries by their chacteristic group of transformations.

Conformal vector fields are important in Riemann - Finsler geometry. To solve problems on conformal vector fields in Riemanna - Finsler geometry as follows:. Let $(M.F)$ be a Finsler manifold. It is known that a vector field $v = v^i \frac{\partial}{\partial x^i}$ on $M$ is a conformal vector field on $F$ with conformal factor $c = c(x)$ if and only if $X_v(F^2) = 4cF^2$, where $X_v = v^i \frac{\partial}{\partial x^i} + y^i \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial y^j}$ [5]. Recently, Shen and Xia [5] have studied conformal vector fields on a Randers manifold with certain curvature properties. They also determine conformal vector fields on a locally projectively flat Randers manifold. Besides they use homothetic vector fields (c = constant) on Randers manifolds to construct new Randers metrics of scalar flag curvature [6]. The theory of Kropina metric was investigated by L.Berwald in connection with a two dimensional Finsler space with rectilinear extremal and were investigated by V.K.Kropina. Randers metrics seem to be among the simplest non - trivial Finsler metrics with many investigation in Physics, Electron optics with a magnetic field, dissipative mechanics, irreversible thermodynamics etc..

In this paper, we shall study the conformal vector fields with Finsler Kropina metric, whose metric is defined in Riemannian metric $\alpha$ and 1-form $\beta$ and its norm and here by inspiring the work of conformal vector fields on some Finsler manifolds. Then we characterize the PDE's of conformal vector fields on Kropina metric. In natural way, we consider the general $(\alpha, \beta)$- metrics are defined as the form:

$$F = \alpha\phi(b^2, \beta^2/\alpha). \tag{1}$$

For example, the Randers metrics and the square metrics are defined by functions $\phi = \phi(b^2, s)$ in the following form:

$$\phi = \frac{\sqrt{1-b^2+s^2}+s}{1-b^2}. \tag{2}$$

$$\phi = \frac{(\sqrt{1-b^2+s^2}+s)}{(1-b^2)^2\sqrt{1-b^2+s^2}}. \tag{3}$$

Based on the some reviews, further we shall study the covariant derivatives of conformal vector field is directly proportional to Kropina metric .

## 2. Preliminaries

Let $M$ be an n-dimensional differential manifold and $TM$ be the tangent bundle. A Finsler metric on $M$ is a function $F = F(x, y) : TM \longrightarrow R$ satisfying the following conditions:

(1) $F(x, y)$ is a $C^\infty$ function on $TM\backslash\{0\}$;
(2) $F(x, y) \geq 0$ and $F(x, y) = 0 \to y = 0$;
(3) $F(x, \lambda y) = \lambda F(x, y), \lambda > 0$;
(4) the fundamental tensor $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}$ is positively defined.

Let

$$C_{ijk} = \tfrac{1}{4}[F^2]_{y^i y^j y^k} = \tfrac{1}{2}\tfrac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $C = C_{ijk}dx^i \bigotimes dx^j \bigotimes dx^k$ on $TM\backslash\{0\}$. We call $C$ the Cartan torison.
Let $F$ be a Finsler metric on an n - dimensional manifold $M$. The canonical geodesic $\sigma(t)$ of $F$ is characterized by

$$\tfrac{d^2\sigma^i(t)}{dt^2} + 2G^i(\sigma(t), \sigma\dot{(t)}) = 0,$$

where $G^i$ are the geodesic coefficients having the expression $G^i = \frac{1}{4}g^{ij}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}$ with $(g^{ij}) = (g_{ij})^{-1}$ and $\dot{\sigma} = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$. A spray on $M$ is a globally $C^\infty$ vector field G on $TM\backslash\{0\}$ which is expressed in local coordinates as follows :

$$G = y^i\tfrac{\partial}{\partial x^i} - 2G^i\tfrac{\partial}{\partial y^i}.$$

Given geodesic coefficients $G^i$, we define the covariant derivatives of a vector field $X = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ by

