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Pseudoparallel Invariant Submanifolds of Lorentzian α-sasakian Manifolds

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In this paper, the study of an invariant submanifold of Lorentzian α-sasakian manifold is carried out and it is shown that, it is also Lorentzian α-sasakian. Further we prove that, if the second fundamental form of an invariant submanifold of Lorentzian α-sasakian manifold is recurrent, 2-recurrent and generalized 2-recurrent then the submanifold is totally geodesic and also an invariant submanifold of Lorentzian α-sasakian manifold with parallel third fundamental form is again totally geodesic. It is proved that pseudoparallel and 2-pseudoparallel invariant submanifolds of Lorentzian α-sasakian manifolds is also totally geodesic. Further, we also show that this property of totally geodesic holds true if $\widetilde{C} \cdot \sigma = L_1 Q(g, \sigma)$ and $\widetilde{C} \cdot \widetilde{\nabla}\sigma = L_1 Q(g, \widetilde{\nabla}\sigma)$.

Key Words : Invariant submanifold, Lorentzian α-sasakian manifold, Pseudoparallel, 2-pseudoparallel and totally geodesic.
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1. Introduction

The study of invariant submanifolds of different contact manifolds is carried out by M. Kon and B.Y. Chen ([7], [14]) from 1970 onwards. Further, it is carried out by H. Endo, D. Chinea, D. Chinea and P.S. Prestelo, B. Ravi and C.S. Bagewadi, K. Yano and M. Kon ([8], [9], [12], [18], [21]) during eighties and nineties. From 2000 onwards there are many papers by C.E. Hretcanu and M. Crasmareasu, recently Aysel Turgut Vanli and Ramazan Sari, S. Sular and C. Ozgur, C. Murathan etal, C. Ozgur and C. Murathan ([5], [13], [15], [16],

[20]) also studied invariant submanifolds of LP-sasakian manifolds. These authors Ahmet Yildiz and Cengizhan Murathan, A.C. Asperti, G.A. Lobos and F. Mercuri, C. Ozgur, S. Sular and C. Murathan ([2], [4], [17]) have studied pseudoparallel invariant submanifolds of contact manifolds. In this paper, we extend the study to invariant submanifolds of Lorentzian α-sasakian. Further, the study of invariant submanifolds of Lorentzian α-sasakian satisfying the conditions $\widetilde{C} \cdot \sigma = L_1 Q(g, \sigma)$ and $\widetilde{C} \cdot \widetilde{\nabla}\sigma = L_1 Q(g, \widetilde{\nabla}\sigma)$, where $\widetilde{C}$ is concircular curvature tensor.

The paper is organized as follows: In section 2, we have given a necessary information about submanifolds. In section 3, some definitions and notions about Lorentzian α-sasakian manifolds and their invariant submanifolds are given. In section 4, deals with an invariant submanifold of Lorentzian α-sasakian manifold whose second fundamental form σ is recurrent, 2-recurrent and generalized 2-recurrent and shown that these type of submanifolds are totally geodesic. Also proved that an invariant submanifold of a Lorentzian α-sasakian manifold with parallel third fundamental form is again totally geodesic. In section 5, pseudoparallel and 2-pseudoparallel invariant submanifolds of Lorentzian α-sasakian manifolds and shown that these type of submanifolds are totally geodesic. In the last section, we have proved that if $\widetilde{C}(X,Y) \cdot \sigma = L_1 Q(g, \sigma)$ and $\widetilde{C}(X,Y) \cdot \widetilde{\nabla}\sigma = L_1 Q(g, \widetilde{\nabla}\sigma)$ then M is totally geodesic.

2. Basic Concepts

The covariant differential of the p^{th} order, $p \geq 1$ of a $(0,k)$-tensor field T, $k \geq 1$ denoted by $\nabla^p T$, defined on a Riemannian manifold (M, g) with the Levi-Civita connection ∇. The tensor T is said to be *recurrent* [19], if the following condition holds on M:

$$(\nabla T)(X_1, ..., X_k; X)T(Y_1, ..., Y_k) = (\nabla T)(Y_1, ..., Y_k; X)T(X_1, ..., X_k) \tag{2.1}$$

respectively.

$$(\nabla^2 T)(X_1, ..., X_k; X, Y)T(Y_1, ..., Y_k) = (\nabla^2 T)(Y_1, ..., Y_k; X, Y)T(X_1, ..., X_k),$$

where $X, Y, X_1, Y_1, ..., X_k, Y_k \in TM$. From (2.1) it follows that at a point $x \in M$, if the tensor T is non-zero then there exists a unique 1-form ϕ respectively, a $(0,2)$-tensor ψ, defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|) \tag{2.2}$$

respectively.

$$\nabla^2 T = T \otimes \psi, \tag{2.3}$$

holds on U, where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T,T)$. The tensor T is said to be *generalized* 2*-recurrent* if

$$\begin{aligned}&((\nabla^2 T)(X_1,...,X_k;X,Y) - (\nabla T\otimes\phi)(X_1,...,X_k;X,Y))T(Y_1,...,Y_k)\\ &= ((\nabla^2 T)(Y_1,...,Y_k;X,Y) - (\nabla T\otimes\phi)(Y_1,...,Y_k;X,Y))T(X_1,...,X_k),\end{aligned}$$

holds on M, where ϕ is a 1-form on M. From this it follows that at a point $x\in M$ if the tensor T is non-zero then there exists a unique $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla^2 T = \nabla T\otimes\phi + T\otimes\psi, \tag{2.4}$$

holds on U.

