J. T. S.
Vol. 6 No.1 (2012), pp.189-201
https://doi.org/10.56424/jts.v6i01.10449

On Semi-Invariant Submanifolds of Nearly Hyperbolic $\beta-$Kenmotsu Manifold

Asha Srivastava and Kamal Bhatia

Department of Mathematics
D.S.B. Campus,Kumaun University, Nainital-263002, India
Email : kamalbhatianainital@gmail.com
(Received: 8 September, 2011)

(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

Semi-invariant submanifold of an almost contact metric manifold were defined and studied by A. Bejancu [1] and later developed by several authors [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] etc.

The purpose of the present paper is to define and study the semi-invariant submanifold of nearly hyperbolic $\beta-$Kenmotsu manifold.

Keywords : Semi-invariant, Hyperbolic, Umbilical, Geodesic, Submanifold.

1. Preliminaries

An n dimensional differential manifold $\overline{M}$ on which there are defined a tensor field F of type (1, 1) a vector field ξ, 1-form u and a Riemannian metric g satisfying for arbitrary vector field $X, Y, Z, \ldots$.

(1.1)
(a) $F^2 = 1 + u \otimes \xi,$

(b) $u(\xi) = -1,$

(c) $u0F = 0,$

(d) $F(\xi) = 0,$

(1.2) $$g(FX, FY) = -g(X, Y) - u(X)u(Y)$$

and

(1.3) $$u(X) = g(X, \xi), \text{ for all } X, Y \in TM$$

is called almost hyperbolic contact metric manifold and the structures (F, ξ, u, g) is almost contact hyperbolic metric structure [2].

Definition 1.1. An almost hyperbolic contact metric structure (F, ξ, u, g) on $\overline{M}$ is called a hyperbolic $\beta-$Kenmotsu manifold if [4]

$$(\overline{\nabla}_X F)Y = \beta(g(FX, Y)\xi - u(Y)FX), \text{ For function } \beta \text{ on } \overline{M}. \tag{1.4}$$

Then we say that hyperbolic $\beta-$Kenmotsu structure is type β. In particular, it is a normal.

Then from (1.1) and (1.4) we can easily obtain

$$\overline{\nabla}_X \xi = -\beta(X + u(X)\xi). \tag{1.5}$$

Thus the structural equation for nearly hyperbolic $\beta-$Kenmotsu manifold are given by

$$(\overline{\nabla}_Y F)Y + (\overline{\nabla}_Y F)X = -\beta(u(X)FY + u(Y)FX) \tag{1.6}$$

and the structural equation for nearly hyperbolic Kenmotsu manifold are defind by

$$(\overline{\nabla}_X F)Y + (\overline{\nabla}_Y F)X = -u(X)FY - u(Y)FX. \tag{1.7}$$

2. Semi-invariant submanifold

Let M be a Riemannian manifold isometrically immersed in a $\beta-$Kenmotsu manifold $\overline{M}$ such that ξ is tangent to M. We denote by same symbol g the Riemannian metric on M. Denote also by ∇ the Levi Civita connection on M with respect to g and $\nabla^\perp$ the linear connection induced by ∇ on the normal bundle $T^\perp M$. Then the equation of Gauss and Weingarten are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.2}$$

where h and A are both called the second fundamental tensors satisfying

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.3}$$

Now, if F is a (1, 1) tensor field on $\overline{M}$, for $X \in TM$ and $N \in T^{\perp}M$, we have

$$(\overline{\nabla}_X F)Y = ((\nabla_X P)Y - A_{QY}X - Bh(X,Y)) + ((\nabla_X Q)Y + h(X,PY) - Ch(X,Y)) \tag{2.4}$$

and

$$(\overline{\nabla}_X F)N = ((\nabla_X B)N - A_{CN}X - PA_N X) + ((\nabla_X C)N + h(X,BN) - QA_N X) \tag{2.5}$$

