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A Semi-Symmetric Metric ξ-Connection in an LP-Sasakian Manifold

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

Yano (1970) investiated a semi-symmetric metric connections in a Riemannian manifold and since then many authors studied this connection. Further Mishra and Pandey (1978) defined a semi-symmetric metric ξ-connection in almost contact manifold and obtained various geometrical properties. Following Mishra and Pandey (1978) we define semi-symmetric metric ξ-connection in Lorentzian Para-Sasakian manifold and study some propeties of curvature tensors.

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1. Introduction

A differential manifold M^n of dimension n is called Lorentzian Para-Sasakian (LP-Sasakian) manifold if it admits a (1, 1)-tensor field ϕ, a covariant vector field ξ, a covariant vector field η and a Lorentzian matric g satisfy

$$\eta(\xi) = -1 \tag{1.1}$$

$$\phi^2(X) = X + \eta(X)\xi \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{1.3}$$

$$g(X, \xi) = \eta(X) \tag{1.4}$$

$$\nabla_X \xi = \phi X \tag{1.5}$$

and

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.6}$$

for arbitrary vector fields X and Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian matric g. (Matsumoto, 1989).

It can be easly seen that in LP-Sasakian manifold, the following relations hold:

$$(a) \quad \phi\xi = 0, \qquad (b) \quad \eta(\phi X) = 0 \tag{1.7}$$

$$\operatorname{rank}\phi = n - 1. \tag{1.8}$$

Further on such an LP-Sasakian manifold with (ϕ, ξ, η, g) structure, the following relation hold (Matsumoto and Mihai, 1988)

$${}'R(X,Y,Z,\xi) = \eta(R(X,Y,Z)) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.9}$$

$$R(\xi, X, Y) = g(X,Y)\xi - \eta(Y)X, \tag{1.10}$$

$$R(\xi, X, \xi) = X + \eta(X), \tag{1.11}$$

$$R(X,Y,\xi) = \eta(Y)X - \eta(X)Y, \tag{1.12}$$

$$Ric(X,\xi) = (n-1)\eta(X), \tag{1.13}$$

$$Ric(\phi X, \phi Y) = Ric(X,Y) - (n-1)\eta(X)\eta(Y), \tag{1.14}$$

for any vector fields X, Y, Z where R(X,Y,Z) and Ric(X, Y) are curvature tensor and Ricci tensor respectively with respect to Riemannian connection ∇ and

$${}'R(X,Y,Z,W) = g(R(X,Y,Z),W). \tag{1.15}$$

Let us put

$$F(X,Y) = g(\phi X, Y). \tag{1.16}$$

Then the tensor field F(X,Y) is symmetric (0, 2)-tensor field and it can be easily seen that

$$F(X,Y) = F(Y,X) = F(\phi X, \phi Y),$$

$$g(\phi X, Y) = g(\phi Y, X), \tag{1.17}$$

$$F(X,Y) = (\nabla_X \eta)Y. \tag{1.18}$$

An LP-Sasakian manifold M^n is said to be an η-Einstein manifold (Yano and Kon, 1984) if its Ricci tensor Ric(X,Y) is of the form

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.19}$$

where a and b are scalars.

An LP-Sasakian manifold M^n is called an Einstein manifold (Sinha, 1982) if its Ricci tensor Ric(X,Y) is of the form

$$Ric(X,Y) = \lambda g(X,Y), \tag{1.20}$$

where λ is in general a function on M^n.

The equation (1.20) can also be written as

$$Ric(X,Y) = \frac{r}{n} g(X,Y), \tag{1.21}$$

where r is the scalar curvature.