$$D_i X(t) = \{X^{\dot{i}}(t) + X^j(t)N^i_j(c(t), c\dot{(t)})\}\frac{\partial}{\partial x^i|_{c(t)}} \ ,$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}, X^{\dot{i}}(t) = \frac{dX^i}{dt}$ and $\dot{c} = \frac{dc^i}{dt}\frac{\partial}{\partial x^i}$.
It is easy to verify that

$$D_{\dot{c}}(X+Y)(t) = D_{\dot{c}}X(t) + D_{\dot{c}}Y(t),$$

$$D_{\dot{c}}(fX)(t) = f^1(t)X(t) + f(t)D_{\dot{c}}X(t) \ .$$

Since $D_{c\dot{(t)}}$ linearly depends on $X(t), D_{\dot{c}}X(t)$ is called the linear covariant derivative. It is easy to see that the canonical geodesic satisfies $D_{\dot{\sigma}} = 0$.
Let $TM$ be the tangent bundle and $\pi : TM\backslash\{0\} \to M$ the natural projection. According to the pulled - back bundle $\pi^*TM$ admits a unique linear connection called the Chern connection.
Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric expressed in terms of a Riemannian metric $\alpha$ and a vector field $V$ on $M$. Consider equation (1) is

$$F = \alpha\phi(b^2, \tfrac{\beta}{\alpha})$$

where $\phi = \phi(b^2, s)$ is a positive smooth function on $[0, b_0) \times (-b_0, b_0)$. It is required that

$$\phi - \phi_2 s > 0, \ \ \phi - \phi_2 s + (b^2 - s^2)\phi_{22} > 0, \tag{4}$$

for $b < b_0$, where,

$$\phi_1 = \tfrac{\partial\phi}{\partial b^2}, \ \phi_2 = \tfrac{\partial\phi}{\partial s},$$
$$\phi_{22} = \tfrac{\partial^2\phi}{\partial s^2}, \ \ \alpha = \tfrac{\sqrt{1-b^2+s^2}}{1-b^2}, \ \beta = \tfrac{s}{1-b^2} \ .$$

We write the function where $\phi = \phi(b^2, s)$ in the following Taylor expansion

$$\phi = p_0 + p_1 s + p_2 s^2 + o(s^3),$$

where

$$p_i = p_i(b^2), \text{ and } p_0 = \tfrac{1}{(1-b^2)^{\frac{1}{2}}}, \ \ p_1 = \tfrac{1}{1-b^2}, p_2 = \tfrac{1}{2(1-b^2)^{3/2}}.$$

Now (2) implies that

$$p_0 > 0, \ \ p_0 + 2b^2 p_2 > 0.$$

But there is no restriction on $p_1$ . If we assume that $p_1 \neq 0$, then $F$ is not reversible. If the Finsler Kropina metric is on the conformal vector field, then Finsler Kropina metric becomes

$$\phi(b^2, s) = \tfrac{1-b^2+s^2}{s(1-b^2)}$$

and (2) and (3) satisfy

$$\frac{1}{2b^2} + \{\frac{p_1^1}{p_1} - \frac{p_0^1}{p_0} + \frac{p_2}{p_0}[2\frac{p_1^1}{p_1} - \frac{p_0^1}{p_0}] - \frac{p_2^1}{p_0}\}b^2 = \frac{1}{2b^2(1-b^2)}. \quad (5)$$

### 2.1. *Definition of Conformal vector fields:*

Let $F$ be a Finsler metric on a manifold $M$, and V be a vector field on $M$. Let $\phi_t$ be the flow generated by $V$. Define $\tilde{\phi} : TM \to TM$ by $\phi_t(x, y) = (\phi_t(x), \phi_t * (y))$. A vector field $V$ is said to be conformal if

$$\phi_t^* \tilde{F} = e^{-2\sigma_t} F, \quad (6)$$

where $\sigma_t$ is a function on $M$ for every $t$.
Differentiating the above equation by t at $t = 0$, we obtain

$$X_v(F) = -2cF, \quad (7)$$

where $c$ is called the conformal factor and we define

$$X_v = V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j}, c = \frac{d}{dt}|_{t=0}\sigma_t. \quad (8)$$