where

$$FX = PX + QX \qquad (PX \in TM, QX \in T^{\perp}M) \tag{2.6}$$

and

$$FN = BN + CN \qquad (BN \in TM, CN \in T^{\perp}M). \tag{2.7}$$

(2.8)
(a) $(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y,$

(b) $(\nabla_X Q)Y = \nabla^{\perp}_{Q} Y - Q\nabla_X Y,$

(c) $(\nabla_X B)N = \nabla_X BN - B\nabla^{\perp}_X N,$

(d) $(\nabla_X C)N = \nabla^{\perp}_X CN - Q\nabla^{\perp}_X N.$

The submanifold M is said to be totally geodesic in $\overline{M}$ is if $h = 0$, minimal in $\overline{M}$ if $H = 0$ and totally umbilical in $\overline{M}$ if

$$h(X,Y) = g(X,Y)H.$$

Applying the distribution D on M, M is called to be $D-$totally geodesic if $\forall\ X, Y \in D$, we have $h(X,Y) = 0$. If for every $X, Y \in D$, we have $h(X,Y) = g(X,Y)K$ for same normal vector K, so M is called $D-$totally umbilical. For two distribution D and S defined on M, M is defined to be $(D,S)-$mixed totally geodesic if for every $X \in D$ and $Y \in S$, we have $h(X,Y) = 0$.

Now we say that D is $S-$ parallel if for all $X \in D$ and $Y \in D$ we have $\nabla_X Y \in D$. If D is D parallel then it is called autoparallel. D is called $X-$parallel for some $X \in TM$ if for all $Y \in D$ we have $\nabla_X Y \in D$. D is said to be parallel if for all $X \in TM$ and $Y \in D$, $\nabla_X Y \in D$.

If a distribution D on M is autoparallel, then it is truly integrable, and from a Gauss formula D is totally geodesic in M. Suppose that D is parallel then the orthogonal complementary distribution $D^{\perp}$ is also parallel, which showing that D is parallel if and only if $D^{\perp}$ is parallel. Therefore M is locally the product of the leaves of D and $D^{\perp}$.

Suppose M be an almost hyperbolic contact metric manifold. $\xi \in TM$ then we write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by $\{\xi\}$ and $\{\xi\}^{\perp}$ is complementary arthogonal distribution of $\{\xi\}$ in M then one gets

(2.9) $$P\xi = 0 = Q\xi, \qquad u0P = 0 = u0Q,$$

(2.10) $$P^2 + BQ = 1 + u \oplus \xi, \qquad QP + CQ = 0,$$

(2.11) $$C^2 + QB = 1, \qquad BC + PB = 0.$$

A submanifold M of almost hyperbolic contact metric manifold $\overline{M}$ with $\xi \in TM$ is called a semi-invariant submanifold of M there exists two differentiable distribution D^1 and D^o on M such that

(1) $TM = D^1 \oplus D^o \oplus (\xi)$,

(2) The distribution D^1 is invariant by F, that is $F(D^1) = D^1$ and

(3) The distribution D^o is invariant by F, that is $F(D^o) \subseteq T^{\perp}M$.

Here

$$D^1_X = \text{Ker } (Q|\xi^{\perp})_X = \{X_X \in |\xi|^{\perp}_X : ||X_X|| = ||PX_X|| = T_XM \cap (T_XM),$$

$$D^o_X = \text{Ker } (P|^{\perp}_{\xi})_X = \{X_X \in |\xi\}^{\perp}_X : ||X_X|| = ||QX_X||\} = T_XM \cap (T_XM),$$

for $X \in TM$, then we have

(2.12) $$X = \Phi^1 X + \Phi^o X + u(X)\xi,$$

where Φ^1 and Φ^o are projection operators of TM on D^1 and D^o respectively.

A semi-invariant submanifold of an almost hyperbolic contact metric manifold [6] becomes an invariant submanifold (resp. Anti-invariant submanifold) if $D^o = \{0\}$ (resp. $D^1 = \{0\}$).