The pseudo $\widetilde{W}_2$ curvature tensor on an LP-Sasakian manifold M^n $(n > 2)$ of type (1,3) with respect to the Riemannian connection ∇ is given by (Prasad and Mourya, 2004)

$$\begin{aligned}\widetilde{W}_2(X,Y,Z) = aR(X,Y,Z) + b[g(Y,Z)QX - g(X,Z)QY] \\ - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.22}$$

where a and b are constant such that $a, b \neq 0$, is the scalar curvature and Q is the (1,1) Ricci tensor with respect to the Riemannian connection ∇ given by

$$g(QX,Y) = Ric(X,Y). \tag{1.23}$$

If $a = 1$ and $b = -\frac{1}{n-1}$ then (1.22) takes the form

$$\begin{aligned}\widetilde{W}_2(X,Y,Z) = &\ R(X,Y,Z) - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY] \\ = &\ W_2(X,Y,Z)\end{aligned}$$

where W_2 is the curvature tensor (Pohhasiyal and Mishra, 1970) with respect to the Riemmannian connection. Hence the W_2-curvature rensor is a particular case of the tensor $\widetilde{W}_2$.

For this reason $\widetilde{W}_2$ is a called a pseudo $\widetilde{W}_2$-curvature tensor.

Then from (1.22), we get

$$\begin{aligned}{}'\widetilde{W}_2(X,Y,Z,W) = a\,{}'R(X,Y,Z,W) + b[g(Y,Z)Ric(X,W) \\ - g(X,Z)Ric(Y,W)] - \frac{r}{n}\left(\frac{a}{n-1} + b\right) \\ [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{1.24}$$

where

$$'\widetilde{W}_2(X,Y,Z,W) = g(\widetilde{W}_2(X,Y,Z),W). \tag{1.25}$$

Further $'\widetilde{W}_2$ satisfies following algebraic properties:

$$'\widetilde{W}_2(X,Y,Z,W) +' \widetilde{W}_2(Y,X,Z,W) = 0, \tag{1.26}$$

$$'\widetilde{W}_2(X,Y,Z,W) + \widetilde{W}_2(Y,Z,X,W) +' \widetilde{W}_2(Z,X,Y,W) = 0. \tag{1.27}$$

Also an LP-Sasakian manifold M^n will be called pseudo $\widetilde{W}_2$ flat if $\widetilde{W}_2$=0.

The pseudo projective curvature tensor $\widetilde{P}$ of an LP-Sasakian manifold M^n (n>2) of type (1,3) with respect to the Riemannian connection ∇ is given by (Prasad, 2000)

$$\begin{aligned} \widetilde{P}(X,Y,Z) =& aR(X,Y,Z) + b[Ric(Y,Z)X - Ric(X,Z)Y] \\ & - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.28}$$

where a and b are constant such that $a, b \neq 0$.

If $a = 1$ and $b = -\frac{1}{n-1}$ then (1.28) takes the form

$$\begin{aligned} \widetilde{P}(X,Y,Z) =& \ R(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y] \\ =& \ P(X,Y,Z), \end{aligned} \tag{1.29}$$

where P(X,Y,Z) is the projective curvature tensor (Chaki, 1987) with respect to Riemannian connection. Hence the projective curvature tensor P is a prticular case of $\widetilde{P}$. For this reasion $\widetilde{P}$ is called pseudo projective curvature tensor.

Furthure satisfies following algebraic properties:

$$\widetilde{P}(X,Y,Z) + \widetilde{P}(Y,X,Z) = 0, \tag{1.30}$$

and

$$\widetilde{P}(X,Y,Z) + \widetilde{P}(Y,Z,X) + \widetilde{P}(Z,X,Y) = 0. \tag{1.31}$$

Also LP-Sasakian manifold M^n will be called pseudo projectively flat if $\widetilde{P} = 0$.

2. ξ-connection in an LP-Sasakian manifold M^n

Let $\overline{\nabla}$ be an affine connection. Then from (1.4), we get

$$g(\overline{\nabla}_X \xi, Y) = (\overline{\nabla}_X \eta)Y \tag{2.1}$$

Theorem 2.1. If $\overline{\nabla}$ is a linear connection in an LP-Sasakian manifold M_n then $\overline{\nabla}_X\xi$=0 if and only if $(\overline{\nabla}_X\eta)Y = 0$.