Let $f:(M,g)\to(\widetilde{M},\widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M,g) into $(n+d)$-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$, $n\geq 2$, $d\geq 1$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.5}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for any tangent vector fields X,Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y),N) = g(A_N X,Y),$$

for tangent vector fields X,Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\widetilde{\nabla}}_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z) \tag{2.7}$$

$$\begin{aligned}(\widetilde{\widetilde{\nabla}}^2\sigma)(Z,W,X,Y) =& (\widetilde{\widetilde{\nabla}}_X\widetilde{\widetilde{\nabla}}_Y\sigma)(Z,W),\\ =& \nabla^{\perp}_X((\widetilde{\widetilde{\nabla}}_Y\sigma)(Z,W)) - (\widetilde{\widetilde{\nabla}}_Y\sigma)(\nabla_X Z,W)\\ &- (\widetilde{\widetilde{\nabla}}_X\sigma)(Z,\nabla_Y W) - (\widetilde{\widetilde{\nabla}}_{\nabla_X Y}\sigma)(Z,W)\end{aligned} \tag{2.8}$$

respectively, where $\widetilde{\widetilde{\nabla}}$ is called the *vander Waerden-Bortolotti connection* of M [7]. If $\widetilde{\widetilde{\nabla}}\sigma = 0$, then M is said to have *parallel second fundamental form* [7] and if $\widetilde{\nabla}\sigma = 0$, then f is called parallel [11]. We next define endomorphisms $R(X, Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y \tag{2.9}$$

respectively. where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0, 2)$-tensor.

Now, for a $(0, k)$-tensor field T, $k \geq 1$ and a $(0, 2)$ -tensor field B on (M, g), we define the tensor $Q(B, T)$ by

$$\begin{aligned} Q(B,T)(X_1, ..., X_k; X, Y) = &-(T(X \wedge_B Y)X_1, ..., X_k) \\ &- \cdots - T(X_1, ..., X_{k-1}, (X \wedge_B Y)X_k). \end{aligned} \tag{2.10}$$

Putting into the above formula $T = \sigma, \widetilde{\nabla}\sigma$ and $B = g$, we obtain the tensor $Q(g, \sigma)$ and $Q(g, \widetilde{\nabla}\sigma)$.

Definition 2.1. An immersion is said to be semiparallel if

$$\widetilde{R}(X,Y) \cdot \sigma = (\widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]})\sigma = 0, \tag{2.11}$$

holds for all vector fields X, Y tangent to M [10], where $\widetilde{R}$ denotes the curvature tensor of the connection $\widetilde{\nabla}$. An immersion is said to be 2-*semiparallel* if $\widetilde{R}(X, Y).\widetilde{\nabla}\sigma = 0$ holds for all vector fields X, Y tangent to M. Further an immersion is said to be pseudoparallel and 2-pseudoparallel [2], if

$$\widetilde{R} \cdot \sigma = L_1 Q(g, \sigma), \tag{2.12}$$

$$\widetilde{R} \cdot \widetilde{\nabla}\sigma = L_1 Q(g, \widetilde{\nabla}\sigma). \tag{2.13}$$

If $Q(g, \sigma) = 0$, $Q(g, \widetilde{\nabla}\sigma) = 0$ and it reduces to (2.11), ie., the definition of semiparallel.

From the Gauss and Weingarten formulas, we obtain

$$(\widetilde{R}(X,Y)Z)^T = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X. \tag{2.14}$$

By (2.11), we have

$$(\widetilde{R}(X,Y) \cdot \sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U, R(X,Y)V), \tag{2.15}$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}. \tag{2.16}$$

Similarly, we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W)=R^{\perp}(X,Y)(\widetilde{\nabla}\sigma)(U,V,W)-(\widetilde{\nabla}\sigma)(R(X,Y)U,V,W)$$
$$-(\widetilde{\nabla}\sigma)(U,R(X,Y)V,W)-(\widetilde{\nabla}\sigma)(U,V,R(X,Y)W), \tag{2.17}$$

for all vector fields X,Y,U,V,W tangent to M, where $(\widetilde{\nabla}\sigma)(U,V,W)=(\widetilde{\nabla}_U\sigma)(V,W)$ [3].

3. Lorentzian α-sasakian manifolds and their submanifolds

A differentiable manifold of dimension $n=(2m+1)$ is called Lorentzian α-sasakian manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \quad \phi(\xi)=0, \quad \eta\cdot\phi=0, \tag{3.1}$$
$$\phi^2 = I+\eta\otimes\xi, \quad g(X,\xi)=\eta(X), \tag{3.2}$$
$$g(\phi X,\phi Y) = g(X,Y)+\eta(X)\eta(Y), \tag{3.3}$$

for all $X,Y\in TM$.