So we have

$$T^{\perp}M = \overline{D}^1 \oplus \overline{D}^o$$

where

$\overline{D}^1 = \text{Ker } (B) = T^{\perp}M \cap F(T^{\perp}M)$,

$\overline{D}^o = \text{Ker } (C) = T^{\perp}M \cap F(TM)$,

$QD^o = \overline{D}^o$, and $B\overline{D}^o = D^o$.

3. Nijenhuis tensor

An almost hyperbolic contact metric manifold is said to be normal if the torsion tensor $N^{(1)}$ vanishes [5]

$$N^1 \equiv [F, F] + 2du \oplus \xi = 0, \tag{3.1}$$

where $[F, F]$ is the Nijenhuis tensor of F and d denotes the exterior derivatives operator.

Now we get the Nijenhuis tensor $[F, F]$ of the structure tensor field F is

$$[F, F](X, Y) = ((\overline{\nabla}_{FX}F)Y - (\overline{\nabla}_{FY}F)X) - F((\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X). \tag{3.2}$$

Lemma 3.1. In an almost hyperbolic metric manifold we have

$$(\overline{\nabla}_Y F)FX = -F(\overline{\nabla}_Y F)X + ((\overline{\nabla}_Y u)X)\xi + u(X)\overline{\nabla}_Y \xi. \tag{3.3}$$

Proof. If $X, Y \in T\overline{M}$, we get

$$(\overline{\nabla}_Y F)FX = \overline{\nabla}_Y(F^2X) - F(\overline{\nabla}_Y FX) + F(F\overline{\nabla}_Y X) - F^2\overline{\nabla}_Y X.$$

$$\begin{aligned}(\overline{\nabla}_Y F)FX &= \overline{\nabla}_Y(X + u(X)\xi) - F(\overline{\nabla}_Y FX) + F(F\overline{\nabla}_Y X) - (\overline{\nabla}_Y X + u(\overline{\nabla}_Y X)\xi) \\ &= ((\overline{\nabla}_Y u)X)\xi + u(X)\overline{\nabla}_Y\xi - F((\overline{\nabla}_Y F)X + F(\overline{\nabla}_Y X)) + F(F\overline{\nabla}_Y X).\end{aligned}$$

$$(\overline{\nabla}_Y F)FX = ((\overline{\nabla}_Y u)X)\xi + u(X)\overline{\nabla}_Y\xi - F(\overline{\nabla}_Y F)X. \tag{3.4}$$

Theorem 3.1. In a nearly hyperbolic $\beta-$Kenmotsu manifold the Nijenhuis tensor $[F, F]$ of F is given by

$$\begin{aligned}[F, F](X, Y) = {} & 4F(\overline{\nabla}_Y F)X + 2du(X, u)\xi - u(X)\overline{\nabla}_Y\xi \\ & -u(Y)\overline{\nabla}_X\xi + \beta(u(Y)F^2X + 3u(X)F^2Y).\end{aligned} \tag{3.5}$$

Proof. In view of Lemma (3.1) and $u0F = 0$ in (1.6)

$$(\overline{\nabla}_{FX}F)Y = -(\overline{\nabla}_Y F)FX - \beta u(Y)F^2X.$$

$$(\overline{\nabla}_{FX}F)Y = F(\overline{\nabla}_Y F)X - (\overline{\nabla}_Y u)X)\xi + u(X)\overline{\nabla}_Y\xi - \beta u(Y)F^2X. \tag{3.6}$$