Proof. Since Y is a arbitrary vector field, from (2.1) it follows that

$$\overline{\nabla}_X\xi = 0, \tag{2.2}$$

if and only if

$$(\overline{\nabla}_X\eta)Y = 0. \tag{2.3}$$

Theorem 2.2. If $\overline{\nabla}$ is a linear connection in an LP-Sasakian manifold M_n then $\overline{\nabla}_X\xi$=0 if and only if

$$(\nabla_X\eta)Y + \eta(\nabla_X Y) - \eta(\overline{\nabla}_X Y) = 0. \tag{2.4}$$

Proof. From theorem (2.1) and $(\overline{\nabla}_X\eta)Y = (\nabla_X\eta)Y + \eta(\nabla_X Y) - \eta(\overline{\nabla}_X Y)$, theorem (2.2) follows.

Remark. From theorem (2.2) we conclude that if a linear connection $\overline{\nabla}$ in an LP-Sasakian manifold M^n satisfies $\overline{\nabla}_X\xi$=0, then M^n admits only those linear connection which satisfies (2.4). Hence we have the following definition.

Definition 2.1. A linear connection $\overline{\nabla}$ in an LP-Sasakian manifold M^n is called a ξ-Connection if it satisfies

$$\overline{\nabla}_X\xi = 0, \tag{2.5}$$

and

$$(\nabla_X\eta)Y + \eta(\nabla_X Y) - \eta(\overline{\nabla}_X Y) = 0. \tag{2.6}$$

3. A semi-symmetric metric ξ-connection in an LP-Sasakian manifold M^n

A linear connection $\overline{\nabla}$ in an LP-Sasakian manifold M^n, is called a semi-symmetric metric connection if its torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X, Y) = \eta(Y)X - \eta(X)Y. \tag{3.1}$$

Further if $\overline{\nabla}$ also satisfies

$$(\overline{\nabla}_X)g(Y, Z) = 0, \tag{3.2}$$

then $\overline{\nabla}$ is called a semi-symmetric connection (Yano, 1970). Let

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y). \tag{3.3}$$

Then it can be seen (Yano, 1970)

$$H(X, Y) = \eta(Y)X - g(X, Y)\xi, \tag{3.4}$$

and

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{3.5}$$

Theorem 3.1. In an LP-Sasakian manifold M^n, the semi-symmetric metric connection $\overline{\nabla}$ given by (3.5) is a semi-symmetric metric ξ-connection if and only if

$$\phi X = X + \eta(X)\xi. \tag{3.6}$$

Proof. Let (3.6) be satiesfied. From (1.1), (1.4), (1.5), (3.5) and (3.6), we have

$$\begin{aligned}\overline{\nabla}_X \xi &= \overline{\nabla}_X \xi + \eta(\xi)X - g(X,\xi)\xi \\ &= \phi X - 1.X - \eta(X)\xi \\ &= X + \eta(X)\xi - X - \eta(X)\xi \\ &= 0.\end{aligned}$$

From (1.1), (1.4), (1.17), (1.18), (2.6), (3.5) and (3.6), we have

$$\begin{aligned}(\nabla_X \eta)Y + \eta(\nabla_X Y) - \eta(\overline{\nabla}_X Y) &= F(X,Y) + \eta(\nabla_X Y + \eta(Y)X - g(X,Y)\xi) \\ &= g(\phi X, Y) + \eta(\nabla_X Y) - \eta(\nabla_X Y) - \eta(Y)\eta X \\ &\quad - g(X,Y)\eta(\xi) \\ &= 0.\end{aligned}$$

Hence $\overline{\nabla}$ is a semi-symmetric metric ξ-connection conversely let $\overline{\nabla}$ be ξ-connection, then (2.5) and (2.6) will be satisfied.