Also a Lorentzian α-sasakian manifold M satisfies [1]

$$\nabla_X\xi = \alpha\phi X, \tag{3.4}$$
$$(\nabla_X\phi)(Y) = \alpha\left\{g(X,Y)\xi+\eta(Y)X\right\}, \tag{3.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Further, on a Lorentzian α-sasakian manifold M the following relations hold [1]:

$$\eta(R(X,Y)Z) = \alpha^2\left\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right\}, \tag{3.6}$$
$$R(\xi,X)Y = \alpha^2\left\{g(X,Y)\xi-\eta(Y)X\right\}, \tag{3.7}$$
$$R(\xi,X)\xi = \alpha^2\left\{\eta(X)\xi+X\right\}, \tag{3.8}$$
$$R(X,Y)\xi = \alpha^2\left\{\eta(Y)X-\eta(X)Y\right\}, \tag{3.9}$$
$$S(X,\xi) = (n-1)\alpha^2\eta(X), \tag{3.10}$$
$$Q\xi = (n-1)\alpha^2\xi, \tag{3.11}$$
$$S(\xi,\xi) = -(n-1)\alpha^2, \tag{3.12}$$

where Q is the Ricci operator, i.e., $g(QX,Y)=S(X,Y)$.

A submanifold M of a Lorentzian α-sasakian manifold $\widetilde{M}$ is called an invariant submanifold of $\widetilde{M}$, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. In an invariant submanifold of a Lorentzian α-sasakian manifold

$$\sigma(X, \xi) = 0, \tag{3.13}$$

for any vector X tangent to M.

Proposition 3.1. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then the following equalities holds on M.

$$\begin{aligned} \nabla_X \xi &= \alpha\phi X, && (3.14)\\ R(X,Y)\xi &= \alpha^2 \{\eta(Y)X - \eta(X)Y\}, && (3.15)\\ Q\xi &= (n-1)\alpha^2\xi, && (3.16)\\ S(X,\xi) &= (n-1)\alpha^2\eta(X), && (3.17)\\ (\nabla_X\phi)(Y) &= \alpha\{g(X,Y)\xi + \eta(Y)X\}, && (3.18)\\ \sigma(X,\phi Y) &= \phi\sigma(X,Y), && (3.19) \end{aligned}$$

where Q denotes the Ricci operator of M defined by $S(X,Y) = g(QX,Y)$.

Proof. Since M is an invariant submanifold of a Lorentzian α-sasakian $\widetilde{M}$. Then ξ is tangent to M

$$\nabla_X \xi = \alpha\phi X. \tag{3.20}$$

Using (2.5), we get

$$\alpha\phi X = \widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi),$$

which gives us

$$\nabla_X \xi = \alpha\phi X, \quad \sigma(X,\xi) = 0,$$

so we get (3.14). Since $\widetilde{M}$ is Lorentzian α-sasakian manifold, we get from (3.5),

$$(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi + \eta(Y)X\}. \tag{3.21}$$

Then, in view of Gauss formula, we have

$$(\widetilde{\nabla}_X\phi)Y = \nabla_X\phi Y + \sigma(X,\phi Y) - \phi\nabla_X Y - \phi\sigma(X,Y). \tag{3.22}$$

Comparing the tangential and normal parts of (3.18) and (3.19), we get

$$(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi + \eta(Y)X\} \quad \text{and} \quad \sigma(X,\phi Y) = \phi\sigma(X,Y).$$

So we obtain (3.18), (3.19). From (2.14), we have

$$\widetilde{R}(X,Y)Z = R^{\perp}(X,Y)\xi + A_{\sigma(X,\xi)}Y - A_{\sigma(Y,\xi)}X.$$

Then using $\sigma(X,\xi)=0$, we find

$$\widetilde{R}(X,Y)Z=R(X,Y)\xi.$$

One can get (3.15). A suitable contraction of (3.15) gives us

$$S(X,\xi)=(n-1)\alpha^2\eta(X), \quad Q\xi=(n-1)\alpha^2\xi.$$

So we can state the following theorem.

Theorem 3.2. An invariant submanifold M of a Lorentzian α-sasakian manifold $\widetilde{M}$ is a Lorentzian α-sasakian manifold.

4. Recurrent invariant submanifolds of Lorentzian α-sasakian manifolds

We consider invariant submanifold of a Lorentzian α-sasakian when σ is recurrent, 2-recurrent, generalized 2-recurrent and M has parallel third fundamental form and prove the following results.

Theorem 4.3. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then σ is recurrent if and only if M is totally geodesic.

Proof. Since σ is recurrent, from (2.2) we get

$$(\widetilde{\nabla}_X\sigma)(Y,Z)=\phi(X)\sigma(Y,Z),$$

where ϕ is a 1-form on M. Then in view of (2.7) in the above equation can be written as

$$\nabla^{\perp}_X\sigma(Y,Z)-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ)=\phi(X)\sigma(Y,Z). \tag{4.1}$$

Taking $Z=\xi$ in (4.1), we have

$$\nabla^{\perp}_X\sigma(Y,\xi)-\sigma(\nabla_XY,\xi)-\sigma(Y,\nabla_X\xi)=\phi(X)\sigma(Y,\xi). \tag{4.2}$$

Making use of relation (3.2), (3.4), (3.13) in (4.2), we get $\alpha\sigma(X,Y)=0$. Since $\alpha\neq 0$, we have $\sigma(X,Y)=0$ i.e., M is totally geodesic. The converse statement is trivial. This completes the proof of the theorem.

Theorem 4.4. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then M has parallel third fundamental form if and only if M is totally geodesic.

