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Study on Kaehlerian Recurrent and Symmetric Spaces of Second Order

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Abstract

Tachibana (1967), Singh (1971) studied and defined the Bochner curvature tensor and Kaehlerian spaces with recurrent Bochner curvature tensor. Further , Negi and Rawat (1994), (1997) studied some bi-recurrence and bi-symmetric properties in a Kaehlerian space and Kaehlerian spaces with recurrent and symmetric Bochner curvature tensor.

In the present paper, we have studied Kaehlerian recurrent and symmetric spaces of second order by taking different curvature tensor and relations between them. Also several theorems have been established therein.

1. Introduction

Let X_{2n} be a $2n-$dimensional almost-complex space and its almost-complex structure, then by definition, we have

$$F_j^s F_s^i = \delta_j^i. \tag{1.1}$$

An almost-complex space with a positive definite Riemannian metric g_{ji} satisfying

$$g_{rs} F_j^r F_i^s = g_{ji} \tag{1.2}$$

is called an almost-Hermitian space. From (1.2) it follows that $F_{ji} = g_{ri} F_j^r$ is skew-symmetric.

If an almost-Hermitian space satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \tag{1.3}$$

where ∇_j denotes the operator of covariant derivative with respect to the symmetric Riemannian connection, then it is called an almost-Kaehlerian space and if it satisfies

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0 \tag{1.4}$$

Then it is called a $K-$space. In an almost-Hermitian space, if

$$\nabla_j F_{ih} = 0. \tag{1.5}$$

Then it is called a Kaehlerian space or briefly a K_n space.

The Riemannian curvature tensor which are denoted by R^h_{ijk} is given by (Weatherburn 1938)

$$R^h_{ijk} = \partial_i \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} h \\ ip \end{matrix} \right\} \left\{ \begin{matrix} p \\ jk \end{matrix} \right\} - \left\{ \begin{matrix} h \\ jp \end{matrix} \right\} \left\{ \begin{matrix} p \\ ik \end{matrix} \right\} \tag{1.6}$$

The Ricci-tensor and scalar curvature are respectively given by

$$R_{ij} = R^a_{aij} \qquad \text{and} \qquad R = R_{ij} g^{ij}.$$

If we define a tensor S_{ij} by

$$S_{ij} = F^a_i R_{aj}, \tag{1.7}$$

Then, we have

$$S_{ij} = -\, S_{ji}, \tag{1.8}$$

and

$$F^a_i S_{aj} = -\, S_{ia}\, F^a_j. \tag{1.9}$$

The holomorphically projective curvature tensor and the H-Concircular curvature tensor are respectively given by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2F^h_k S_{ij}) \tag{1.10}$$

and

$$C^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{1.11}$$

The equation (1.10), in view of (1.11) may be expressed as

$$P^h_{ijk} = C^h_{ijk} + \frac{1}{n(n+2)}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k) -$$
$$- \frac{R}{(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{1.12}$$

If we put

$$L_{ij} = R_{ij} - \frac{R}{n} g_{ij} \tag{1.13}$$

and

$$M_{ij} = F^a_i S_{aj} = S_{ij} - \frac{R}{n} F_{ij}, \tag{1.14}$$

Then (1.12) reduces to the form

$$P^h_{ijk} = C^h_{ijk} + \frac{R}{n(n+2)}(L_{ik}\delta^h_j - L_{jk}\delta^h_i + M_{ik}F^h_j - M_{jk}F^h_i + 2M_{ij}F^h_k). \quad (1.15)$$

Now, we have the following :

2. Kaehlerian Recurrent Space of Second Order

Definition (2.1) : A Kaehler space K_n satisfying the relation

$$\nabla_b\nabla_a R^h_{ijk} = \lambda_{ab} R^h_{ijk}, \quad (2.1)$$

For some non- zero tensor λ_{ab}, will be called a Kaehlerian recurrent space of second order and is called Ricci-recurrent (or, semi-recurrent) space of second order, if it satisfies

$$\nabla_b\nabla_a R_{ij} = \lambda_{ab} R_{ij}, \quad (2.2)$$

Multiplying the above equation by g^{ij}, we have

$$\nabla_b\nabla_a R = \lambda_{ab} R, \quad (2.3)$$

Remark (2.1) : From (2.1) and (2.2), it follows that every Kaehlerian recurrent space of second order is Ricci-recurrent space of second order but the converse is not necessarily true.

