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Differential Sandwich Results for Certain Subclasses of Analytic Multivalent Functions

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In the present investigation, we study applications of the theory of differential subordination and superordination, that are connected to Wright's generalized hypergeometric function. Relevant connections of the results are noted and the new results are also pointed out.

Keywords and Phrases : Univalent functions, subordination, superordination, Hadamard product (convolution), Wright's generalized hypergeometric functions.
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1. Introduction

Let $\mathbb{H}$ be the class of analytic functions in the unit disk $\mathbb{U} := \{z : |z| < 1\}$ and let $\mathbb{H}[a, p]$ be the subclass of $\mathbb{H}$ consisting of functions of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots, \quad p \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

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Let $\mathbb{A}(p)$ be the subclass of $\mathbb{H}$ consisting of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in \mathbb{N}. \tag{1.1}$$

For simplicity, let $\mathbb{H}[a] = \mathbb{H}[a, 1]$. Also, let $\mathbb{A}(1) \equiv A$ be the subclass of $\mathbb{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

For f and F be members of the function class $\mathbb{H}$, the function f is said to be subordinate to F in $\mathbb{U}$, or the function F is said to be superordinate to f in $\mathbb{U}$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = F(w(z))$ $(z \in \mathbb{U})$. In such a case, we write $f \prec F$ $(z \in \mathbb{U})$ or $F \prec f$ $(z \in \mathbb{U})$. Furthermore if the function F is univalent in $\mathbb{U}$, then we have that the following equivalence holds (see [10, 17]);

$$f \prec F(z \in \mathbb{U}) \Leftrightarrow f(0) = F(0) \,\text{and}\, f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination

$$\phi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \ (z \in \mathbb{U}), \tag{1.3}$$

then p is called a solution of the differential subordination (1.3). The univalent function q is called a dominant of the solutions of the differential subordination (1.3) or, more simply, a dominant if $p \prec q$ $(z \in \mathbb{U})$ for all p satisfying (1.3) is said to be the best dominant. A dominant $\widetilde{q}$ that satisfies the subordination relationship $\widetilde{q} \prec p$ $(z \in \mathbb{U})$ for all dominants q of (1.3) is said to be the best dominant.

Let $\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in $\mathbb{U}$ and satisfying the following differential superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \ (z \in \mathbb{U}), \tag{1.4}$$

then p is called a solution of the differential superordination (1.4). An analytic function q is called a subordinant of the solutions of the differential superordination (1.4) or, more simply, a subordinant if $q \prec p$ $(z \in \mathbb{U})$ for all p satisfying (1.4). A univalent subordinant $\widetilde{q}$ that satisfies the subordination relationship $q \prec \widetilde{q}$ $(z \in \mathbb{U})$ for all subordinants q of (1.4) is said to be the best subordinant.

Recently, Bulboacă [6] (see also [5]) considered certain classes of first order differential superordinations as well as superordination-preserving integral operators by using the results of Miller and Mocanu[18]. Further, many researchers ([16, 19, 23]) have obtained sufficient conditions on normalized analytic functions f(z) by means of differential subordinations and superordinations. Also the results have been extended and discussed for multivalent functions by Aouf and Bulboacă [1], Aouf et al., [4], Cho et al., [7] [9] and Goyal et al., [13] and others.

For functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.5}$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ $(l, m \in \mathbb{N} = 1, 2, 3, ...)$ such that

$$1 + \sum_{n=1}^{m} B_m - \sum_{n=1}^{l} A_n \geq 0 \tag{1.6}$$

the Wright generalization [26]

$${}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z]$$

$$= {}_l\Psi_m[(\alpha_n, A_n)_{1,l}(\beta_n, B_n)_{1,m}; z]$$

of the hypergeometric function ${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$ is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \sum_{n=0}^{\infty}\{\prod_{t=0}^{l}\Gamma(\alpha_t + nA_t\}\{\prod_{t=0}^{m}\Gamma(\beta_t + nB_t\}^{-1}\frac{z^n}{n!}, z \in \mathbb{U}.$$

If $A_t = 1(t = 1, 2, ..., l)$ and $B_t = 1(t = 1, 2, ..., m)$ we have the relationship:

$$\begin{aligned} \Omega_l\Psi_m[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z] &\equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \end{aligned} \tag{1.7}$$

$(l \leq m + 1;\ l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U})$ is the generalized hypergeometric function (see for details [26]) where $\mathbb{N}$ denotes the set of all positive integers and $(\lambda)_n$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t)\right). \tag{1.8}$$

By using the generalized hypergeometric function Dziok and Srivastava [11] introduced the linear operator which was subsequently extended by Dziok and Raina [12] (see also [3]) by using the Wright generalized hypergeometric function.