Proof. Suppose that M has parallel third fundamental form. Then we can write

$$(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z, W) = 0.$$

Replacing W with ξ in the above equation and using (2.8), we have

$$\nabla^{\perp}_X((\widetilde{\nabla}_Y \sigma)(Z, \xi)) - (\widetilde{\nabla}_Y \sigma)(\nabla_X Z, \xi) - (\widetilde{\nabla}_X \sigma)(Z, \nabla_Y \xi) - (\widetilde{\nabla}_{\nabla_X Y} \sigma)(Z, \xi) = 0. \tag{4.3}$$

Taking account of (2.7) in (4.3) and using (3.13) we get

$$\begin{aligned} 0 &= -\nabla^{\perp}_X \alpha\sigma(Z, \phi Y) + \alpha\sigma(\nabla_X Z, \phi Y) - \nabla^{\perp}_X \alpha\sigma(Z, \phi Y) \\ &\quad + \alpha\sigma(\nabla_X Z, \phi Y) + \sigma(Z, \nabla_X \alpha\phi Y) + \alpha\sigma(Z, \phi\nabla_X Y). \end{aligned} \tag{4.4}$$

Putting $Y = \xi$ in (4.4) and taking account of (3.2), (3.4), (3.13), we get $\alpha^2\sigma(X, Z) = 0$. Since $\alpha \neq 0$, we have $\sigma(X, Z) = 0$ i.e., M is totally geodesic. The converse statement is trivial. Hence, the proof of the theorem is completed.

Corollary 4.1. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then σ is 2-recurrent if and only if M is totally geodesic.

Proof. Since σ is 2-recurrent, from (2.3), we have

$$(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z, W) = \sigma(Z, W)\phi(X, Y). \tag{4.5}$$

Taking $W = \xi$ in (4.5) and using the proof of the Theorem 4.4, we get $\alpha^2\sigma(X, Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This completes the proof of the corollary.

Theorem 4.5. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then σ is generalized 2-recurrent if and only if M is totally geodesic.

Proof. Since σ is generalized 2-recurrent, from (2.4), then one can write

$$(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z, W) = \psi(X, Y)\sigma(Z, W) + \phi(X)(\widetilde{\nabla}_Y \sigma)(Z, W), \tag{4.6}$$

where ψ and ϕ are 2-recurrent and 1-form, respectively. Taking $W = \xi$ in (4.6) and taking account of the equation (3.13), we get

$$(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z, \xi) = \phi(X)(\widetilde{\nabla}_Y \sigma)(Z, \xi).$$

Then making use of (2.7) and (2.8) in above equation and in view of (3.13), we have

$$\begin{aligned}&-\nabla^{\perp}_{X}\alpha\sigma(Z,\phi Y)+\alpha\sigma(\nabla_X Z,\phi Y)-\nabla^{\perp}_{X}\alpha\sigma(Z,\phi Y)+\alpha\sigma(\nabla_X Z,\phi Y)\\&+\sigma(Z,\nabla_X\alpha\phi Y)+\alpha\sigma(Z,\phi\nabla_X Y)=-\alpha\phi(X)\sigma(Z,\phi Y).\end{aligned}$$

Putting $Y=\xi$ in the above equation and using (3.2), (3.4), (3.13), we get $\alpha^2\sigma(X,Z)=0$. Since $\alpha\neq 0$, we have $\sigma(X,Z)=0$ i.e., M is totally geodesic. The converse statement is trivial. Thus our theorem is proved.

5. Pseudoparallel and 2-Pseudoparallel Invariant submanifolds of Lorentzian α-sasakian manifolds

We consider invariant submanifolds of Lorentzian α-sasakian manifolds satisfying the conditions $\widetilde{R}\cdot\sigma=L_1Q(g,\sigma)$ and $\widetilde{R}\cdot\widetilde{\nabla}\sigma=L_1Q(g,\widetilde{\nabla}\sigma)$.

Theorem 5.6. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then the condition $\widetilde{R}\cdot\sigma=L_1Q(g,\sigma)$ and $\widetilde{R}\cdot\widetilde{\nabla}\sigma=L_1Q(g,\widetilde{\nabla}\sigma)$ hold on M. i.e., M is (i) pseudoparallel, if $L_1\neq\alpha^2$, (ii) 2-pseudoparallel if and only if M is totally geodesic.

Proof. (i) Since M is pseudoparallel, i.e., $\widetilde{R}\cdot\sigma=L_1Q(g,\sigma)$. Put $X=V=\xi$ and using (3.1), (3.13) in (2.10), (2.15), we get

$$R^{\perp}(\xi,Y)\sigma(U,\xi)-\sigma(R(\xi,Y)U,\xi)-\sigma(U,R(\xi,Y)\xi)=L_1\sigma(U,Y). \tag{5.1}$$