Definition (2.2) :A Kaehler space K_n satisfying the condition

$$\nabla_b\nabla_a P^h_{ijk} = \lambda_{ab} P^h_{ijk}, \quad (2.4)$$

For some non-zero tensor λ_{ab}, will be called a Kaehlerian $H-$Projective recurrent space of second order or, briefly a $K_n - P$ space.

Definition (2.3) : A Kaehler space K_n satisfying the relation

$$\nabla_b\nabla_a C^h_{ijk} = \lambda_{ab} C^h_{ijk}, \quad (2.5)$$

For some non-zero tensor λ_{ab}, will be called a Kaehlerian $H-$Concircular recurrent space of second order or, briefly a $K_n - C$ space.

Theorem (2.1) : Every Kaehlerian recurrent space of second order is $K_n - C$ space.

Proof : Differentiating (1.11) covariantly with respect to x^a, again differentiate the result thus obtained covariantly with respect to x^b, we have

$$\nabla_b\nabla_a C^h_{ijk} = \nabla_b\nabla_a R^h_{ijk} + \frac{\nabla_b\nabla_a R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{2.6}$$

Multiplying (1.11) by λ_{ab}, then subtracting from (2.6), we obtain

$$\nabla_b\nabla_a C^h_{ijk} - \lambda_{ab}C^h_{ijk} = \nabla_b\nabla_a R^h_{ijk} - \lambda_{ab}R^h_{ijk} + \frac{(\nabla_b\nabla_a R - \lambda_{ab}R)}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i$$
$$+F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{2.7}$$

Now, let the space be Kaehlerian recurrent space of second order, then equation (2.7) with the help of equations (2.1) and (2.3) becomes

$$\nabla_b\nabla_a C^h_{ijk} - \lambda_{ab}C^h_{ijk} = 0,$$

Or,

$$\nabla_b\nabla_a C^h_{ijk} = \lambda_{ab}C^h_{ijk},$$

Which shows that the space is $K_n - C$ space.

Similarly, in view of equations (1.10),(2.1),(2.2) and (1.7), we have the following :

Theorem (2.2) : Every Kaehlerian recurrent space of second order is $K_n - P$ space.

Theorem (2.3) : The necessary and sufficient condition for a $K_n - C$ space to be a $K_n - P$ space is that

$$(\nabla_b\nabla_a L_{ik} - \lambda_{ab}L_{ik})\delta^h_j - (\nabla_b\nabla_a L_{jk} - \lambda_{ab}L_{jk})\delta^h_i + (\nabla_b\nabla_a M_{ik} - \lambda_{ab}M_{ik})F^h_j$$
$$-(\nabla_b\nabla_a M_{jk} - \lambda_{ab}M_{jk})F^h_i + 2(\nabla_b\nabla_a M_{ij} - \lambda_{ab}M_{ij})F^h_k = 0. \tag{2.8}$$

Proof : Suppose $K_n - C$ space is a $K_n - P$ space.

Differentiating (1.15) covariantly w.r.t. x^a, again differentiate the result thus obtained covariantly w.r.t. x^b, we have

$$\nabla_b\nabla_a P^h_{ijk} = \nabla_b\nabla_a C^h_{ijk} + \frac{1}{(n+2)}(\nabla_b\nabla_a L_{ik}\delta^h_j - \nabla_b\nabla_a L_{jk}\delta^h_i + \nabla_b\nabla_a M_{ik}F^h_j$$
$$- \nabla_b\nabla_a M_{jk}F^h_i + 2\nabla_b\nabla_a M_{ij}F^h_k) \tag{2.9}$$

Transvecting (1.15) by λ_{ab} and subtracting from the above equation (2.9), we have

$$\nabla_b\nabla_a P^h_{ijk} - \lambda_{ab}P^h_{ijk} = \nabla_b\nabla_a C^h_{ijk} - \lambda_{ab}C^h_{ijk} + \frac{1}{(n+2)}[(\nabla_b\nabla_a L_{ik} - \lambda_{ab}L_{ik})\delta^h_j$$
$$-(\nabla_b\nabla_a L_{jk} - \lambda_{ab}L_{jk})\delta^h_i + (\nabla_b\nabla_a M_{ik} - \lambda_{ab}M_{ik})F^h_j$$
$$-(\nabla_b\nabla_a M_{jk} - \lambda_{ab}M_{jk})F^h_i + 2(\nabla_b\nabla_a M_{ij} - \lambda_{ab}M_{ij})F^h_k] \qquad (2.10)$$

Since a $K_n - C$ space is a $K_n - P$ space, then equation (2.10), in view of (2.4) and (2.5) reduces to (2.8).