Therefore

$$
\begin{aligned}
[F,F](X,Y) &= ((\overline{\nabla}_{FX}F)Y - (\overline{\nabla}_{FY}F)X) - F((\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X) \\
&= F(\overline{\nabla}_Y F)X - ((\overline{\nabla}_Y u)X)\xi - u(X)\overline{\nabla}_Y\xi - \beta(u(FX)^o FY \\
&\quad + u(y)F^2X) - F(\overline{\nabla}_X F)Y + ((\overline{\nabla}_X u)Y)\xi - u(Y)\overline{\nabla}_X\xi \\
&\quad + \beta(u(FY)^o FX + u(X)F^2Y) + F(\overline{\nabla}_X F)Y + F(\overline{\nabla}_Y F)
\end{aligned}
$$

$$
\begin{aligned}
[F,F](X,Y) &= 2F[(\overline{\nabla}_Y F)X - (\overline{\nabla}_X F)Y] - ((\overline{\nabla}_Y u) + ((\overline{\nabla}_X u)Y)\xi \\
&\quad - u(X)\overline{\nabla}_Y\xi + u(Y)\overline{\nabla}_X\xi - \beta(u(Y)F^2X - u(X)F^2Y)
\end{aligned}
$$

$$
\begin{aligned}
[F,F](X,Y) &= 2F(\overline{\nabla}_Y F)X - 2F(\overline{\nabla}_X F)Y + 2du(X,Y)\xi \\
&\quad - u(X)\overline{\nabla}_Y\xi + u(Y)\overline{\nabla}_X\xi - \beta(u(Y)F^2X - u(X)F^2Y).
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
[F,F](X,Y) &= 2F(\overline{\nabla}_Y F)X - 2F(\overline{\nabla}_Y F)X - \beta(u(X)FY + u(Y)FX) \\
&\quad + 2du(X,Y)\xi - u(X)\overline{\nabla}_Y\xi + u(Y)\overline{\nabla}_X\xi - \beta(u(Y)F^2X - u(X)F^2Y) \\
&= 4F((\overline{\nabla}_Y F)X) + 2F\beta(u(X)FY - u(Y)FX) + 2du(X,Y)\xi \\
&\quad - u(X)\overline{\nabla}_Y\xi + u(Y)\overline{\nabla}_X\xi - \beta(u(Y)F^2X - u(X)F^2Y)
\end{aligned}
$$

$$
\begin{aligned}
[F,F](X,Y) &= 4F((\overline{\nabla}_Y F)X) + 2du(X,Y)\xi - u(X)\overline{\nabla}_Y\xi \\
&\quad + u(Y)\overline{\nabla}_X\xi + \beta(u(Y)F^2X + 3u(X)F^2Y).
\end{aligned}
$$

From (3.5) we obtain

$$
\begin{aligned}
u(N^1(X,Y) &= u([F,F](X,Y) + 2du(X,Y)\xi) \\
&= u[4F((\overline{\nabla}_Y F)X) + 2du(X,Y)\xi - u(X)\overline{\nabla}_Y\xi \\
&\quad + u(Y)\overline{\nabla}_X\xi + \beta(u(Y)F^2X + 3u(X)F^2Y) + 2du(X,Y)\xi
\end{aligned}
$$

$$
\xi u(N^1(X,Y) = -4du(X,Y).
$$

In particular, if X and Y are perpendicular to ξ then (3.6) gives

$$
[F,F](X,Y) = 4F(\overline{\nabla}_Y F)X - 2u(|X,Y|)\xi, \qquad \forall X, Y^{\perp}\xi.
$$

4. Some basic results

Taken M be a submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. Applying (2.4) and (2.6) in (1.6) $\forall X, Y \in TM$ then we find

$$
\begin{aligned}
&(\overline{\nabla}_X P)Y - A_{QY}X - Bh(X,Y) + ((\overline{\nabla}_X Q)Y + h(X,PY) - Ch(X,Y) + ((\overline{\nabla}_Y P)X \\
&\quad - A_{QX}Y - Bh(X,Y)) + ((\overline{\nabla}_Y Q)X + h(PX,Y) - Ch(X,Y) \\
&\quad = -\beta(u(X)PY + u(X)QY + u(Y)PX + u(Y)QX).
\end{aligned}
$$

$$(\nabla_X P)Y - A_{QY}X - 2Bh(X,Y) + \overline{\nabla}_X Q)Y - 2Ch(X,Y) + (\overline{\nabla}_Y P)X - A_{QX}Y$$
$$+(\nabla_Y Q)X + h(PX,Y) + h(X,PY)$$
$$= -\beta(y(X)PY + u(Y)PX + u(X)QY + u(Y)QX).$$