From (1.1), (1.4), (1.5), (2.5) and (3.5), we have

$$\phi X - X - \eta(X)\xi = 0,$$

which gives

$$\phi X = X + \eta(X)\xi$$

4. Curvature tensor of semi-symmetric metric ξ-connection in LP-Sasakian manifold

Let $\overline{R}$ be curvature tensor of $\overline{\nabla}$. Then

$$\overline{R}(X,Y,Z) = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z. \tag{4.1}$$

From (1.4),(1.17) and (3.6), we have

$$\begin{aligned}(\nabla_X \eta)Y &= F(X,Y) = g(\phi X, Y) = g(X + \eta(X)\xi, Y) \\ &= g(X,Y) + \eta(X)\eta(Y)\end{aligned} \tag{4.2}$$

In consequences of (1.1), (1.4), (1.5), (1.18), (3.5), (3.6), (4.1) and (4.2), we have

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + g(X,Z)Y - g(Y,Z)X. \tag{4.3}$$

where

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4.4}$$

is the curvature tensor with respect to ∇.

Contracting (4.3) with respect to X, we get

$$\overline{R}(Y,Z) = Ric(Y,Z) - (n-1)g(Y,Z). \tag{4.5}$$

Contracting (4.5), we get

$$\overline{r} = r - n(n-1). \tag{4.6}$$

where $\overline{R}ic$(Y,Z) and $\overline{r}$ are Ricci tensor and scalar curvature tensor with respect to $\overline{\nabla}$.

Let

$$'\overline{R}(X,Y,Z,W) = g(R(X,Y,Z),W). \tag{4.7}$$

Then from (4.3) and (4.7), we have

$$\begin{aligned} '\overline{R}(X,Y,Z,W) = &\overline{R}(X,Y,Z,W) + g(X,Z)g(Y,W) \\ &- g(Y,Z)g(X,W). \end{aligned} \tag{4.8}$$

Theorem 4.1. In an LP-Sasakian manifold with semi-symmetric metric ξ-connection, we have

$$\begin{aligned} &(a) \quad '\overline{R}(X,Y,Z,\xi) = 0, \\ &(b) \quad '\overline{R}(T,Y,Z,\xi) = 0, \\ &(c) \quad '\overline{R}(X,Y,\xi) = 0, \\ &(d) \quad '\overline{R}(\xi,X,\xi) = 0, \\ &(e) \quad '\overline{R}ic(X,\xi) = 0. \end{aligned} \tag{4.9}$$

Proof. In consequences of equation (1.4), (1.9), (1.10), (1.11), (1.12), (1.13), (4.3), (4.5), (4.8), equation (4.9) follow.

Theorem 4.2. If in LP-Sasakian manifold M^n, the curvature tensor with respect to the semi-symmetric metric ξ-connection, vanishes, then

$$\begin{aligned} &(a) \quad R(X,Y,\xi) = X - \eta(X)Y, \\ &(b) \quad \phi(R(X,Y,\xi)) = \eta(Y)\phi X - \eta(X)\phi Y, \\ &(c) \quad \eta(R(X,Y,\xi)) = 0, \\ &(d) \quad \eta(R(\xi,Y,Z) + F(Y,Z) = 0. \end{aligned} \tag{4.10}$$

Proof. Let $\overline{R}(X, Y, Z) = 0$, then from (4.3), we get

$$R(X, Y, Z) = g(Y, Z)X - g(X, Z)Y. \tag{4.11}$$

From (4.11) we get

$$R(X, Y, \xi) = \eta(Y)X - \eta(X)Y,$$

which proves (a).

Proof of (b) and (c) follow from (a).

From (4.11), we get

$$R(\xi, Y, Z) = g(Y, Z)\xi - \eta(Z)Y,$$

or

$$\eta(R(\xi, Y, Z)) = -g(Y, Z) - \eta(Z)\eta(Y),$$

which in view of (4.2) gives (d).