Conversely, if $K_n - C$ space satisfies the condition (2.8), then (2.10) in view of (2.5) reduces to

$$\nabla_b\nabla_a P^h_{ijk} - \lambda_{ab}P^h_{ijk} = 0,$$

which shows that the space is $K_n - P$ space.

This completes the proof.

Theorem (2.4) : If in a Kaehler space satisfying any two of the following properties :

(i) the space is Kaehlerian Ricci- recurrent space of second order,

(ii) the space is Kaehlerian Projective recurrent space of second order,

(iii) the space is H-Concircular recurrent space of second order , then it must also satisfies third.

Proof : Differentiating (1.12) covariantly w.r.t. x^a, again differentiate the result thus obtained covariantly w.r.t. x^b, we have

$$\nabla_b\nabla_a P^h_{ijk} = \nabla_b\nabla_a C^h_{ijk} + \frac{1}{(n+2)}(\nabla_b\nabla_a R_{ik}\delta^h_j - \nabla_b\nabla_a R_{jk}\delta^h_i + \nabla_b\nabla_a S_{ik}F^h_j$$
$$-\nabla_b\nabla_a S_{jk}F^h_i + 2\nabla_b\nabla_a S_{ij}F^h_k - \frac{\nabla_b\nabla_a R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i$$
$$+F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k), \qquad (2.11)$$

Multiplying (1.12) by λ_{ab} and subtracting the result from (2.11), we have

Kaehlerian Ricci-recurrent space of second order, Kaehlerian Projective recurrent space of second order and Kaehlerian $H-$Concircular recurrent space of second order are respectively characterized by the equations (2.2), (2.4) and (2.5).

The statement of the above theorem follows in view of equations (2.2), (2.4), (2.5) and (2.12).

3. Kaehlerian Symmetric Space of Second Order

Definition (3.1) : A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a R^h_{ijk} = 0, \qquad \text{or equivalently} \qquad \nabla_b \nabla_a R_{ijkl} = 0, \tag{3.1}$$

Will be called Kaehlerian symmetric space of second order and is called Kaehlerian Ricci-symmetric or (semi-symmetric) space of second order, if it satisfies

$$\nabla_b \nabla_a R_{ij} = 0, \tag{3.2}$$

Multiplying the above equation by g^{ij}, we have

$$\nabla_b \nabla_a R = 0, \tag{3.3}$$

Remark (3.1) : From (3.1) and (3.2), it follows that every Kaehlerian symmetric space of second order is Kaehlerian Ricci-symmetric space of second order , but the converse is not necessarily true.

Definition (3.2) : A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a P^h_{ijk} = 0, \qquad \text{or equivalently} \qquad \nabla_b \nabla_a P_{ijkl} = 0, \tag{3.4}$$

will be called a Kaehlerian $H-$Projective symmetric space of second order or, briefly a $^*K_n - P$ space.

Definition (3.3) : A Kaehler space K_n satisfying the condition

$$\nabla_b \nabla_a C^h_{ijk} = 0, \qquad \text{or equivalently} \qquad \nabla_b \nabla_a C_{ijkl} = 0, \tag{3.5}$$

will be called a Kaehlerian $H-$Concircular symmetric space of second order or, briefly $^*K_n - C$ space.

Theorem (3.1) : The necessary and sufficient condition for a $^*K_n - C$ space to be a $^*K_n - P$ space is that

$$\nabla_b \nabla_a L_{ik}\delta^h_j - \nabla_b \nabla_a L_{jk}\delta^h_i + \nabla_b \nabla_a M_{ik}F^h_j - \nabla_b \nabla_a M_{jk}F^h_i + 2\nabla_b \nabla_a M_{ij}F^h_k = 0. \tag{3.6}$$

Proof : From equations (1.5), (2.9) and (3.5), we have

$$\nabla_b \nabla_a P^h_{ijk} = \frac{1}{(n+2)}(\nabla_b \nabla_a L_{ik}\delta^h_j - \nabla_b \nabla_a L_{jk}\delta^h_i + \nabla_b \nabla_a M_{ik}F^h_j - \nabla_b \nabla_a M_{jk}F^h_i$$

$$+2\nabla_b \nabla_a M_{ij}F^h_k = 0. \tag{3.7}$$

Since ${}^*K_n - C$ space is a ${}^*K_n - P$ space, hence equation (3.7) reduces to the form

$$\nabla_b\nabla_a L_{ik}\delta_j^h - \nabla_b\nabla_a L_{jk}\delta_i^h + \nabla_b\nabla_a M_{ik}F_j^h - \nabla_b\nabla_a M_{jk}F_i^h + 2\nabla_b\nabla_a M_{ij}F_k^h = 0. \tag{3.8}$$