Using (3.8) and (3.13) in (5.1), we get $(L_1-\alpha^2)\sigma(U,Y)=0$. Since $L_1\neq\alpha^2$, we have $\sigma(U,Y)=0$ i.e., M is totally geodesic. The converse statement is trivial.
(ii) Since M is 2-pseudoparallel, i.e., $\widetilde{R}\cdot\widetilde{\nabla}\sigma=L_1Q(g,\widetilde{\nabla}\sigma)$. Put $X=V=\xi$ and using (3.1), (3.4), (3.13) in (2.10), (2.17), we get

$$\begin{aligned}&R^{\perp}(\xi,Y)(\widetilde{\nabla}\sigma)(U,\xi,W)-(\widetilde{\nabla}\sigma)(R(\xi,Y)U,\xi,W)-(\widetilde{\nabla}\sigma)(U,R(\xi,Y)\xi,W)\\&-(\widetilde{\nabla}\sigma)(U,\xi,R(\xi,Y)W)=-L_1[\eta(W)\Big\{\nabla^{\perp}_{\xi}\sigma(Y,U)-\sigma(\nabla_\xi Y,U)\\&-\sigma(Y,\nabla_\xi U)\Big\}+\nabla^{\perp}_{W}\sigma(Y,U)-\sigma(\nabla_W Y,U)-\sigma(Y,\nabla_W U)\\&-\eta(Y)\Big\{\nabla^{\perp}_{\xi}\sigma(W,U)-\sigma(\nabla_\xi W,U)-\sigma(W,\nabla_\xi U)\Big\}\\&-\eta(U)\Big\{\nabla^{\perp}_{\xi}\sigma(Y,W)-\sigma(\nabla_\xi Y,W)-\sigma(Y,\nabla_\xi W)\Big\}].\end{aligned} \tag{5.2}$$

In view of (2.7), (3.4) and (3.13), we have

$$\begin{aligned}&(\widetilde{\nabla}\sigma)(U,\xi,W)=(\widetilde{\nabla}_U\sigma)(\xi,W)\\&=\nabla^{\perp}_{U}\sigma(\xi,W)-\sigma\left(\nabla_U\xi,W\right)-\sigma\left(\xi,\nabla_U W\right)=-\alpha\sigma(\phi U,W).\end{aligned} \tag{5.3}$$

Also, in view of (2.7), (3.4), (3.7), (3.8) and (3.13), we have the following equalities:

$$\begin{aligned}(\widetilde{\nabla}\sigma)(R(\xi,Y)U,\xi,W) &= (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,W), \\ &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,W) - \sigma(\nabla_{R(\xi,Y)U}\xi,W) - \sigma(\xi,\nabla_{R(\xi,Y)U}W), \\ &= \alpha^3\eta(U)\sigma(\phi Y,W),\end{aligned} \tag{5.4}$$

$$\begin{aligned}(\widetilde{\nabla}\sigma)(U,R(\xi,Y)\xi,W) &= (\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W), \\ &= \nabla^{\perp}_U\sigma(R(\xi,Y)\xi,W) - \sigma(\nabla_U R(\xi,Y)\xi,W) - \sigma(R(\xi,Y)\xi,\nabla_U W), \\ &= \nabla^{\perp}_U\sigma\left(\alpha^2\left\{\eta(Y)\xi+Y\right\},W\right) - \sigma\left(\nabla_U\alpha^2\left\{\eta(Y)\xi+Y\right\},W\right) \\ &\quad -\alpha^2\sigma(Y,\nabla_U W)\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}(\widetilde{\nabla}\sigma)(U,\xi,R(\xi,Y)W) &= (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)W), \\ &= \nabla^{\perp}_U\sigma(\xi,R(\xi,Y)W) - \sigma(\nabla_U\xi,R(\xi,Y)W) - \sigma(\xi,\nabla_U R(\xi,Y)W), \\ &= -\alpha^3\left\{\sigma(\phi U,g(Y,W)\xi) - \eta(W)\sigma(\phi U,Y)\right\}.\end{aligned} \tag{5.6}$$

Substituting (5.3) – (5.6) into (5.2), we obtain

$$\begin{aligned}&-\alpha R^{\perp}(\xi,Y)\sigma(\phi U,W) - \alpha^3\eta(U)\sigma(\phi Y,W) - \nabla^{\perp}_U\sigma\left(\alpha^2\left\{\eta(Y)\xi+Y\right\},W\right) \\ &+\sigma\left(\nabla_U\alpha^2\left\{\eta(Y)\xi+Y\right\},W\right) + \alpha^2\sigma(Y,\nabla_U W) + \alpha^3\left\{\sigma(\phi U,g(Y,W)\xi)\right. \\ &\left.-\ \eta(W)\sigma(\phi U,Y)\right\} = -L_1[\eta(W)\left\{\nabla^{\perp}_\xi\sigma(Y,U) - \sigma(\nabla_\xi Y,U) - \sigma(Y,\nabla_\xi U)\right\} \\ &+\nabla^{\perp}_W\sigma(Y,U) - \sigma(\nabla_W Y,U) - \sigma(Y,\nabla_W U) \\ &-\eta(Y)\left\{\nabla^{\perp}_\xi\sigma(W,U) - \sigma(\nabla_\xi W,U) - \sigma(W,\nabla_\xi U)\right\} \\ &-\eta(U)\left\{\nabla^{\perp}_\xi\sigma(Y,W) - \sigma(\nabla_\xi Y,W) - \sigma(Y,\nabla_\xi W)\right\}].\end{aligned} \tag{5.7}$$

If $W=\xi$ and using (3.1), (3.4), (3.13) in (5.7), we get $2\alpha^3\sigma(U,Y)=0$. Since $\alpha\neq 0$, we have $\sigma(U,Y)=0$ i.e., M is totally geodesic. The converse statement is trivial. This proves the theorem.