Theorem 4.3. For an LP-Saakian manifold M^n with respect to the semi-symmetric metric ξ-connection, we have

$$\begin{aligned}
&\text{(a)} \quad {}'\overline{R}(X, Y, Z, W) + {}'\overline{R}(Y, X, Z, W) = 0,\\
&\text{(b)} \quad {}'\overline{R}(X, Y, Z, W) + {}'\overline{R}(Y, X, Z, W) + {}'\overline{R}(Z, X, Y, W) = 0,\\
&\text{(c)} \quad {}'\overline{R}(X, Y, Z, W) + {}'\overline{R}(X, Y, W, Z) = 0,\\
&\text{(d)} \quad {}'\overline{R}(X, Y, Z, W) - {}'\overline{R}(Z, W, X, Y) = 0,\\
&\text{(e)} \quad (\overline{\nabla}_X \overline{R})(Y, Z) + (\overline{\nabla}_Y \overline{R})(Z, X) + (\overline{\nabla}_Z \overline{R})(X, Y) =\\
&\qquad\qquad 2\eta(X)\overline{R}(Y, Z) + 2\eta(Y)\overline{R}(Z, X) + 2\eta(Z)\overline{R}(X, Y).
\end{aligned} \tag{4.12}$$

Proof. Proof of (a), (b), (c) and (d) follow from (4.8) and Bianchi first identity.

Bianchi second identity for a linear connection is given by (Sinha, 1982)

$$\begin{aligned}
&(\overline{\nabla}_X \overline{R})(Y, Z) + (\overline{\nabla}_Y \overline{R})(Z, X) + (\overline{\nabla}_Z \overline{R})(X, Y) =\\
&\qquad - \overline{R}(\overline{T}(X, Y), Z) - \overline{R}(\overline{T}(Y, Z), X) - \overline{R}(\overline{T}(Z, X), Y).
\end{aligned}$$

Using (3.1) and (4.12a) in above expression, we get

$$\begin{aligned}
(\overline{\nabla}_X \overline{R})(Y, Z) + (\overline{\nabla}_Y \overline{R})(Z, X) + (\overline{\nabla}_Z \overline{R})(X, Y) &= -\overline{R}(\eta(Y)X - \eta(X)Y, Z)\\
&\quad - \overline{R}(\eta(X)Z - \eta(Z)X, Y) - \overline{R}(\eta(Z)Y - \eta(Y)Z, X)\\
&= 2\eta(X)\overline{R}(Y, Z) + 2\eta(Y)\overline{R}(Z, X) + 2\eta(Z)\overline{R}(X, Y).
\end{aligned}$$

which gives (4.12)e.

Theorem 4.4. If in LP-Sasakian manifold the curvature tensor of semi-symmetric metric ξ-connection vanish, then it has a constant curvature +1.

Proof. If $\overline{R}(X, Y, Z) = 0$, then (4.3) gives

$$R(X, Y, Z) = g(Y, Z)X - g(X, Z)Y$$

which proves the statement.

Theorem 4.5. Let M^n be an LP-Sasakian manifold with semi-symmetric metric $\xi-$connection. If Ricci tensor with respect to semi-symmetric metric $\xi-$connection vanish, then the manifold becomes an $\eta-$Einstein manifold with associated constants a = n −1 and b = 0.

Proof. If $\overline{R}ic$(Y,Z)=0, then from (4.5), we get

$$Ric(Y, Z) = (n-1)g(Y, Z),$$

which is of the form (1.19) with associated constants a = n − 1 and b = 0. Hence the theorem.

Theorem 4.6. Let M^n be an LP-Sasakian manifold with semi-symmetric metric $\xi-$connection. Then the manifold is an Einstein manifold with respect to semi-symmetric metric ξ connection, if and only if it is an Einstien manifold with respect to Riemannian connection.