Differentiating the above equation by t at $t = 0$, we obtain

$$X_v(F) = -2cF, \quad (9)$$

where $c$ is called the conformal factor and we define

$$X_v = V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j}, c = \frac{d}{dt}|_{t=0}\sigma_t. \quad (10)$$

## 3. Conformal vector fields on Finsler- Kropina metric

In this section we shall study the conformal vector field on Kropina metric with (2). Let $V$ be a conformal vector field of $F$ with conformal factor $c(x)$.

$$i.e., X_v(F^2) = 4cF^2. \quad (11)$$

Now we are in the position from (2) and to solve the above with the Kropina metric, we have

$$F = \frac{\alpha^2}{\beta} = \frac{1 - b^2 + s^2}{s(1 - b^2)},$$

then (11) implies

$$X_v(F^2) = \phi^2 X_v(\alpha^2) + \alpha^2 X_v(\phi^2),$$

$$X_v(F^2) = \phi^2 X_v(\alpha^2) + 2\phi\alpha^2\phi_1 X_v(b^2) + 2\phi\phi_2\alpha X_v(\beta) - 2\phi\phi_2\beta X_v(\alpha),$$

$$\begin{aligned} &X_v(F^2) = (1 + b^4 + s^4 - 2b^2 - 2s^2b^2)X_v(\alpha^2) \\ &+ 2\alpha^2 X_v(b^2)(s^3 - s^3b^2 + s^5) + 2\alpha X_v(\beta)(b^2 + b^4 + b^6 + s^4 - s^4b^2) \\ &- 2\beta X_v(\alpha)(b^6 + 2b^4 - b^2 + s^4 - s^4b^2 + 2s^2b^2 \\ &- b^4s^2 - s^2)/s^3(1 - b^2)^3 \end{aligned}$$

$$\begin{aligned} X_v(F^2) &= A_0[A_1 X_v(\alpha^2) + 2\alpha^2 X_v(b^2)A_2 \\ &+ 2\alpha X_v(\beta)A_3 - 2\beta X_v(\alpha)A_4], \end{aligned} \tag{12}$$

where,

$$\begin{aligned} A_0 &= \frac{1}{s^3(1 - b^2)^3}, \\ A_1 &= 1 + b^4 + s^4 - 2b^2 - 2s^2b^2, \\ A_2 &= s^3 - s^3b^2 + s^5, \\ A_3 &= b^4(1 + b^2) + b^2(1 - s^4) + s^4, \\ A_4 &= b^6 + b^4(2 - s^2) + s^2(2b^2 - s^4 - 1) - b^2 + s^4, \\ X_v(\alpha^2) &= 2V_{0;0}, \quad X_v(\beta) = (V^j b_{i;j} + b^j V_{j;i})y^i. \end{aligned} \tag{13}$$

Then equation (12) equivalent to

$$(\phi - \phi_2 s)V_{0;0} + \alpha\phi_2(V^j b_{i;j} + b^j V_{j;i})y^i(\phi_1 X_v(b^2) - 2c\phi)\alpha^2 = 0,$$

$$\begin{aligned} &\left(\frac{1 - b^2 + s^2}{s(1 - b^2)} - (\frac{s^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2})s\right) V_{0;0} \\ &+ \alpha\left(\frac{s^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right)\left(V^j b_{i;j} + b^j v_{j;i}\right) y^i \\ &+ \left[(\frac{s^3}{s^2(1 - b^2)^2})X_v(b^2) - 2c(\frac{1 - b^2 + s^2}{s(1 - b^2)})\right]\alpha^2 = 0 \end{aligned}$$

$$\begin{aligned} &\frac{(1 - b^2 + s^2)(1 - b^2) - (s^2 - s^2b^2 + 2b^2 - sb^4 - 1)}{s(1 - b^2)^2} V_{0;j} \\ &+ \alpha\left(\left(\frac{s^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right)(V^j b_{i;j} + b^j V_{j;i})\right) y^i \\ &+ \frac{s\alpha^2 X_v(b^2) - 2c\alpha^2(1 - b^2 + s^2)}{s(1 - b^2)} = 0 \end{aligned}$$

$$\begin{aligned} &\frac{b^4 + sb^4 - 4b^2}{s(1 - b^2)} V_{0;j} + \alpha\left\{\left(\frac{\alpha^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right) V^j b_{i;j} + b^j V_{j;i}\right\} y^i \\ &+ s^2\alpha^2 X_v(b^2) - \frac{2c\alpha^2(1 - b^2 + s^2)}{s(1 - b^2)} = 0. \end{aligned}$$

