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Einstein-Kaehlerian Recurrent Space of Second Order

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Abstract

Walker (1950) and Roter (1964) studied and defined Ruse's spaces of recurrent curvature and second order recurrent spaces respectively.

In the present paper, we have studied and defined Einstein-Kaehlerian recurrent space of second order and several theorems have been established therein.

1. Introduction

An $n(= 2m)$ dimensional Kaehlerian space K^n is an even dimensional Riemannian space, with a mixed tensor field F_i^h and with Riemannian metric g_{ij} satisfying the following conditions

$$F_i^h F_j^i = -\delta_j^h, \tag{1.1}$$

$$F_{ij} = -F_{ji}, \ (F_{ij} = F_i^a g_{aj}) \tag{1.2}$$

and

$$F_{i,j}^h = 0, \tag{1.3}$$

where the $(,)$ followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, which we denote by R_{ijk}^h is given by

$$R_{ijk}^h = \partial_i \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} h \\ ii \end{matrix} \right\} \left\{ \begin{matrix} l \\ jk \end{matrix} \right\} - \left\{ \begin{matrix} h \\ jl \end{matrix} \right\} \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} \tag{1.4}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denote real local coordinates.

The Ricci-tensor and the scalar curvature are respectively given by

$$R_{ij} = R_{aij}^a \qquad \text{and} \qquad R = R_{ij} g^{ij}.$$

It is well known that these tensors satisfies the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j}, \tag{1.5}$$

$$R_{,i} = 2R^a_{i,a} \tag{1.6}$$

$$F^a_i R_{aj} = - R_{ia} F^a_j, \tag{1.7}$$

and

$$F^a_i R^j_a = R^a_i F^j_a \tag{1.8}$$

Let R_{hijk} be the components of the Riemannian curvature tensor.

We define a bi-recurrent space as a non-flat Riemannian V_n, the Riemannian Curvature tensor of which satisfies a relation of the form

$$R_{hijk,ab} = \lambda_{ab} R_{hijk} \tag{1.9}$$

where λ_{ab} is a non-zero tensor of the second order called the tensor of recurrence or recurrence tensor.

A Kaehlerian space K^n is said to be Kaehlerian recurrent space of second order if the curvature tensor field satisfy the condition

$$R_{hijk,ab} - \lambda_{ab} R_{hijk} = 0 \tag{1.10}$$

for some non-zero recurrence tensor λ_{ab}.

THe space is said to be Kaehlerian Ricci recurrent space of second order, if it satisfies the condition

$$R_{ij,ab} - \lambda_{ab} R_{ij} = 0 \tag{1.11}$$

Multiplying the above equation by g^{ij}, we get

$$R_{,ab} - \lambda_{ab} R = 0 \tag{1.12}$$

An immediate consequence of (1.9) and Bianchi identity

$$R_{hijk,a} + R_{hika,j} + R_{hiaj,k} = 0$$

gives for a bi-recurrent space

$$\lambda_{ab} R_{hijk} + \lambda_{jb} R_{hika} + \lambda_{kb} R_{hiaj} = 0 \tag{1.13}$$

In the case

$$R_{hijk,ab} = 0$$

(1.9) and (1.13) are satisfied for $\lambda_{ij} = 0$ and the space may or may not satisfy (1.13) for some non-zero tensor λ_{ij}

Let us suppose that a Kaehlerian space is an Einstein one, then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n}\, g_{ij}, \tag{1.14}$$

at every point of the space.

Theorem 1. If a recurrent space of second order (or bi-recurrent space) be Einstein, then the Ricci-curvature tensor vanishes.

Proof. Considering (1.13), transvecting by $g^{hk}g^{ij}$, we get

$$\lambda_{ab}R - \lambda_{jb}g^{ij}R_{ia} - \lambda_{kb}g^{hk}R_{ha} = 0$$

i.e.

$$\lambda_{ab}R - 2\lambda_{jb}g^{ij}R_{ia} = 0$$

Let a bi-recurrent space be Einstein one. Then making use of (1.14), in (1.15), we obtain

$$\lambda_{ab}R - 2\lambda_{jb}g^{ij}\frac{R}{n}\, g_{ia} = 0$$

whence

$$(n-2)\lambda_{ab}R = 0.$$

Since $\lambda_{ab} \neq 0$ and $n > 2$, $R = 0$ which is equivalent in an Einstein space to saying that $R_{ij} = 0$. This completes the proof.

Theorem 2. In an Einstein recurrent space of second order, the scalar $g^{rs}\lambda_{rs}$ vanishes.

Proof. Transvecting (1.13) by g^{hk} and with the aid of $R_{ij} = 0$, we get

$$\lambda_{kb}R^{k}_{iaj} = 0 \tag{1.16}$$

Transvecting (1.13) again by g^{ab} yields

$$\phi R_{hijk} - \lambda_{jb}g^{ab}R_{akhi} + \lambda_{kb}g^{ab}R_{ajhi} = 0 \tag{1.17}$$

where we have put the scalar $g^{ab}\lambda_{ab} = \phi$. Simplifying (1.17), we get

$$\phi R_{hijk} = \lambda_{jb}R^{b}_{khi} - \lambda_{kb}R^{b}_{jhi}.$$

This, by virtue of (1.16), gives

$$\phi R_{hijk} = 0.$$

Hence, either $\phi = 0$ or $R_{hijk} = 0$. But $R_{hijk} \neq 0$, because the case of flatness contradicts the definition of a recurrent space of second order (or, bi-recurrent space).

Therefore $\phi = 0$, i.e., $g^{ab}\lambda_{ab} = 0$ or, $g^{rs}\lambda_{rs} = 0$.

Which completes the proof of the theorem.

2. Condition for recurrent space of second order to be recurrent

We know the definition of a recurrent space. Evidently, a recurrent space is bi-recurrent or recurrent space of second order, but the converse is not true. It will however be shown in the form of a theorem that under certain conditions a recurrent space of second order (or, bi-recurrent space) becomes recurrent.

Theorem 3. A recurrent space of second order (or, bi-recurrent space) with $\lambda^{rs}\lambda_{rs} = 0$, $g^{rs}\lambda_{rs} \neq 0$ is recurrent when and only when the space is Ricci-recurrent.

Proof. If a recurrent space of second order is recurrent, then the space is Ricci-recurrent. Conversely, if $\lambda^{rs}\lambda_{rs} = 0$ and $g^{rs}\lambda_{rs} \neq 0$, then as shown by Roter [2], the curvature tensor of a recurrent space of second order (bi-recurrent space) has the following form

$$R_{hijk} = \frac{2}{R}(R_{hk}R_{ij} - R_{hj}R_{ik}), \tag{2.1}$$

we then consider those recurrent spaces of second order which are Ricci-recurrent having β_l as vector of recurrence.

Equation (2.1) thus yields

$$R_{hijk,a} = \frac{4}{R}\beta_l(R_{hk}R_{ij} - R_{hj}R_{ik}) - \frac{2}{R}\beta_l(R_{hk}R_{ij} - R_{hj}R_{ik} = \frac{2}{R}\beta_l(R_{hk}R_{ij} - R_{hj}R_{ik})$$
$$= \beta_l R_{hijk}.$$

Therefore, the space is recurrent.

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