As a result, we have

Proposition 4.1. Let M be a submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. Then for all $X, Y \in TM$ we have

$$(\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2Bh(X,Y) = -\beta(u(X)PY + u(Y)PX). \tag{4.1}$$

$$(\nabla_X Q)Y + (\nabla_Y Q)X + h(X,PY) + h(PX,Y) - 2Ch(X,Y) = -\beta(u(X)QY + u(Y)QX). \tag{4.2}$$

Therefore, we have

Proposition 4.2. Let M be a submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. Then for all $X, Y \in TM$, we find that

$$\overline{\nabla}_X FY - \overline{\nabla}_Y FX - F[X,Y] \tag{4.3}$$
$$= 2((\nabla_X P)Y - A_{QY}X - Bh(X,Y)) + 2((\nabla_X Q)Y + h(X,PY)$$
$$-(h(X,Y)) + \beta(u(Y)PX + u(X)PY) + \beta(u(Y)QX + u(X)QY).$$

Therefore, we have

$$P[X,Y] = -\nabla_X PY - \nabla_Y PX + A_{QX}Y + A_{QY}X + 2P\nabla_X Y + 2Bh(X,Y) + \beta(u(Y)PX + u(X)PY), \tag{4.4}$$

$$Q[X,Y] = -\nabla^{\perp}_X QY - \nabla^{\perp}_Y QX - h(X,PY) - h(PX,Y) + 2Q\nabla_X Y + 2(h(X,Y) - \beta(u(Y)QX + u(X)QY). \tag{4.5}$$

Proposition 4.3. Let M be a semi-invariant submanifold of nearly hyperbolic $\beta-$Kenmotsu manifold. Then (P, ξ, u, g) is a nearly hyperbolic $\beta-$Kenmotsu structure on the distribution $D^1 \oplus \{\xi\}$ if $Bh(X,Y) = 0$, for all $X, Y \in D^1 \oplus \{\xi\}$.

Proof. In view of $D^1 \oplus \{\xi\} = \ker\ (F)$ and (2.10), we get $P^2 = 1 + u \oplus \xi$ on $D^1 \oplus \{\xi\}$ consequently we have

$$P\xi = 0,\ u(\xi) = -1,\ uoP = 0.$$

Applying $D^1 \oplus \{\xi\} = \ker\ (F)$ and $Bh(X,Y) = 0$ in (4.1), we obtain

(4.6) $$(\nabla_X P)Y + (\nabla_Y P)X = -\beta(u(X)PY + u(Y)PX).$$

Theorem 4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. we have

(a) If $D^o \oplus \{\xi\}$ is autoparallel then

$$A_{QX}Y + A_{QY}X + 2Bh(X,Y) = 0, \qquad X, Y \in D^o\{\xi\},$$

(b) If $D^1 \oplus \{\xi\}$ is autoparallel then

$$h(X, PY) + h(PX, Y) = 2Ch(X, Y), \qquad X, Y \in D^1 \oplus \{\xi\}.$$

Proof. From (4.1) and autoparallel of $D^o \oplus \{\xi\}$ we get (a). Consequently from (4.2) and autoparallel of $D^1 \oplus \{\xi\}$ we get (b).

Moreover proposition (4.3) and theorem (4.1), we have

Theorem 4.2. Let M be a submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold with $\xi \in TM$. If M is invariant then M is nearly hyperbolic $\beta-$Kenmotsu. Therefore

$$h(X, PY) + h(PX, Y) - 3Ch(X, Y) = 0, \qquad X, Y \in TM$$

Proof. From $D^1 \oplus \{\xi\} = \ker\ (F)$ and equation (4.2).