Which implies,

$$B_1V_{0;0} + \alpha B_2(V^j b_{i;j} + b^j V_{j;i})y^i + s^2\alpha^2 X_v(b^2) - 2B_3 c\alpha^2 = 0, \tag{14}$$

where,

$$\begin{aligned} B_1 &= \frac{b^4(1+s^2) - 4b^2}{s(1-b^2)^2}, \\ B_2 &= \frac{\alpha^2 - b^4 - b^2(s^2-2) - 1}{s^2(1-b^2)^2}, \\ B_3 &= \frac{1-b^2+s^2}{s(1-b^2)}. \end{aligned} \tag{15}$$

To simplify the computation, we fixed point $x \in M$ and make a co-ordinate change such that

$$y = \frac{s}{\sqrt{b^2-s^2}}\overline{\alpha}, \quad \alpha = \frac{b}{b^2-s^2}\overline{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2-s^2}}\overline{\alpha}, \quad \overline{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}.$$

Then we have

$$V_{0;0} = V_{1;1}\frac{s^2}{b^2-s^2}\overline{\alpha^2} + (\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + \overline{V}_{0;0}, \tag{16}$$

$$V^j b_i + b^j V_{j;i} y^i = (V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + (V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}), \tag{17}$$

where,

$$\overline{V}_{1;0} + \overline{V}_{0;1} = \sum_{a=2}^{n}(V_{1;p} + V_{p;1})y^p, \quad \overline{V}_{0;0} = \sum_{p,q=0}^{n} V_{p;q}y^p y^q, \tag{18}$$

$$V^j\overline{b}_{0;j} + b^j\overline{V}_{j;0} = \sum_{p=2}^{n}(V^j b_{p;j} + b^j V_{j;p})y^p.$$

From (16) and (17) in to (14), which yields

$$\begin{aligned} &(\phi - \phi_2 s)\left\{V_{1;1}\frac{s^2}{b^2-s^2}\overline{\alpha^2} + (V_{1;0}+V_{0;1})\frac{s}{\sqrt{b^2-s^2}\overline{\alpha}} + V_{0;0}\right\} \\ &+ \phi_2\frac{b}{\sqrt{b^2-s^2}}\overline{\alpha}\left\{(V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + (V^j\overline{b}_{0;i} + b^j V_{j;0})\right\} \end{aligned}$$

$$\begin{aligned} &B_1\{V_{1;1}\frac{s^2}{b^2-s^2}\overline{\alpha^2} + (\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + \overline{V}_{0;0}\}\} \\ &+ B_2\frac{b}{\sqrt{b^2=s^2}}\overline{\alpha}\{(V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + (V^j\overline{b}_{0;j} \\ &+ b^j\overline{V}_{j;0})\} + [s^2 X_v(b^2) - 2B_3 c]\frac{b^2}{b^2-s^2}\overline{\alpha^2} = 0. \end{aligned} \tag{19}$$

Consider the polynomial

$$\phi = p_0 + p_1 s + p_2 s^2 + o(s^3)$$

with $p_i = p_i(b^2)$ then we have,

$$\phi_1 = p_0^1 + p_1^1 s + p_2^1 s^2 + o(s^2).$$

$$\tfrac{s}{(1-b^2)^2} = p_0^1 + p_1^1 s + p_2^1 s^2 + o(s^2).$$

By letting $s = 0$ in (19) then

$$p_0 \overline{V}_{0;0} + p_1 (V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}) \overline{\alpha} + \{p_0^1 X_v(b^2) - 2cp_0\} \overline{\alpha^2} = 0. \tag{20}$$