Using Theorems 4.3, 4.4, 4.5, 5.6 and corollary 4.1, we have the following result

Corollary 5.2. Let M be an invariant submanifold of Lorentzian α-sasakian manifold $\widetilde{M}$. Then the following statements are equivalent:

(1) σ is recurrent;
(2) σ is 2-recurrent;

(3) M has parallel third fundamental form;
(4) M is pseudoparallel and $L_1 \neq \alpha^2$;
(5) M is 2-pseudoparallel;
(6) M is totally geodesic.

6. Pseudoparallelism and 2-Pseudoparallelism with respect to concircular curvature tensor

We consider invariant submanifold of a Lorentzian α-sasakian manifold satisfying the conditions $\widetilde{C}(X,Y)\cdot\sigma = L_1Q(g,\sigma)$ and $\widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma = L_1Q(g,\widetilde{\nabla}\sigma)$, where $\widetilde{C}$ is concircular curvature tensor.

Definition 6.1. An immersion $f:(M,g)\to(\widetilde{M},\widetilde{g})$ is said to be pseudoparallel with respect to concircular curvature tensor if $\widetilde{C}(X,Y)\cdot\sigma = L_1Q(g,\sigma)$ and 2-pseudoparallel $\widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma = L_1Q(g,\widetilde{\nabla}\sigma)$.

Now study of invariant submanifolds of Lorentzian α-sasakian manifolds which satisfy the following

$$\widetilde{C}(X,Y)\cdot\sigma = L_1Q(g,\sigma) \quad\text{and}\quad \widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma = L_1Q(g,\widetilde{\nabla}\sigma).$$

Theorem 6.7. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Prove that M is (i) pseudoparallel and $L_1 \neq (\frac{r}{n(n-1)}-\alpha^2)$, $\alpha^2 \neq \frac{r}{n(n-1)}$, (ii) 2-pseudoparallel with respect to **concircular** curvature tensor if and only if M is totally geodesic.

Proof. The concircular curvature tensor is given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \left[\frac{r}{n(n-1)}\right]\left[g(Y,Z)X - g(X,Z)Y\right], \tag{6.1}$$

where r is the scalar curvature.

Similar to (2.15) and (2.17) the tensors $\widetilde{C}(X,Y)\cdot\sigma$ and $\widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma$ are defined by

$$\begin{aligned}(\widetilde{C}(X,Y)\cdot\sigma)(U,V) &= R^{\perp}(X,Y)\sigma(U,V) - \sigma(\widetilde{C}(X,Y)U,V)\\ &\quad -\sigma(U,\widetilde{C}(X,Y)V),\end{aligned} \tag{6.2}$$

$$\begin{aligned}(\widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W) &= R^{\perp}(X,Y)(\widetilde{\nabla}\sigma)(U,V,W) - (\widetilde{\nabla}\sigma)(\widetilde{C}(X,Y)U,V,W)\\ &\quad -(\widetilde{\nabla}\sigma)(U,\widetilde{C}(X,Y)V,W) - (\widetilde{\nabla}\sigma)(U,V,\widetilde{C}(X,Y)W)\end{aligned} \tag{6.3}$$

respectively.

Putting $X=\xi$, $Y=X$, $Z=Y$ and $X=\xi$, $Y=X$, $Z=\xi$ in (6.1) and using (3.7), (3.8), we have

$$\widetilde{C}(\xi,X)Y = \left[\alpha^2-\frac{r}{n(n-1)}\right]\{g(X,Y)\xi-\eta(Y)X\}, \tag{6.4}$$

$$\widetilde{C}(\xi,X)\xi = \left[\alpha^2-\frac{r}{n(n-1)}\right]\{\eta(X)\xi+X\}. \tag{6.5}$$

(i) Since M satisfies the condition $\widetilde{C}(X,Y)\cdot\sigma=L_1Q(g,\sigma)$. Put $X=V=\xi$ and using (3.1), (3.13) in (2.10) and (6.2), we get

$$R^{\perp}(\xi,Y)\sigma(U,\xi)-\sigma(\widetilde{C}(\xi,Y)U,\xi)-\sigma(U,\widetilde{C}(\xi,Y)\xi)=L_1\sigma(U,Y). \tag{6.6}$$

So, using (3.13) the above equation is reduces to

$$-\sigma(U,\widetilde{C}(\xi,Y)\xi)=L_1\sigma(U,Y).$$

Making use of (6.5) in above equation, we have

$$-\sigma\left(U,\left[\alpha^2-\frac{r}{n(n-1)}\right]\{\eta(Y)\xi+Y\}\right)=L_1\sigma(U,Y).$$

In view of (3.13), we obtain $[L_1+(\alpha^2-\frac{r}{n(n-1)})]\sigma(U,Y)=0$. Suppose $L_1\neq(\frac{r}{n(n-1)}-\alpha^2)$ then $\sigma(U,Y)=0$ i.e., M is totally geodesic. The converse statement is trivial.