5. Integrability of the distribution $D^1 \oplus \{\xi\}$

Lemma 5.1. Let M be a semi-invariant submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. For $X, Y \in D^1 \oplus \{\xi\}$ we get

(5.1) $$Q[X, Y] = -h(X, PY) - h(PX, Y) + 2Q\nabla_X Y + 2Ch(X, Y).$$

Consequently

(5.2) $$-h(X, PX) + Q\nabla_X X + Ch(X, X) = 0.$$

Proof. Using $D^1 \oplus \{\xi\} = \ker\ \{Q\}$ and (4.5) we get equation (5.1) and applying $X = Y$ in equation (5.1) we get (5.2).

From (5.2) and $D^1 \oplus \{\xi\} = \ker\ \{Q\}$, we present the following theorem:

Theorem (5.1) : The distribution $D^1 \oplus \{\xi\}$ on a semi-invariant submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold is integrable if and only if

$$h(X, PY) + h(PX, Y) = 2(Q\nabla_X Y + Ch(X, Y)).$$

Then we required the following :

Definition 5.1. Let M be a Riemannian manifold with the Riemannian connection ∇. A distribution D on M will be called nearly autoparallel if for all $X, Y \in D$ we have $(\nabla_X Y + \nabla_Y X) \in D$ or equivalently $\nabla_X X \in D$.

Thus we get following flow chart

Parallel $\Rightarrow$ Autoparallel $\Rightarrow$ nearly autoparallel,

Parallel $\Rightarrow$ Integrable,

Autoparallel $\Rightarrow$ Integrable and

Nearly autoparallel + Integrable $\Rightarrow$ Autoparallel.

Theorem 5.2. Let M be a semi-invariant submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. Then the following four statements holds

(a) the distribution $D^1 \oplus \{\xi\}$ is autoparallel,

(b) $h(X, PY) + h(PX, Y) = 2Ch(X, Y), X, Y \in D^1 \oplus \{\xi\}$,

(c) $h(X, PX) = 2Ch(X, X)$, $X \in D^1 \oplus \{\xi\}$,

(d) the distribution $D^1 \oplus \{\xi\}$, is nearly autoparallel are related by (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). In particular if $D^1 \oplus \{\xi\}$ is integrable then the above four statements are equivalent

Proof. (a) $\Rightarrow$ (b) follows from theorem (5.1) putting $X = Y$ in (b) $\Rightarrow$ (c), from (4.8) we get (c) $\Rightarrow$ (d).

This completes the proof of theorem.

6. Integrability of the distribution $D^o \oplus \{\xi\}$

Lemma 6.1. Suppose M be a semi-invariant submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. Then

$$3(A_{FX}Y - A_{FY}X) = P[X, Y], \qquad \text{for } X, Y \in D^o \oplus \{\xi\}. \tag{6.1}$$

Proof. Suppose $X, Y \in D^o \oplus \{\xi\}$ and $Z \in TM$. Then we have

$$-A_{FX}Z + \nabla^{\perp}_Z FX = \overline{\nabla}_Z FX = (\overline{\nabla}_Z F)X + F\overline{\nabla}_Z X$$

$$= (\overline{\nabla}_X F)Z - u(X)FZ - u(Z)FX + F\nabla_Z X + Fh(Z, X).$$

Therefore

$$Fh(Z, X) = -A_{FX}Z + \nabla^{\perp}_Z FX + u(X)FZ - u(Z)FX - F\nabla_Z X + (\overline{\nabla}_X F)Z$$

and hence we have

$g(Fh(Z,X),Y) = -g(A_{FX}Z,Y)+g((\overline{\nabla}_X F)Z,Y)-g(A_{FX}Y,Z)-g((\overline{\nabla}_X F)Y,Z).$