According to the irrationality of $\overline{\alpha}$, the (19) is equivalent to

$$p_1 (V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}) = 0, \tag{21}$$

$$p_0 (\overline{V}_{0;0} + p_0^1 X_v(b^2) - 2cp_0) \overline{\alpha^2} = 0. \tag{22}$$

Therefore, the equation (22) yields

$$(V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}) = 0,$$

$$V^j b_{p;j} + b^j \overline{V}_{j;p} = 0. \tag{23}$$

By (23) we have

$$V_{p;q} + V_{q;p} = -2\{\frac{p_0^1}{p_0} X_v(b^2) - 2c\} \delta_{pq}, \quad 2 \leq p, q \leq n. \tag{24}$$

Again irrationality of $\overline{\alpha}$ from (14) we get

$$B_1 (\overline{V}_{1;0} + \overline{V}_{0;1}) \frac{s}{\sqrt{b^2 - s^2}} \overline{\alpha} = 0. \tag{25}$$

$$\begin{aligned} &B_1 \{V_{1;1} \frac{b^2}{b^2 - s^2} \overline{\alpha^2} + \overline{V}_{0;0}\} + B_2 \frac{bs}{b^2 - s^2} \overline{\alpha^2} (V^j b_{1;j} + b^j V_{j;1}) \\ &+ \{s^2 X_v(b^2) - 2cB_2\} \frac{b^2}{b^2 - s^2} \overline{\alpha^2} = 0. \end{aligned} \tag{26}$$

From (24) we get

$$\overline{V}_{1;0} + \overline{V}_{0;1} = 0.$$

This equivalent to

$$V_{1;p} + V_{p;1} = 0. \tag{27}$$

Solving (22) for $\overline{V}_{0;0}$ and plugging it in to (25) we have

$$\begin{aligned} &B_1 s^2 \{V_{1;1} \frac{p_0^1}{p_0} (X_v(b^2) - 2c)\} - \{\frac{p_0^1}{p_0} X_v(b^2) - 2c\} B_1(b^2) \\ &+ B_3 sb (V^j b_{1;j} + b^j V_{j;1}) + B_2 b^2 X_v(b^2) \\ &- 2cb^2 \frac{(1 - b^2 - s^2)}{1 - b^2} = 0. \end{aligned} \tag{28}$$

By Taylor series, expansion of $\phi(b^2, s)$ then plugging it in to (25) and by the coefficients of $s$ we have.

$$bp_1(V^j b_{1;j} + b^j V_{j;1}) + b^2 X_v(b^2)\frac{\partial p_1}{\partial b^2} - 2cb^2 p_1 = 0. \tag{29}$$

Then

$$V^j b_{1;j} + b^j V_{j;1} = -(\frac{p_1^1}{p_1} X_v(b^2) - 2c)b_i. \tag{30}$$

Then by (23) and (30) we have

$$V^j b_{i;j} + b^j V_{j;i} = -(\frac{p_1^1}{p_1} X_v(b^2) - 2c)b_i. \tag{31}$$

Substituting (30) in (29), we have

$$\begin{aligned} &B_1 s^2\{V_{1;j} + (\frac{p_1^1}{p_0} X_v(b^2) - 2c)\} \\ &- b^2 X_v(b^2)\{\frac{p_0^1}{p_0} B_1 - B_2 + B_3 s\frac{p_1^1}{p_1}\} = 0. \end{aligned} \tag{32}$$

The coefficients of all powers of s must vanish in (32). In particular, the coefficients of $s^2$ vanishes.
We have

$$V_{1;1} + \frac{p_0^1}{p_0} X_v(b^2) - 2cb = -b^2 X_v(b^2) R_0, \tag{33}$$

where

$$R_0 = [\frac{p_0^1}{p_0}\frac{p_2}{p_0} + \frac{p_0^1}{p_0} - 2\frac{p_1^1}{p_1}\frac{p_2}{p_0}].$$