Let $\Theta_p[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : \mathbb{A}(p) \to \mathbb{A}(p)$ be a linear operator defined by

$$\Theta_p[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) \quad := \quad z^p \; \Omega_l\Psi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z).$$

We observe that, for $f(z)$ of the form (1.1), we have

$$\Theta_p[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) = z^p + \sum_{n=p+1}^{\infty} \sigma_{n,p}(\alpha_1) \; a_n z^n \tag{1.9}$$

where $\sigma_{n,p}(\alpha_1)$ is defined by

$$\sigma_{n,p}(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + A_1(n-p)) \ldots \Gamma(\alpha_l + A_l(n-p))}{(n-p)!\Gamma(\beta_1 + B_1(n-p)) \ldots \Gamma(\beta_m + B_m(n-p))} \tag{1.10}$$

and Ω is given by (1.8).

If, for convenience, we write

$$\Theta_p[\alpha_1]f(z) = \Theta_p[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z) \tag{1.11}$$

introduced by Dziok and Raina [12]. In view of (1.9), we get,

$$zA_1(\Theta_p[\alpha_1]f(z))' = \alpha_1\Theta_p[\alpha_1 + 1]f(z) - (\alpha_1 - pA_1)\Theta_p[\alpha_1]f(z), \;\; A_1 > 0. \tag{1.12}$$

We observe that for $A_t = 1(t = 1, 2, ..., l)$ and $B_t = 1(t = 1, 2, ..., m)$, we obtain Dziok-Srivastava linear operator [11]. Also for $f \in \mathbb{A}$, the linear operator $\Theta_1[\alpha_1] = \Theta[\alpha_1]$ was introduced by Dziok and Raina [12] and extensively studied by others [3, 4, 16].

Further, we note that for $f \in \mathbb{A}(p)$, $A_t = 1(t = 1, 2, ..., l)$, $B_t = 1(t = 1, 2, ..., m)$, $l = 2$ and $m = 1$, we have the following well known operators.

(1) For $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$, we have $\Theta_p[a, 1; c]f(z) = L_p(a, c)f(z)$ $(a > 0; c > 0; p \in \mathbb{N})$ introduced and studied by [22].

(2) For $\alpha_1 = \delta + p$, $\alpha_2 = p$ and $\beta_1 = p$, we get $\Theta_p[\delta + p, p; p]f(z) = D^{\delta+p-1}f(z)$ $(\delta > -p; p \in \mathbb{N})$ where $D^{\delta+p-1}$ is $\delta + p - 1$-th order of Ruscheweyh derivative, introduced and studied by [14].

(3) For $\alpha_1 = 1+p$, $\alpha_2 = 1$ and $\beta_1 = 1+p-\delta$, we get $\Theta_p[1+p, 1; 1+p-\delta]f(z) = \Omega_z^{\delta,p}f(z)$ where $\Omega_z^{\delta,p}$ was introduced and studied by Srivastava and Aouf [24], defined by

$$\Omega_z^{\delta,p}f(z) = z^{\delta}D_z^{\delta} \quad (0 \leq \delta < 1; p \in \mathbb{N}),$$

where D_z^{δ} is the fractional derivative operator.

(4) For $\alpha_1 = c + p$, $\alpha_2 = 1$ and $\beta_1 = c + p + 1$, we get $\Theta_p[c + p, 1; c + p + 1]f(z) = J_{c,p}f(z)$ where $J_{c,p}$ is the generalized Bernardi-Libera-Livingston-integral operator, defined by

$$J_{c,p}f(z) := \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c > -p; p \in \mathbb{N}).$$

(5) For $\alpha_1 = p+1$, $\alpha_2 = 1$ and $\beta_1 = n+p$, we have $\Theta_p[p+1, 1; n+p]f(z) = I_{n,p}f(z)$ $(n \in Z; n > -p; p \in \mathbb{N})$ where the operator $I_{n,p}$ was introduced and studied by Liu and Noor [15].

(6) For $\alpha_1 = \eta + p$, $\alpha_2 = c$ and $\beta_1 = a$, we have $\Theta_p[\eta + p, c; a]f(z) = I_p^{\eta}f(z)$ $(a, c \in R \setminus \overline{Z}_0; \eta > -p; p \in \mathbb{N})$ where the operator I_n^{η} is Cho-Kwon-Srivastava operator [8].