It means that

$$g(Fh(Z,X),Y) = -g(h(Z,X),FY) = -g(A_{FY}X,Z).$$

Therefore from the above two relations , we get

$$g(A_{FX}X,Z) = g(A_{FX}Y,Z) + g((\overline{\nabla}_X F)Y,Z), \tag{6.2}$$

for $X,Y \in D^o \oplus \{\xi\}$ we drive $(\overline{\nabla}_X F)Y$ as followed

$$\overline{\nabla}_X FY - \overline{\nabla}_Y FX = A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX$$

and

$$\overline{\nabla}_X FY - \overline{\nabla}_Y FX = (\overline{\nabla}_X F)Y - (\overline{\nabla}_X F)X + F[X,Y].$$

We get

$$(\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X = A_{FX}Y - A_{FY}X + \overline{\nabla}^{\perp}_X FX - F[X,Y].$$

From (1.7) we get

(6.3) $(\overline{\nabla}_X F)Y = \frac{1}{2}(A_{FX}Y) - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y] - u(Y)FX - u(X)FY).$

In view of (6.2) and (6.3) we get the required result (6.1).

Theorem 6.1. Let M be a semi-invariant submanifold of nearly hyperbolic $\beta-$Kenmotsu manifold. Then the distribution $D^o \oplus \{\xi\}$, is integrable if and only if

$$A_{FX}Y = A_{FY}X, \qquad X,Y \in D^o \oplus \{\xi\}.$$

Proof. Taking (6.1) in (6.2) for $X,Y \in D^o \oplus \{\xi\}$ we obtain $A_{FX}Y = A_{FY}X$.

Corollary 6.1. Let M be a semi-invariat submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold. Then the distribution $D^o \oplus \{\xi\}$ is integrable.

7. Totally umbilical and totally geodesic submanifolds

Lemma 7.1. Let M be a submanifold of nearly hyperbolic $\beta-$Kenmotsu manifold, tangent to ξ. Then the integral curve of ξ in M is a geodesic in M, and ξ is an asymptotic direction.

Proof. Since in a nearly hyperbolic $\beta-$Kenmotsu manifold we have $\overline{\nabla}_\xi \xi = 0$. Now in view of equation (2.1) we get $h(\xi,,\xi) = 0$, this completes the proof.

Proposition 7.1. Let D be a distribution on a submanifold M of nearly hyperbolic $\beta-$Kenmotsu such that $\xi \in TM$. If $MD-$totally umbilical then M is $D-$totally geodesic.

Proof. Since $D-$totally umbilical we get

$$h(X,Y) = g(X,Y)K, \qquad \forall X,Y \in D$$

A direction ξ at a point of M is an asymptotic direction if normal vector field $K = 0$, which implies that $h(X,Y) = 0$, which shows that M is totally geodesic. In view of this proposition we get the theorem:

Theorem 7.1. Every totally umbilical submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold, tangent to u, is totally geodesic.

8. Totally hyperbolic contact umbilical and totally hyperbolic contact geodesic submanifolds

Let M be a submanifold of an almost hyperbolic contact metric manifolds, tangent to ξ. In this case $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by $\{u\}$ and $\{u\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M.

Definition 8.1. A submanifold M of an almost hyperbolic contact metric manifold, tangent to ξ is called [16].

(1) Totally hyperbolic contact umbilical if it is $\xi^{\perp}$ totally umbilical and

(2) Totally hyperbolic contact geodesic if it is $\{\xi\}^{\perp}$ totally geodesic.

The condition of totally hyperbolic contact umbilical and totally hyperbolic contact geodesic are respectively

$$h(F^2X, F^2Y) = g(F^2X, F^2Y)K, \qquad \forall X,Y \in TM, \tag{8.1}$$

$$h(F^2X, F^2Y) = 0, \qquad \forall X,Y \in TM \tag{8.2}$$

where K is normal vector field.