(ii) Since M satisfies the condition $\widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma=L_1Q(g,\widetilde{\nabla}\sigma)$, Put $X=V=\xi$ and using (3.1), (3.4), (3.13) in (2.10) and (6.3), we get

$$\begin{aligned}
&R^{\perp}(\xi,Y)(\widetilde{\nabla}\sigma)(U,\xi,W)-(\widetilde{\nabla}\sigma)(\widetilde{C}(\xi,Y)U,\xi,W)-(\widetilde{\nabla}\sigma)(U,\widetilde{C}(\xi,Y)\xi,W)\\
&-(\widetilde{\nabla}\sigma)(U,\xi,\widetilde{C}(\xi,Y)W)=-L_1[\eta(W)\left\{\nabla^{\perp}_{\xi}\sigma(Y,U)-\sigma(\nabla_{\xi}Y,U)\right.\\
&\left.-\sigma(Y,\nabla_{\xi}U)\right\}+\nabla^{\perp}_{W}\sigma(Y,U)-\sigma(\nabla_WY,U)-\sigma(Y,\nabla_WU)\\
&-\eta(Y)\left\{\nabla^{\perp}_{\xi}\sigma(W,U)-\sigma(\nabla_{\xi}W,U)-\sigma(W,\nabla_{\xi}U)\right\}\\
&-\eta(U)\left\{\nabla^{\perp}_{\xi}\sigma(Y,W)-\sigma(\nabla_{\xi}Y,W)-\sigma(Y,\nabla_{\xi}W)\right\}].
\end{aligned} \tag{6.7}$$

In view of (2.7), (3.4), (6.4), (6.5) and (3.13), we have the following equalities:

$$
\begin{aligned}
&(\widetilde{\nabla}\sigma)(\widetilde{C}(\xi,Y)U,\xi,W) = (\widetilde{\nabla}_{\widetilde{C}(\xi,Y)U}\sigma)(\xi,W), \qquad (6.8)\\
&= \nabla^{\perp}_{\widetilde{C}(\xi,Y)U}\sigma(\xi,W) - \sigma(\nabla_{\widetilde{C}(\xi,Y)U}\xi,W) - \sigma(\xi,\nabla_{\widetilde{C}(\xi,Y)U}W),\\
&= \alpha\left[\alpha^2 - \frac{r}{n(n-1)}\right]\eta(U)\sigma(\phi Y,W),
\end{aligned}
$$

$$
\begin{aligned}
&(\widetilde{\nabla}\sigma)(U,\widetilde{C}(\xi,Y)\xi,W) = (\widetilde{\nabla}_U\sigma)(\widetilde{C}(\xi,Y)\xi,W), \qquad (6.9)\\
&= \nabla^{\perp}_U\sigma(\widetilde{C}(\xi,Y)\xi,W) - \sigma(\nabla_U\widetilde{C}(\xi,Y)\xi,W) - \sigma(\widetilde{C}(\xi,Y)\xi,\nabla_U W),\\
&= \nabla^{\perp}_U\sigma\left(\left[\alpha^2 - \frac{r}{n(n-1)}\right]\{\eta(Y)\xi+Y\},W\right) - \sigma\left(\nabla_U\left[\alpha^2 - \frac{r}{n(n-1)}\right]\right.\\
&\left.\times\{\eta(Y)\xi+Y\},W\right) - \left[\alpha^2 - \frac{r}{n(n-1)}\right]\sigma(Y,\nabla_U W)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\widetilde{\nabla}\sigma)(U,\xi,\widetilde{C}(\xi,Y)W) = (\widetilde{\nabla}_U\sigma)(\xi,\widetilde{C}(\xi,Y)W), \qquad (6.10)\\
&= \nabla^{\perp}_U\sigma(\xi,\widetilde{C}(\xi,Y)W) - \sigma(\nabla_U\xi,\widetilde{C}(\xi,Y)W) - \sigma(\xi,\nabla_U\widetilde{C}(\xi,Y)W),\\
&= -\alpha\left[\alpha^2 - \frac{r}{n(n-1)}\right]\{\sigma(\phi U,g(Y,W)\xi) - \eta(W)\sigma(\phi U,Y)\}.
\end{aligned}
$$

Substituting (5.3) and (6.8) – (6.10) into (6.7), we obtain

$$
\begin{aligned}
&-\alpha R^{\perp}(\xi,Y)\sigma(\phi U,W) - \alpha\left[\alpha^2 - \frac{r}{n(n-1)}\right]\eta(U)\sigma(\phi Y,W)\\
&-\nabla^{\perp}_U\sigma\left(\left[\alpha^2 - \frac{r}{n(n-1)}\right]\{\eta(Y)\xi+Y\},W\right) + \sigma\left(\nabla_U\left[\alpha^2 - \frac{r}{n(n-1)}\right]\right.\\
&\left.\times\{\eta(Y)\xi+Y\},W\right) + \left[\alpha^2 - \frac{r}{n(n-1)}\right]\sigma(Y,\nabla_U W) + \alpha\left[\alpha^2 - \frac{r}{n(n-1)}\right]\\
&\times\{\sigma(\phi U,g(Y,W)\xi) - \eta(W)\sigma(\phi U,Y)\} = -L_1[\eta(W)\left\{\nabla^{\perp}_\xi\sigma(Y,U)\right.\\
&\left.-\sigma(\nabla_\xi Y,U) - \sigma(Y,\nabla_\xi U)\right\} + \nabla^{\perp}_W\sigma(Y,U) - \sigma(\nabla_W Y,U) - \sigma(Y,\nabla_W U)\\
&-\eta(Y)\left\{\nabla^{\perp}_\xi\sigma(W,U) - \sigma(\nabla_\xi W,U) - \sigma(W,\nabla_\xi U)\right\}\\
&-\eta(U)\left\{\nabla^{\perp}_\xi\sigma(Y,W) - \sigma(\nabla_\xi Y,W) - \sigma(Y,\nabla_\xi W)\right\}]. \qquad (6.11)
\end{aligned}
$$

Taking $W=\xi$ and using (3.1), (3.4), (3.13) in (6.11), we get $\alpha[\alpha^2-\frac{r}{n(n-1)}]\sigma(U,Y) = 0$. Suppose $[\alpha^2 - \frac{r}{n(n-1)}] \neq 0$ then $\sigma(U,Y)=0$ i.e., M is totally geodesic. The converse statement is trivial. Hence the theorem is proved.