The main object of the present paper is to find a sufficient condition for certain normalized analytic functions $f(z)$ in $\mathbb{U}$ such that $(f * \Psi)(z) \neq 0$ and f satisfy

$$q_1(z) \prec \left(\frac{\alpha\Theta_p[\alpha_1 + 1](f * \Phi)(z) + \beta\Theta_p[\alpha_1](f * \Psi)(z)}{(\alpha + \beta)z^p}\right)^{\mu} \prec q_2(z), \tag{1.13}$$

where q_1, q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = 1$, $q_2(0) = 1$ and $\Phi(z) = z^p + \sum\limits_{n=p+1}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=p+1}^{\infty} \mu_n z^n$ are analytic functions in $\mathbb{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Also, we obtain the number of known results as their special cases.

For our present investigation, we shall need the following:

Lemma 1.1. [21, p.159, Theorem 6.2] The function $L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = +\infty$, is a subordination chain if

$$\Re\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}>0, z\in\mathbb{U},\ t\geq 0.$$

Definition 1.2. [18, p.817, Definition 2] Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}-\mathbb{E}(f)$, where

$$\mathbb{E}(f)=\{\zeta\in\partial\mathbb{U}:\lim_{z\to\zeta}f(z)=\infty\}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial\mathbb{U}-\mathbb{E}(f)$.

Lemma 1.3. [17, p.132, Theorem 3.4h] Let q be univalent in the unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w)\neq 0$ when $w\in q(\mathbb{U})$. Set

$$Q(z):=zq'(z)\phi(q(z))\quad \textit{and}\quad h(z):=\theta(q(z))+Q(z).$$

Suppose that

(1) $Q(z)$ is starlike univalent in $\mathbb{U}$ and
(2) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\}>0$ for $z\in\mathbb{U}$.

If p is analytic with $p(0)=q(0)$, $p(\mathbb{U})\subseteq\mathbb{D}$ and

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z)), \tag{1.14}$$

then

$$p(z)\prec q(z)$$

and q is the best dominant.

Lemma 1.4. [6, p.289, Corollary 3.2] Let q be convex univalent in the unit disk $\mathbb{U}$ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$. Suppose that

(1) $\Re\{\vartheta'(q(z))/\varphi(q(z))\}>0$ for $z\in\mathbb{U}$ and
(2) $\psi(z)=zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z)\in\mathbb{H}[q(0),1]\cap Q$, with $p(\mathbb{U})\subseteq\mathbb{D}$, and $\vartheta(p(z))+zp'(z)\varphi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z))+zq'(z)\varphi(q(z))\prec\vartheta(p(z))+zp'(z)\varphi(p(z)), \tag{1.15}$$

then $q(z)\prec p(z)$ and q is the best subordinant.

2. Subordination results

Using Lemma 1.3, we first prove the following theorem.

Theorem 2.1. Let $\Phi, \Psi \in \mathbb{A}(p)$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathbb{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and assume that

$$\Re \left\{ \frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0 \quad (z \in \mathbb{U}). \tag{2.1}$$

If $f \in \mathbb{A}(p)$ satisfies

$$\begin{aligned} &\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) = \\ &\qquad \Delta(f,\ \Phi, \Psi,\ \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) \\ &:= \begin{cases} \gamma_1 + \gamma_2 \left(\dfrac{\alpha\Theta_p[\alpha_1 + 1](f * \Phi)(z) + \beta\Theta_p[\alpha_1](f * \Psi)(z)}{(\alpha + \beta)z^p} \right)^{2\mu} \\ +\gamma_3 \left(\dfrac{\alpha\Theta_p[\alpha_1 + 1](f * \Phi)(z) + \beta\Theta_p[\alpha_1](f * \Psi)(z)}{(\alpha + \beta)z^p} \right)^{\mu} \\ +\dfrac{\gamma_4 \mu}{A_1} \left(\dfrac{\alpha(\alpha_1 + 1)[\Theta_p[\alpha_1 + 2](f * \Phi)(z) - \Theta_p[\alpha_1 + 1](f * \Phi)(z)]}{\alpha\Theta_p[\alpha_1 + 1](f * \Phi)(z) + \beta\Theta_p[\alpha_1](f * \Psi)(z)} \right. \\ \left. \dfrac{+\beta\alpha_1[\Theta_p[\alpha_1 + 1](f * \Psi)(z) - \Theta_p[\alpha_1](f * \Psi)(z)]}{} \right), \end{cases} \end{aligned} \tag{2.3}$$

and $A_1 > 0$, then

$$\left(\frac{\alpha\Theta_p[\alpha_1 + 1](f * \Phi)(z) + \beta\Theta_p[\alpha_1](f * \Psi)(z)}{(\alpha + \beta)z^p} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) := \left(\frac{\alpha\Theta_p[\alpha_1 + 1](f * \Phi)(z) + \beta\Theta_p[\alpha_1](f * \Psi)(z)}{(\alpha + \beta)z^p} \right)^{\mu} \quad (z \in \mathbb{U}). \tag{2.4}$$