From (1) in (8.1), we have

$$h(X,Y) = -g(FX,FY)K - u(X)h(Y,\xi) - u(Y)h(X,\xi) \tag{8.3}$$

and

$$h(X,Y) = -u(X)h(Y,\xi) - u(Y)h(X,\xi) \text{ respectively.} \tag{8.4}$$

Theorem 8.1. It M is a totally hyperbolic contact umbilical semi-invariant submanifold of nearly hyperbolic $\beta-$Kenmotsu manifold then M is (D^1, D^o) mixed totally geodesic.

Proof. We have $h(X,Y) = g(X,Y)K$ and for $X, Y \in \{u\}^{\perp}$, $h(\xi,,\xi) = g(\xi,,\xi)K$

$$g(\xi,,\xi)K = 0, \text{ by using gauss equation } \Rightarrow K = 0.$$

Therefore M is (D^1, D^o) mixed totally geodesic.

Theorem 8.2. Let M be a totally hyperbolic contact umbilical submanifold of a nearly hyperbolic $\beta-$Kenmotsu manifold, then either $D^o = \{0\}$ or (Dim $D^o) = 1$ or the normal vector field K is orthogonal to FD^o.

Proof : If Dim $(D^o) > 1$, for each $H \in D^o$, $\exists X \in D^o$. Such that $g(X, H) = 0$ and $||X|| = 1$. Then

$$\begin{aligned} g(K, H) &= g(h(X, X), Qh) = g(A_{QH}X, X) \\ &= g(A_{QX}H, X) = g(h(X, H)QX) = 0. \end{aligned}$$

This prove theorem.

References

[1] Bejancu, A. : On semi-invariant submanifolds of an almost contact metric manifold., An. Stint. Univ. "Al. L. Cuza" Lasi Sect. la Mab., 27 (1987), 17.

[2] Srivastava, Asha and Bhatia, Kamal : CR-submanifold of a hyperbolic trans Sasakian manifold (under publication).

[3] Srivastava, Asha : invariant submanifolds in hyperbolic Riemannian manifolds, J.T.S.I., 21 (2003).

[4] Parsad, R., Sharma Shiv and Rai, A. K. : On Semi-invariant submanifolds of nearly trans-Sasakian manifolds., Bull. Cal. Math. Soc., 98(4) (2006), 347-366 .

[5] Blair, D. E. : Contact manifold in Riemannian geometry, lecture notes in Math, 509, Springer Verlag (1976).

[6] Bejancu, A. : Geometry of CR-submanifold, D.Reidel Publi.Co., Holland (1986).

[7] Matsumoto, K. Shahid, M. H. and Mihai, I. : Semi-invariant submanifolds of certain almost contact manifold. Bull. Yamagata Univ. Natur. Sci. 13-3 (1994), 183.

[8] Srivastava, Asha and Uperti, Jaya : CR-Submanifold of a hyperbolic Kahler manifold, J.T.S.I, 17 (1999).

[9] Blair, D. E., Showers, D. K. and Yano, K. : Nearly Sasakian structures, Kodai Math. Sem. Rep., 27 (1976),175.

[10] Matsumoto, K. : Semi-invariant submanifolds of certain class of almost contact manifold., Tensor, 43 (1986), 28.

[11] Dube, K. K. and Bhatt, Latika : On CR-submanifold of trans hyperbolic Sasakian manifold. Acta ciencia indica, XXIXM (2003), 091.

[12] Mishra, R. S. : Almost contact metric manifolds, Monograph1, Tensor Soc. of india, Lucknow (1991).

[13] Blair, D. E. and Oubina, J. A. : Conformal and related changes of metric on the product of two almost contact metric manifolds., Publications Matematiqes, 34 (1990), 199.

[14] Calin, C. : Contact CR-submanifolds of a nearly Sasakian manifold, Demonstratio math., 29 (1996), 337.

[15] Shahid, M. H. : CR-submanifold of trans Sasakian manifold., Indian J. Pure Appl. Math., 22(12)(1991), 1007-1012.

[16] Tripathi, M. M. : A note on certain Legendre curves in a Kenmotsu manifold, Ganita, 51, No. 1 (2000), 57.