Conversely, if a ${}^*K_n - C$ space satisfies equation (3.6), then (3.7) reduces to the form

$$\nabla_b\nabla_a P_{ijk}^h = 0$$

which shows that the space is ${}^*K_n - P$ space.

Theorem (3.2) :A necessary and sufficient condition for a $H-$Concircular symmetric space of second order to be Kaehlerian-Ricci symmetric space of second order is that

$$\nabla_b\nabla_a R_{ijk}^h + \lambda_{ab}[C_{ijk}^h - R_{ijk}^h - \frac{R}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)] = 0 \tag{3.9}$$

Proof : If the space is a $H-$Concircular symmetric space of second order, then equation (2.7) in view of (3.5) reduces to the form

$$\nabla_b\nabla_a R_{ijk}^h - \lambda_{ab}R_{ijk}^h + \lambda_{ab}C_{ijk}^h + \frac{(\nabla_b\nabla_a R - \lambda_{ab}R)}{n(n+2)}[g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)] = 0 \tag{3.10}$$

Now, if the space is Kaehlerian- Ricci symmetric space of second order then (3.2) is satisfied and equation (3.10), in view of (3.2) reduces to (3.9).

Conversely, if $H-$Concircular symmetric space of second order satisfies the condition (3.9), then equation (2.7) gives

$$\frac{\nabla_b\nabla_a R}{n(n+2)}[g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)] = 0$$

which gives $\nabla_b\nabla_a R = 0$

$\nabla_b\nabla_a g^{ij}R_{ij} = 0$ since $R = R_{ij}g^{ij}$

Or $\nabla_b\nabla_a R_{ij} = 0$ since $g^{ij} \neq 0$

which shows that the space is Kaehlerian Ricci-symmetric space of second order.

References

1. Tachibana, S. : On the Bochner curvature tensor, Nat. Sci. Rep. Ochanomizu University, 18(1)(1967), 15-16 .
2. Lal, K. B. and Singh, S. S. : On Kaehlerian spaces withrecurrent Bochner curvature , Acc. Naz. Lincei, Vol. LI, No's. 3-4 (1971), 143-150.
3. Yano, K. : Differential Geometry on Complex and almost complex spaces, Pergamon Press (1965).
4. Negi, D. S. and Rawat, K. S. : Some investigations in Kaehlerian space, Acta Ciencia Indica,Vol. XXIII M, No. 4 (1997), 255-258.
5. Negi, D. S. and Rawat, K. S. : Theorems on Kaehlerian spaces with recurrent and symmetric Bochner curvature tensor , Acta Ciencia Ind. , Vol. XXIII M , No. 3 (1997), 239-242.
6. Rawat, K. S. and Silswal, G. P. : The study of Einstien-Kaehlerian Bi-recurrent and bi-symmetric spaces, Acta Ciencia Indica, Vol. XXXII M, No.1 (2006), 417-422.
7. Rawat, K. S. and Girish Dobhal : On the bi- recurrent Bochner curvature tensor, Jour. Of the tensor society, 1 (2007), 33-40.
8. Rawat, K.S. and Kunwar Singh : Some bi-recurrence properties in a Kaehlerian space, Jour. PAS, 14 (2008) (Mathematical Science), 199-205.
9. Rawat, K. S. and Girish Dobhal : Study of the decomposition of recurrent curvature tensor fields in a Kaehlerian recurrent space, Jour. Pure and Applied Mathematica Sciences, LXVIII, No. 1-2, Sept. (2008).
10. Rawat , K.S. and Girish Dobhal : Some investigations in Kaehlerian manifold, Impact Jour. Sci. Tech. FIJI ISLANDS, 2(3)(2008),129-134.
11. Bagwati, C. S. : On Kaehlerian n-recurrent spaces, J. Karnatak University Science, 21 (1976), 110-122.
12. Ravi, B. and Bagwati, C. S. : On n-symmetric in recurrent Kaehlerian manifold, Ganita, 38 (1987) no. 1-2, 49-60.