Then the function p is analytic in $\mathbb{U}$ and $p(0) = 1$. Therefore, by making use of (2.4) and (1.12), we obtain

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)}$$

$$= \begin{cases} \gamma_1 + \gamma_2 \left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z) + \beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p} \right)^{2\mu} \\ +\gamma_3 \left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z) + \beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p} \right)^{\mu} \\ +\frac{\gamma_4\mu}{A_1} \left(\frac{\alpha(\alpha_1+1)[\Theta_p[\alpha_1+2](f*\Phi)(z) - \Theta_p[\alpha_1+1](f*\Phi)(z)]}{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z) + \beta\Theta_p[\alpha_1](f*\Psi)(z)} \right. \\ \left. \frac{+\beta\alpha_1[\Theta_p[\alpha_1+1](f*\Psi)(z) - \Theta_p[\alpha_1](f*\Psi)(z)]}{} \right). \end{cases} \tag{2.5}$$

By using (2.5) in (2.2), we have

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)} \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}. \tag{2.6}$$

By setting

$$\theta(w) := \gamma_1 + \gamma_2 \omega^2(z) + \gamma_3 \omega \quad \textit{and} \quad \phi(\omega) := \frac{\gamma_4}{w},$$

it can be easily observed that $\theta(w)$, $\phi(w)$ are analytic in $\mathbb{C} - \{0\}$ and $\phi(w) \neq 0$. Also we see that

$$Q(z) := zq'(z)\phi(q(z)) = \gamma_4 \frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$ and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

By the hypothesis of Theorem 2.1, the result now follows by an application of Lemma 1.3.

By fixing $\Phi(z) = \frac{z^p}{1-z}$ and $\Psi(z) = \frac{z^p}{1-z}$ $(or, \mu_n = \lambda_n = 1, n \geq p+1, p \in \mathbb{N})$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. Let $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathbb{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (2.1) holds true. If $f \in \mathbb{A}(p)$ satisfies

$$\gamma_1 + \gamma_2 \left(\frac{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p} \right)^{2\mu}$$
$$+\gamma_3 \left(\frac{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p} \right)^{\mu}$$
$$+\frac{\gamma_4\mu}{A_1} \left(\frac{\alpha(\alpha_1+1)[\Theta_p[\alpha_1+2]f(z) - \Theta_p[\alpha_1+1]f(z)] + \beta\alpha_1[\Theta_p[\alpha_1+1]f(z) - \Theta_p[\alpha_1]f(z)]}{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)} \right)$$
$$\prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)},$$

and $A_1 > 0$, then

$$\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

By taking $A_t = 1$ $(t = 1, \ldots, l)$ and $B_t = 1$ $(t = 1, \ldots, m)$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \Psi(z) = \frac{z^p}{1-z}$ in Theorem 2.1, we state the following corollary.

Corollary 2.3. Let $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathbb{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (2.1) holds true. If $f \in \mathbb{A}(p)$ satisfies

$$\gamma_1 + \gamma_2 \left(\frac{\alpha z f'(z) + \beta f(z)}{(\alpha+\beta)z^p} \right)^{2\mu} + \gamma_3 \left(\frac{\alpha z f'(z) + \beta f(z)}{(\alpha+\beta)z^p} \right)^{\mu}$$
$$+\gamma_4\mu \left(\frac{\alpha[z^2 f''(z) + \beta[zf'(z) - f(z)]}{\alpha z f'(z) + \beta f(z)} \right)$$
$$\prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\left(\frac{\alpha z f'(z) + \beta f(z)}{(\alpha+\beta)z^p} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

By fixing $\alpha = 1$ and $\beta = 0$ in Corollary 2.3, we obtain the following corollary.