By (24),(26) and (33), we have

$$V_{i;j} + V_{j;i} = 4cp_{ij} - 2X_v(b^2)\{\frac{p_0^1}{p_0} p_{ij} + R_0 b_i b_j\}. \tag{34}$$

It equivalent to

$$v_{i;j} + v_{j;i} = 4c\alpha - 2X_v(b^2)\{\frac{p_0^1}{p_0}\alpha + R_0\beta\}. \tag{35}$$

Contracting (26) with $b^i$ and $b^j$ yields

$$V_{i;j} b^i b^j = 2cb^2 - b^2 X_v(b^2)\{\frac{p_0^1}{p_0} + R_0 b^2\}. \tag{36}$$

This equivalent to

$$V_{i;j} b^i b^j = 2c\beta^2 - b^2 X_v(b^2).$$

Contracting 26 with $b^i$ and $b^j$ yields

$$V_{i;j}b^ib^j = 2cb^2 - b^2X_v(b^2)\{\frac{1}{2b^2} + \frac{p_1^1}{p_1}\}. \tag{37}$$

Here,we used the fact that $X_v(b^2) = 2b_{i;k}b^iV^k$ . Then comparing (33) with (38) yields

$$X_v(b^2)\{R_1 - R_0b^2\} = 0, \tag{38}$$

where $R_1 = \frac{1}{2b^2} + \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0}$.

Now, (38) reduced to

$$X_v(b^2)\{R_1 + R_2b^2\} = 0. \tag{39}$$

Here, two cases arises : Case 1: If

$$R_1 + R_2b^2 \neq 0, \tag{40}$$

where, $R_2 = \frac{p_0^1}{p_0}\frac{p_2^1}{p_0} + \frac{p_2^1}{p_0} - 2\frac{p_1^1}{p_1}\frac{p_2}{p_0}$.
It follows from (40) that $X_v(b^2) = 0$ and in (30) and we have

$$V_{i;j} + V_{j;i} = 4c\alpha, \quad V^jb_{i;j} + b^jV_{j;i} = 2c\beta. \tag{41}$$

Notice that if $X_v(b^2) = 0$ and (41) holds then $V$ satisfies (12) and $V$ is an conformal vector field.
Therefore, we obtain

**Theorem-1**

Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an $n$-dimensional manifold M ($n \geq 3$) and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field.Then $V$ is a conformal vector field of $F$ with conformal factor $c = c(x)$ if and only if $X_v(b^2) = 0$ and

$$V_{i;j} + V_{j;i} = 4c\alpha, \qquad V^jb_{i;j} + b^jV_{j;i} = 2c\beta. \tag{42}$$

Case 2: If

$$R_1 + R_2b^2 = 0. \tag{43}$$

In this case $X_v(b^2) \neq 0$. Then obviously, we have

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1b_ib_j, \tag{44}$$

$$V^jb_{i;j} + V_{j;i}b^j = 2\bar{c}\beta. \tag{45}$$

Since $V$ is conformal vectror field and above equation then (12) is reduced to

$$X_v(b^2)\{B_1b^{-1}[(b^2 - s^2)R_1^*] + B_2 - (\frac{1 - b^2 + s^2}{s(1 - b^2)})\frac{p_1^1}{p_1} = 0. \tag{46}$$

Therefore it follows we obtain

**Theorem-2**

Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M ($n \geq 3$) and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field.Then ,V is a conformal vector field of F with conformal factor $c = c(x)$ if and only if

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1 b_i b_j,$$

$$V^j b_{i;j} + V_{j;i} = 2\bar{c}\beta, \tag{47}$$

$$X_v(b^2)\{B_1 b^{-1}[(b^2 - s^2)R_1^*] + B_2 - (\frac{1 - b^2 + s^2}{s(1 - b^2)})\frac{p_1^1}{p_1} = 0, \tag{48}$$

where,

$$R_1 = (\frac{1}{2b^2} + \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0}),$$

$$R_1^* = (\frac{p_1^1}{p_1} - \frac{p_0^1}{p_0})\frac{s^2}{2b^2},$$

$$B_1 = \frac{b^4(1 + s^2) - 4b^2}{s(1 - b^2)^2},$$

$$B_2 = \frac{\alpha^2 - b^4 - b^2(s^2 - 2) - 1}{s^2(1 - b^2)^2},$$

and $\bar{c} = c - \frac{1}{2}X_v(b^2)\frac{p_0^1}{p_0}$.