Proof. From (4.6), we get

$$n - 1 = \frac{r - \overline{r}}{n} \tag{4.13}$$

Again from (4.5) and (4.13), we get

$$\overline{Ric}(Y, Z) = Ric(Y, Z) - \frac{(r - \overline{r})}{n} g(Y, Z), \tag{4.14}$$

Hence from (1.21) and (4.14), theorem follows.

5. The Pseudo $\overset{*}{W}_2$-Curvature Tensor of LP-Sasakian manifold with semi-symmetric metric ξ-connection

Let the Pseudo $\overset{*}{W}_2$-curvature tensor with respect semi-symmetric metric ξ-connection, then $\overset{*}{W}_2$ is given by

$$\begin{aligned} {}'\overset{*}{W}_2\,(X, Y, Z, W) = a\,{}'\overline{R}(X, Y, Z, W) + b[g(Y, Z)\overline{R}ic(X, W) \\ - g(X, Z)\overline{R}ic(Y, W)] - \frac{\overline{r}}{n}\left(\frac{a}{n-1} + b\right) \\ [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned} \tag{5.1}$$

where

$$'\overset{*}{W}_2(X,Y,Z,W) = g(\overset{*}{W}_2(X,Y,Z),W)). \tag{5.2}$$

Theorem 5.1. If M^n be an LP-Sasakian manifold admitting a semi-symmetric metric ξ-connection, then Pseudo $\overset{*}{W}_2$-curvature tensor with respect semi-symmetric metric ξ-connection is equal to the Pseudo $\widetilde{W}_2$-curvature tensor with respect to the Riemannian connection. That is

$$'\widetilde{W}_2(X,Y,Z,W) = \overset{*}{W}_2(X,Y,Z,W)$$

Proof. From (1.24), (4.5), (4.6), (4.8) and (5.2), we get

$$'\widetilde{W}_2(X,Y,Z,W) = \overset{*}{W}_2(X,Y,Z,W)$$

Theorem 5.2. If M^n be an LP-Sasakian manifold with vanishing Ricci tensor with respect to semi-symmetric metric ξ-connection, then M^n is $\overset{*}{W}_2$ flat if and only if curvature tensor with respect to semi-symmetric metric ξ-connection vanishes.

Proof. If

$$\overline{Ric}(Y,Z) = 0, \tag{5.3}$$

then

$$\overline{r} = 0. \tag{5.4}$$

From (4.5), (4.6), (5.3) and (5.4), we get

$$Ric(Y,Z) = (n-1)g(Y,Z), \tag{5.5}$$

and

$$r = n(n-1). \tag{5.6}$$

From (1.25),(4.8), (5.5) and (5.6), we get

$$\begin{aligned} '\overset{*}{W}_2(X,Y,Z,W) &= aR(X,Y,Z,W) + b(n-1)[g(Y,Z)g(X,W) \\ &\qquad - g(X,Z)g(Y,W)] - \frac{n(n-1)}{n}\left(\frac{a}{n-1} + b\right) \\ &\qquad \times [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &= a\,'\overline{R}(X,Y,Z,W). \end{aligned} \tag{5.7}$$

Hence the theorem follows from (5.7).

Theorem 5.3. If M^n be an LP-Sasakian manifold with vanishing curvature tensor with respect to semi-symmetric metric ξ-connection, then M^n is $\overset{*}{W}_2$ flat.