Corollary 2.4. Let $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu \in \mathbb{C}$ such that $\mu \neq 0$ and q be convex univalent with $q(0) = 1$, and (2.1) holds true. If $f \in \mathbb{A}(p)$ satisfies

$$\gamma_1 + \gamma_2 \left(f'(z)\right)^{2\mu} + \gamma_3 \left(f'(z)\right)^{\mu} + \gamma_4 \mu \frac{zf''(z)}{f'(z)} \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\left(\frac{f'(z)}{z^{p-1}}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.1, we have the following corollary.

Corollary 2.5. Let $\Phi, \Psi \in \mathbb{A}(p)$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathbb{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$. Assume that

$$\Re\left\{\frac{\gamma_3}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\gamma_2}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{1-ABz^2}{(1+Az)(1+Bz)}\right\} > 0.$$

If $f \in \mathbb{A}(p)$ and

$$\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi) \prec \gamma_1 + \gamma_2\left(\frac{1+Az}{1+Bz}\right)^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z) + \beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

3. Superordination results

Now, by applying Lemma 1.4, we prove the following theorem.

Theorem 3.1. Let $\Phi, \Psi \in \mathbb{A}(p)$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathbb{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and assume that

$$\Re\left\{\frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z)\right\} \geq 0. \tag{3.1}$$

If $f \in \mathbb{A}(p)$, $\left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu} \in \mathbb{H}[q(0),1] \cap Q$. Let $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ be univalent in $\mathbb{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi), \tag{3.2}$$

where $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ is given by (2.3), then

$$q(z) \prec \left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu}$$

and q is the best subordinant.

Proof. Define the function p by

$$p(z) := \left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu}. \tag{3.3}$$

With simple computation from (3.3), we get,

$$\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)}.$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w^2 + \gamma_3 w$ and $\phi(w) = \frac{\gamma_4}{w}$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(w) \neq 0$.

If we let

$$\begin{aligned} L(z,t) &= \vartheta(q(z)) + \phi(q(z))tzq'(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 t\frac{zq'(z)}{q(z)} \\ &= a_1(t)z + \dots \end{aligned} \tag{3.4}$$

Differentiating (3.4) with respect to z and t, we have

$$\begin{aligned} \frac{\partial L(z,t)}{\partial z} &= 2\gamma_2 q(z)q'(z) + \gamma_3 q'(z) + t\gamma_4\left[\frac{zq''(z)}{q(z)} + \frac{q'(z)}{q(z)} - z\left(\frac{q'(z)}{q(z)}\right)^2\right] \\ &= a_1(t) + \dots \end{aligned}$$

and

$$\frac{\partial L(z,t)}{\partial t} = \gamma_4 \frac{zq'(z)}{q(z)}.$$

Also,

$$\frac{\partial L(0,t)}{\partial z} = \gamma_4 q'(0)\left[\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(0) + t\frac{1}{q(0)}\right].$$

From the univalence of q we have $q'(0) \neq 0$ and $q(0) = 1$, it follows that $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = +\infty$.

A simple computation yields,

$$\Re\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - zq'(z)\right)\right\}.$$

Using the fact that q is convex univalent function in $\mathbb{U}$ and $\gamma_4 \neq 0$, we have,

$$\Re\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \text{ if } \Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z)\right\} > 0, \quad z \in \mathbb{U}, \ t \geq 0.$$

Now Theorem 3.1 follows by applying Lemma 1.4.

By fixing $\Phi(z) = \frac{z^p}{1-z}$ and $\Psi(z) = \frac{z^p}{1-z}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. Let $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathbb{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (3.1) holds true. If $f \in \mathbb{A}(p)$, $\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z)+\beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p}\right)^\mu \in \mathbb{H}[q(0), 1] \cap Q$. Let

$$\gamma_1 + \gamma_2\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p}\right)^{2\mu}$$

$$+\gamma_3\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p}\right)^{\mu}$$

$$+\frac{\gamma_4\mu}{A_1}\left(\frac{\alpha(\alpha_1+1)[\Theta_p[\alpha_1+2]f(z) - \Theta_p[\alpha_1+1]f(z)]}{\alpha\Theta_p[\alpha_1+1]f(z) + \beta\Theta_p[\alpha_1]f(z)}\right.$$

$$\left.\frac{+\beta\alpha_1[\Theta_p[\alpha_1+1]f(z) - \Theta_p[\alpha_1]f(z)]}{}\right)$$