**References**


1. Shen ZhongMin and Yuan MinGao *Conformal vector fields on some Finsler manifolds* Sci China Math,2016,59:107-114.doi.10.1007/s11425-5099-9.
2. Shen Z,Xia Q. On conformal vector fields on Randers manifolds. Sci China Math,2012,55:1869-1882.
3. Shen Z,Xia Q. A class of Randers metrics of scalar flag curvature.Int J Math,2013,24: 146-155.
4. Shen Z,Xia Q. Conformal vector fields on a locally projectively flat Randers manifold.PublMathDebrecen,2014,84:463-474.
5. Shen Z,Z.Xing H. On Randers metrics of isotropic S-curvature.Acta math Sin Engl Ser,2008,24:789-796.
6. Yu C,Zhu H. On a new class of Finsler metrics.Differential Geom Appl, 2011,29:244-254.
7. TIAN Huang Jia ,Projective vector fields on Finsler Manifolds, Appl Math J-Chiese Univ 2014,29(2): 217-229.
8. U.P.Singh , A.K.Singh , Coformal transformations of Kropina metric.
9. Z.Shen, On projectively flat $(\alpha, \beta)$- metrics, Canadian Math. Bull., 52(1) (2009), 132-144
10. Z.Shen and G.Yang, On Square metrics of scalar flag curvature, Israel J.Mayh.,2016 (Accepted).

11. B.Li and Z.Shen, On a class of projectively flat Finsler metrics with constant flag curvature, Int. J. of Math., 18(7)(2007),1-12.
12. L.Kang, On Conformal vector fields of $(\alpha, \beta)$ - spaces,preprint.
13. L. Kang, On conformally flat Randers metrics (in Chinese), Sci. Sin. Math., 41(5)(2011), 439-446.
14. M.K. Roopa And S.K. Narasimhamurthy, On Einstein Exponential Finsler Metric With Constant Killing Form Bull. Cal. Math. Soc., 109, (4) 295–304 (2017)
15. Xinyue cheng, Z. Shen, *A class of Finsler metrics with isotropic s- curvature,* Israel Journal of Mathematics, November 2009. preprint.
16. Mehdi Rafie-Rad and Bahman REZAEI, *On the Projective Algebra of Randers Metric of Constant Flag Curvature* Symmetry, Integrability and Geometry: Methods and Applications , SIGMA 7 (2011), 085, 12 pages.
17. A. Shirafkan and Rae.Rad,*On the C- projective vector fields on Randers spaces* ResearchGate, preprint 27 November 2018.
18. H. Najafi, A. Tayebi,*A new quantity in Finsler geometry*C H Acad. sci. paris, ser I, volume 349, Issues 1-2 (2011) 81-83.
19. A. Tayebi, E. Peyghan and H. Sadeghi,*On a class of locally dually flat Finsler metrics with isotropic s- curvature* $IJST$ (2012) $A_3$ (Special issue- Mathematics): 377-382 Tranian Journal of Science and Technology.
20. N. Natesh, S.K. Narasimhamurthy and Roopa M.K., Conformal vector fields on Finsler space with special $(\alpha, \beta)$ - metric, Journal of Advances in Mathematics and computerscience, 31(4): 1-8, ISSN : 2456-9968, 2019.
21. Z. Shen,*On Some Non-Riemannian Quantities in Finsler Geometry* , Canad. Math. Bull. 56(1) (2013), 184-193.
22. Shen Z., Differential geometry of spray and Finsler spaces, Kluwer Journal of Advances in Mathematics and computerscience, 31(4): 1-8, ISSN : 2456-9968, 2019.
23. G. Yang, On a class of Finsler metrics of scalar flag curvature, preprint.