Proof. If
$$\overline{R}(X,Y,Z)=0, \tag{5.8}$$
then
$$\overline{R}ic(Y,Z)=0, \tag{5.9}$$
and
$$\overline{r}=0. \tag{5.10}$$
From (4.5), (4.6), (4.8), (5.8), (5.9) and (5.10), we get
$$Ric(Y,Z) = (n-1)g(Y,Z), \tag{5.11}$$
$$r = n(n-1) \tag{5.12}$$
$$'R(X,Y,Z,W) = g(Y,Z)g(X,W)-g(X,Z)g(Y,W) \tag{5.13}$$
From (1.24), (5.11), (5.12) and (5.13),we get
$$\begin{aligned}'\overset{*}{W}_2(X,Y,Z,W) &= a[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]+b(n-1)[g(Y,Z)\\ &\quad\times g(X,W)-g(X,Z)g(Y,W)]-\frac{n(n-1)}{n}\left(\frac{a}{n-1}+b\right)\\ &\quad\times[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &=0.\end{aligned}$$
which proves the theorem.

Theorem 5.4. The pseudo $\overset{*}{W}_2$ curvature tensopr with respect to semi-symmetric metric ξ-connection satiesfies following algebraic properties:
$$'\overset{*}{W}_2(X,Y,Z,W)+{}'\overset{*}{W}_2(Y,X,Z,W)=0. \tag{5.14}$$
and
$$'\overset{*}{W}_2(X,Y,Z,W)+{}'\overset{*}{W}_2(Y,X,Z,W)+{}'\overset{*}{W}_2(Z,X,Y,W)=0. \tag{5.15}$$

Proof. Proof follows from (1.26),(1.27) and theorem (5.1).

6. Pseudo projective curvature tensor $\overset{*}{P}$ of an LP-Sasakian manifold with respect to semi-symmetric metric ξ-connection

Theorem 6.1. If M^n is an LP-Sasakian manifold admitting a semi-symmetric metric ξ-connection then pseudo projective curvature tensor with respect to semi-symmetric metric ξ-connection $\overset{*}{P}$ is equal to pseudo projective curvature with respect to Riemannian connection $\widetilde{P}$.

Proof. Let $\overset{*}{P}$ be pseudo projective curvature tensor of an LP-Sasakian manifold with respect to semi-symmetric metric ξ-connection, then $\overset{*}{P}$ is given by

$$\begin{aligned} \overset{*}{P}(X,Y,Z) &= a\overline{R}(X,Y,Z) + b[\overline{R}ic(Y,Z)X - \overline{R}ic(X,Z)Y] \\ &\quad - \frac{\overline{r}}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{6.1}$$

From (1.28), (4.3), (4.5), (4.6) and (6.1), we get

$$\begin{aligned} \overset{*}{P}(X,Y,Z) &= a[R(X,Y,Z) + g(X,Z)Y - g(Y,Z)X] \\ &\quad + b[\{Ric(Y,Z) - (n-1)g(Y,Z)\}\,X \\ &\quad - \{Ric(X,Z) - (n-1)g(X,Z)\}\,Y] - \frac{r - n(n-1)}{n} \\ &\quad \times \left(\frac{a}{n-1} + b\right)[g(Y,Z)X) - g(X,Z)Y] \\ &= \widetilde{P}(X,Y,Z), \end{aligned} \tag{6.2}$$

which proves the theorem.

Theorem 6.2. If M^n be an LP-Sasakian manifold with vanishing Ricci curvature tensor with respect to semi-symmetric metric ξ-connection, then M^n is pseudo projectively flat if and only if curvature tensor with respect to semi-symmetric metric ξ-connection vanishing.

Proof. If

$$\overline{R}ic(Y,Z) = 0 \tag{6.3}$$

then

$$\overline{r} = 0. \tag{6.4}$$

From (4.5), (4.6), (6.3) and (6.4), we get

$$Ric(Y,Z) = (n-1)g(Y,Z), \tag{6.5}$$

and

$$r = n(n-1). \tag{6.6}$$

From (1.28), (4.3), (6.5) and (6.6), we get

$$\begin{aligned} \overset{*}{P}(X,Y,Z) &= aR(X,Y,Z) + b(n-1)[g(Y,Z)X - g(X,Z)Y] \\ &\quad - \frac{n(n-1)}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{6.7}$$

From (6.7) theorem follows.