be univalent in $\mathbb{U}$ and

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4\frac{zq'(z)}{q(z)}\prec$$
$$\gamma_1+\gamma_2\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z)+\beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p}\right)^{2\mu}$$
$$+\gamma_3\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z)+\beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p}\right)^{\mu}$$
$$+\frac{\gamma_4\mu}{A_1}\left(\frac{\alpha(\alpha_1+1)[\Theta_p[\alpha_1+2]f(z)-\Theta_p[\alpha_1+1]f(z)]}{\alpha\Theta_p[\alpha_1+1]f(z)+\beta\Theta_p[\alpha_1]f(z)}\right.$$
$$\left.\frac{+\beta\alpha_1[\Theta_p[\alpha_1+1]f(z)-\Theta_p[\alpha_1]f(z)]}{}\right),$$

then
$$q(z)\prec\left(\frac{\alpha\Theta_p[\alpha_1+1]f(z)+\beta\Theta_p[\alpha_1]f(z)}{(\alpha+\beta)z^p}\right)^{\mu}$$
and q is the best subordinant.

When $A_t=1$ $(t=1,\ldots,l)$ and $B_t=1$ $(t=1,\ldots,m)$, $l=2$, $m=1$, $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$ $\Phi(z^p)=\Psi(z^p)=\frac{z}{1-z}$ in Theorem 3.1 with $\alpha=1$ and $\beta=0$, we derive the following corollary.

Corollary 3.3. Let $\gamma_i\in\mathbb{C}$ $(i=1,\ldots,4)(\gamma_4\neq 0)$, $0\neq\mu\in\mathbb{C}$ and q be convex univalent with $q(0)=1$, and (3.1) holds true. If $f\in\mathbb{A}(p)$, $(f'(z))^{\mu}\in\mathbb{H}[q(0),1]\cap Q$. Let $\gamma_1+\gamma_2\left(f'(z)\right)^{2\mu}+\gamma_3\left(f'(z)\right)^{\mu}+\gamma_4\mu\frac{zf''(z)}{f'(z)}$ be univalent in $\mathbb{U}$ and

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4\frac{zq'(z)}{q(z)}\prec\gamma_1+\gamma_2\left(f'(z)\right)^{2\mu}+\gamma_3\left(f'(z)\right)^{\mu}+\gamma_4\mu\frac{zf''(z)}{f'(z)},$$

then $q(z)\prec\left(\frac{f'(z)}{z^{p-1}}\right)^{\mu}$ and q is the best subordinant.

By taking $q(z)=(1+Az)/(1+Bz)$ $(-1\leq B<A\leq 1)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.4. Let $\Phi,\Psi\in\mathbb{A}(p)$, $\gamma_i\in\mathbb{C}$ $(i=1,\ldots,4)(\gamma_4\neq 0)$, $\mu,\alpha,\beta\in\mathbb{C}$ such that $\mu\neq 0$ and $\alpha+\beta\neq 0$, and q be convex univalent with $q(0)=1$, and $\Re\left\{\frac{\gamma_3}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right)+\frac{2\gamma_2}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right)^2\right\}>0$. If

$$f\in\mathbb{A}(p),\ \left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu}\in\mathbb{H}[q(0),1]\cap Q.$$

Let $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ be univalent in $\mathbb{U}$ and

$$\gamma_1+\gamma_2(\frac{1+Az}{1+Bz})^2+\gamma_3\frac{1+Az}{1+Bz}+\gamma_4\frac{(A-B)z}{(1+Az)(1+Bz)}\prec\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^\mu$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

4. Sandwich results

There is a complete analog of Theorem 2.1 for differential subordination and Theorem 3.1 for differential superordination. We can combine the results of Theorem 2.1 with Theorem 3.1 and obtain the following sandwich theorem.

Theorem 4.1. Let q_1 and q_2 be convex univalent in $\mathbb{U}$, $\gamma_i\in\mathbb{C}$ $(i=1,\ldots,4)(\gamma_4\neq 0)$, $\mu,\alpha,\beta\in\mathbb{C}$ such that $\mu\neq 0$ and $\alpha+\beta\neq 0$, and let q_2 satisfy (2.1) and q_1 satisfy (3.1). For $f,\Phi,\Psi\in\mathbb{A}(p)$, let $\left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^\mu\in\mathbb{H}[1,1]\cap Q$ and $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ defined by (2.3) be univalent in $\mathbb{U}$ satisfying

$$\gamma_1+\gamma_2q_1^2(z)+\gamma_3q_1(z)+\gamma_4\frac{zq_1'(z)}{q_1(z)}\prec\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)\prec\gamma_1+\gamma_2q_2^2(z)+\gamma_3q_2(z)+\gamma_4\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\left(\frac{\alpha\Theta_p[\alpha_1+1](f*\Phi)(z)+\beta\Theta_p[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^\mu\prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and best dominant.