Using corollary 5.2 and Theorem 6.7, we have the following result

Corollary 6.3. Let M be an invariant submanifold of a Lorentzian α-sasakian manifold $\widetilde{M}$. Then the following statements are equivalent:

(1) σ is recurrent;
(2) σ is 2-recurrent;
(3) M has parallel third fundamental form;
(4) M is pseudoparallel, if $L_1 \neq \alpha^2$;
(5) M is 2-pseudoparallel;
(6) M satisfies the condition $\widetilde{C}(X,Y)\cdot\sigma = L_1 Q(g,\sigma)$ and $L_1 \neq (\frac{r}{n(n-1)} - \alpha^2)$, $\alpha^2 \neq \frac{r}{n(n-1)}$;
(7) M satisfies the condition $\widetilde{C}(X,Y)\cdot\widetilde{\nabla}\sigma = L_1 Q(g,\widetilde{\nabla}\sigma)$ and $\alpha \neq 0$, $\alpha^2 \neq \frac{r}{n(n-1)}$;
(8) M is totally geodesic.

References

[1] Ahmet Yildiz, Mine Turan and Bilal Eftal Acet : On Three-Dimensional Lorentzian α-Sasakian manifolds, Bulletin of Mathematical Analysis and Applications., 1(1) (2009), 90-98.

[2] Yildiz, Ahmet and Murathan, Cengizhan : Invariant submanifolds of Sasakian space forms, Kyungpook Math. J., 95 (2009), 135-150.

[3] Arslan, K., Lumiste, U., Murathan, C. and Ozgur, C. : 2-semi-parallel surfaces in space forms. I. Two particular cases, Proc. Estonian Acad. Sci. Phys. math., 49(3) (2000), 139-148.

[4] Asperti, A.C., Lobos, G. A. and Mercuri, F. : Pseudo-parallel immersions in space forms, Math. Contemp., 17 (1999), 59-70.

[5] Aysel Turgut Vanli and Ramazan Sari, : Invariant submanifolds of trans-sasakian manifolds, DIFFERENTIAL GEOMETRY-DYNAMICAL SYSTEMS., Geometry Balkan Press (2010).

[6] Blair, D. E. : Contact manifolds in Riemannian Geometry, Lecture Notes in Math., 509, Springer-Verlag, Berlin, (1976).

[7] Chen, B. Y. : Geometry of submanifolds and its applications, Science University of Tokyo, Tokyo., (1981).

[8] Chinea, D. : Invariant submanifolds of a quasi-sasakian manifold, Riv. Mat. Univ. Parma., 11 (1985), 25-29.

[9] Chinea, D. and Prestelo, P. S. : Invariant submanifolds of a trans-sasakian manifold, Publ. Mat. Debrecen., $38/1-2$ (1991), 103-109.

[10] Deprez, J. : Semi-parallel surfaces in the Euclidean space, Journal of Geometry., 11 (1985), 192-200.

[11] Dillen, F. : Semi-parallel Hypersurfaces of a Real Space form, Israel J. Math., 75 (1991), 193-202.

[12] Endo, H. : Invariant submanifolds in contact metric manifolds, Tensor (N.S)., 43(1) (1986), 83-87.

[13] Hretcanu, C. E. and Crasmareanu,M. : On some invariant submanifolds in a Riemannian manifold with golden structure, Ann. Mat. Pura Appl., 123(4) (1980), 35-58.

[14] Kon, M. : Invariant submanifolds in normal contact metric manifolds, Kodai Math., 25 (1973), 330-336.

[15] Murathan, C. : On invariant submanifolds satisfying a nullity condition, An. Univ. Bucuresh. Mat. Inform., 50(162) (2001), 103-113.

[16] C. Ozgur and C. Murathan, : On invariant submanifolds of Lorentzian para-sasakian manifolds, The Arabian J. Sci. Engg., 34(2A) (2010), 177-185.

[17] Ozgur, C., Sular, S. and Murathan, C. : On pseudoparallel invariant submanifolds of contact metric manifolds, Bull. Transilv. Univ. Bra sov ser. B (N.S)., 14(49 Suppl.) (2007), 227-234.

[18] Ravi, B. and Bagewadi, C. S. : Invariant submanifolds in a conformal K-contact Riemannian manifolds, Indian J. Pure appl. Math., 20(11) (1989), 1119-1125.

[19] Roter, W. : On conformally recurrent Ricci-recurrent manifolds, Colloq Math., 46(1) (1982), 45-57.

[20] Sular, Sibel and Ozgur, Cihan : On some submanifolds of Kenmotsu manifolds, Chaos, Solitons and Fractals., 42 (2009), 1990-1995.

[21] Yano, K. and Kon, M. : Structures on manifolds, World Scientific Publishing., (1984).