Theorem 6.3. If M^n be an LP-Sasakian manifold with vanishing curvature tensor with respect to semi-symmetric metric ξ-connection, then M^n is pseudo projectively flat.

Proof. If

$$\overline{R}(X, Y, Z) = 0, \tag{6.8}$$

then

$$\overline{Ric}(Y, Z) = 0, \tag{6.9}$$

and

$$\overline{r} = 0. \tag{6.10}$$

From (4.3), (4.5), (4.6), (5.8), (6.9) and (6.10),we get

$$R(X, Y, Z) = g(Y, Z)X - g(X, Z)Y, \tag{6.11}$$

$$Ric(Y, Z) = (n - 1)g(Y, Z), \tag{6.12}$$

$$r = n(n - 1) \tag{6.13}$$

From (1.28), (6.11), (6.12) and (6.13),we get

$$\begin{aligned}\widetilde{P}(X, Y, Z) &= a[g(Y, Z)X - g(X, Z)Y] + b(n - 1)[g(Y, Z)X - g(X, Z)Y] \\ &\quad - \frac{n(n - 1)}{n}\left(\frac{a}{n - 1} + b\right)[g(Y, Z)X - g(X, Z)Y] \\ &= 0.\end{aligned}$$

which proves the theorem.

Theorem 6.4. The pseudo projective curvature tensor tensor $\overset{*}{P}$ with respect to semi-symmetric metric ξ-connection satisfy following algebraic:

$$\overset{*}{P}(X, Y, Z) + \overset{*}{P}(Y, X, Z) = 0, \tag{6.14}$$

and

$$\overset{*}{P}(X, Y, Z) + \overset{*}{P}(Y, X, Z) + \overset{*}{P}(Z, X, Y) = 0. \tag{6.15}$$

7. Group Manifold of LP-Sasakian manifold with respect to semi-symmetric matric ξ-connection

Theorem 7.1. An LP-Sasakian manifold M^n is a group manifold with respect to semi-symmetric metric ξ-connection if and only if curvature tensor with respect to semi-symmetric metric ξ-connection vanishes.

Proof. An LP-Sasakian manifold M^n is a group manifold with respect to semi-symmetric metric ξ-connection if (Agashe and Chafle, 1992)

$$\overline{R}(X, Y, Z) = 0, \tag{7.1}$$

and

$$(\overline{\nabla}_X \overline{T})(Y, Z) = 0. \tag{7.2}$$

From (2.6) and (3.1),we get

$$\begin{aligned}(\overline{\nabla}_X \overline{T})(Y, Z) &= \overline{\nabla}_X(\overline{T}(Y, Z)) - \overline{T}(\overline{\nabla}_X Y, Z) - \overline{T}(Y, \overline{\nabla}_X Z)\\ &= \overline{\nabla}_X(\eta(Y)Z - \eta(Z)Y) - \overline{T}(\overline{\nabla}_X Y, Z) - \overline{T}(Y, \overline{\nabla}_X Z)\\ &= ((\overline{\nabla}_X \eta)Y)Z - ((\overline{\nabla}_X \eta)Z)Y\\ &= 0.\end{aligned} \tag{7.3}$$

Therefore, it follows that manifold M^n will be a group manifold if and only if

$$\overline{R}(X, Y, Z) = 0. \tag{7.4}$$

which proves the theorem.

Theorem 7.2. An LP-Sasakian manifold M^n is a group manifold with respect to semi-symmetric metric ξ-connection if and only if

$$R(X, Y, Z) = g(Y, Z)X - g(X, Z)Y. \tag{7.5}$$

Proof. Proof follows from (4.3) and (7.4)

Theorem 7.3. An LP-Sasakian manifold M^n is a group manifold with respect to semi-symmetric metric ξ-connection, then M^n is pseudo projectively flat.

Proof. Proof follows from theorem (6.3) and theorem (7.1).

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