By taking $q_1(z)=\frac{1+A_1z}{1+B_1z}$ $(-1\leq B_1<A_1\leq 1)$ and $q_2(z)=\frac{1+A_2z}{1+B_2z}$ $(-1\leq B_2<A_2\leq 1)$ in Theorem 4.1 we obtain the following result.

Corollary 4.2. For $f,\Phi,\Psi\in\mathbb{A}(p)$, let $\left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^\mu\in\mathbb{H}[1,1]\cap Q$ and $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ defined by (2.3) be univalent in $\mathbb{U}$ satisfying

$$\gamma_1+\gamma_2(\frac{1+A_1z}{1+B_1z})^2+\gamma_3\frac{1+A_1z}{1+B_1z}+\gamma_4\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}\prec\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)\prec\gamma_1+\gamma_2(\frac{1+A_2z}{1+B_2z})^2+\gamma_3\frac{1+A_2z}{1+B_2z}+\gamma_4\frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$$

then

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z^p}\right)^{\mu} \prec \frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ are respectively the best subordinant and best dominant.

5. Remarks and Conclusion

We remark that, one can easily restate Theorem 4.1 for the different choices of $\Phi(z)$, $\Psi(z)$, A_t, B_t, l, m, $\alpha_1, \alpha_2, \ldots \alpha_l$, $\beta_1, \beta_2, \ldots \beta_m$ and for $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

Remark 5.1.

(1) Putting $p = 1$ $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\alpha = 1$, $\beta = 0$, $q(z) = \frac{1}{(1-z)^{2ab}}$ $(b \in \mathbb{C} \setminus \{0\})$, $\mu = a$ and $\gamma_4 = \frac{1}{b}$ in Corollary 2.3, we get the result obtained by Obradovič et al., [20, Theorem 1].

(2) Putting $p = 1$ $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$ $\alpha = 0$, $\beta = 1$, $q(z) = \frac{1}{(1-z)^{2b}}$ $(b \in \mathbb{C} \setminus \{0\})$, $\mu = 1$ and $\gamma_4 = \frac{1}{b}$ in Corollary 2.3 and then combining this together with Lemma 1.3, we obtain the result of Srivastava and Lashin [25, Theorem 3].

(3) Taking $p = 1$ $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$ $\alpha = 0$, $\beta = 1$, $\gamma_4 = \frac{e^{i\lambda}}{abcos\lambda}$ $(a, b \in \mathbb{C}, |\lambda| < \frac{\pi}{2})$, $\mu = a$ and $q(z) = (1-z)^{-2abcos\lambda e^{-i\lambda}}$ in Corollary 2.3, we obtain the result of Aouf et al. [2, Theorem 1].

(4) By taking $A_t = 1$ $(t = 1, \ldots, l)$ and $B_t = 1$ $(t = 1, \ldots, m)$, $p = \beta = \gamma_1 = 1$, and $\gamma_2 = \gamma_3 = \alpha = 0$, in Corollary 2.2, we have the result obtained by the second author [19, Theorem 3.5].

(5) By setting taking $A_t = 1$ $(t = 1, \ldots, l)$ and $B_t = 1$ $(t = 1, \ldots, m)$, $l = 2$, $m = 1$, $\alpha_1 = \alpha_2 = \beta_1 = 1$ $\alpha = \gamma_2 = \gamma_3 = 0, \beta = \gamma_1 = p = 1$, $\Psi(z) = \frac{z}{1-z}$ and $q(z) = (1+Bz)^{\mu(A-B)/B}$ in Corollary 2.5, we get the result obtained by Goyal et al., [13, Corollary 3.6].

We conclude this paper by remarking that in view of the function class defined by the subordination relation (1.13) and expressed in terms of the convolution (1.5) involving arbitrary coefficients, the main results would lead to additional new results. In fact, by appropriately selecting the arbitrary sequences ($\Phi(z)$ and $\Psi(z)$) and specializing the parameters $A_t = 1$ $(t = 1, \ldots, l)$ and $B_t = 1$ $(t = 1, \ldots, m)$, l, m, α, β, μ, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 and the function $q(z)$ the results presented in this paper would find further applications for the classes which incorporate generalized forms of linear operators in Theorem 2.1, Theorem 3.1 and Theorem 4.1 would eventually lead further new results. These considerations can fruitfully be worked out and we skip the details in this regard.

References

[1] Aouf, M. K. and Bulboacă, T. : Subordination and superordination properties of multivalent functions defined by certain integral operator, J. Franklin Inst., 347 no. 3 (2010), 641–653.

[2] Aouf, M. K., Al-Oboudi, F. M. and Haidan, M. M. : On some results for λ-spirallike and λ-Robertson functions of complex order, Publ. Inst. Math., (Beograd) (N.S.) 77(91) (2005), 93–98.

[3] Aouf, M. K. and Dziok, J. : Certain class of analytic functions associated with the Wright generalized hypergeometric function, J. Math. Appl., 30 (2008), 23–32.

[4] Aouf, M. K., Shamandy, A., Mostafa, A. O. and Madian, S. M. : Sandwich theorems for analytic functions involving Wrights generalized hypergeometric function, Acta Universitatis Apulensis, 26 (2011), 267–288.

[5] Bulboacă, T. : A class of superordination-preserving integral operators, Indag. Math. (N.S.), 13 no. 3 (2002), 301–311.

[6] Bulboacă, T. : Classes of first-order differential superordinations, Demonstratio Math., 35 no. 2 (2002), 287–292.

[7] Cho, N. E., Kim, I. H. and Srivastava, H. M. : Sandwich-type theorems for multivalent functions associated with the Srivastava-Attiya operator, Appl. Math. Comput., 217 no. 2 (2010), 918–928.

[8] Cho, N. E., Kwon, O. S. and Srivastava, H. M. : Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292 no. 2 (2004), 470–483.

[9] Cho, N. E., et al., : Subordination and superordination for multivalent functions associated with the Dziok-Srivastava operator, J. Inequal. Appl., 2011, Art. ID 486595, 17 pp.

[10] Duren, P. L. : Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[11] Dziok, J. and Srivastava, H. M. : Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput., 103 no. 1 (1999), 1–13.

[12] Dziok, J. and Raina, R. K. : Families of analytic functions associated with the Wright generalized hypergeometric function, Demonstratio Math., 37 no. 3, (2004), 533–542.

[13] Goyal, S. P., Goswami, P. and Silverman, H. : Subordination and superordination results for a class of analytic multivalent functions, Int. J. Math. Math. Sci., 2008, Art. ID 561638, 12 pp.

[14] Kumar, V. and Shukla, S. L. : Multivalent functions defined by Ruscheweyh derivatives. I, II, Indian J. Pure Appl. Math., 15 no. 11 (1984), 1216–1227, 1228–1238.

[15] Liu, J.-L. and Noor, K. I. : Some properties of Noor integral operator, J. Nat. Geom., 21 no. 1-2, (2002), 81–90.

[16] Magesh, N. : Differential sandwich results for certain subclasses of analytic functions, Mathematical and Computer Modelling, 54 (2011) 803–814.

[17] Miller, S. S. and Mocanu, P. T. : Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[18] Miller, S. S. and Mocanu, P. T. : Subordinants of differential superordinations, Complex Var. Theory Appl., 48 no. 10, (2003), 815–826.

[19] Murugusundaramoorthy, G. and Magesh, N. : Differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator, JIPAM. J. Inequal. Pure Appl. Math., 7 no. 4 (2006), Article 152, 9 pp.

[20] Obradović, M., Aouf, M. K. and Owa, S. : On some results for starlike functions of complex order, Publ. Inst. Math. (Beograd) (N.S.), 46(60) (1989), 79–85.

[21] Pommerenke, C. : Univalent functions, Vandenhoeck & Ruprecht, Göttingen, 1975.

[22] Saitoh, H. : A linear operator and its applications of first order differential subordinations, Math. Japon., 44 no. 1 (1996), 31–38.

[23] Shanmugam, T. N., Ravichandran, V. and Sivasubramanian, S. : Differential sandwich theorems for some subclasses of analytic functions, Aust. J. Math. Anal. Appl., 3 no. 1 (2006), Art. 8, 11 pp.

[24] Srivastava, H. M. and Aouf, M. K. : A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. II, J. Math. Anal. Appl., 192 no. 3 (1995), 673–688.

[25] Srivastava, H. M. and Lashin, A. Y. : Some applications of the Briot-Bouquet differential subordination, JIPAM. J. Inequal. Pure Appl. Math., 6 no. 2 (2005), Article 41, 7 pp.

[26] Wright, E. M. : The asymptotic expansion of the generalized hypergeometric function, Proc. London Math. Soc., (2), 46 (1940), 389–408.