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On Conformal Tranformation of Finsler Spaces with the Metric $ds = \left(g_{ij}(y)\, y^i\, y^j\right)^{1/2} + d_i(x,y)\, y^i$

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In the present paper we treat the Finsler space F^{*n} equipped with the fundamental function L^* which is obtained by the conformal transformation of the metric $ds = \left(g_{ij}(y)\, y^i\, y^j\right)^{1/2} + d_i(x,y)\, y^i$. We have here expressed the Cartan's connection of $F^{*n} = (M^n, L^*)$ in terms of the one of $F^n = (M^n, L)$ using the difference tensor D^i_{jk}. Also, we have obtained the v-curvature tensor S^*_{hijk} of the Finsler space F^{*n}.

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1. Introduction

G. Rander's in 1941 introduced a Finsler space with metric $ds = \alpha + \beta$, where $\alpha^2 = g_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form. The geometrical properties of the Rander's space have been studied by various authors [7], [8] etc. In all these works the covariant vector b_i is assumed to be the function of positional co-ordinates (x) only. Later in 1971, M.Matsumoto [5] studied the properties of a Finsler space $F'^n = (M^n, L')$ obtained from a locally Minkowskian space $F^n = (M^n, L)$ by the following transformation:

$$L'(x,y) = L(y) + b_i(x)\, y^i.$$

In 1980, H. Izumi [2] introduced the h-vector while studying the conformal transformation of Finsler spaces. The h-vector d_i is assumed to be v-covariantly constant with respect to Cartan's connection $C\Gamma$ and is not only the function of positional co-ordinates but it is also the function of directional arguments.

The conformal theory of Finsler metric has been defined by Hashiguchi [1] and Knebelman [3]. According to the conformal theory two metric functions are said to be conformal if the length of an arbitrary vector in one is proportional to the length in the other.

This paper deals with a metric:

$$L^*(x,y) = e^{\sigma}\left(L(y) + \delta(x,y)\right) \tag{1.1}$$

where $\sigma(x)$ is a scalar field, $L = \left(g_{ij}(y)y^i y^j\right)^{1/2}$ is a Minkowski metric and $\delta = d_i(x,y)y^i$, d_i being h-vector. The metric $\overline{L} = L(y) + \delta$ in equation (1.1) has been studied by Prasad [6]. It is similar to the Rander's one but with different tensor properties in a way that the Riemmanian space with metric $\alpha^2 = g_{ij}(x)y^i y^j$ is characterized by $C^i_{jk} = 0$, on the other hand the Minkowski space is characterized by $R_{hijk} = 0$, $C_{hij\|k} = 0$. Besides this the covariant vector b_i of the one-form vanishes on differentiation while the h-vector d_i satisfies $\dot{\partial}_j d_i = \rho L^{-1} h_{ij}$.

The purpose of the present paper is to study the conformally transformed Finsler space with metric defined in (1.l). We shall here determine the Cartan's connection $C\Gamma$ of the Finsler space F^{*n} in terms of the one of $F^n = (M^n, L)$. Also the v-curvature tensor S^*_{hijk} of the space F^{*n} has been obtained.

2. Preliminaries

Let M^n be an n-dimensional smooth manifold and $F^{*n} = (M^n, L^*)$ be an n- dimensional Finsler space equipped with the fundamental function $L^*(x,y)$ defined as :

$$L^*(x,y) = e^{\sigma}\left\{\left(g_{ij}(y)y^i y^j\right)^{1/2} + d_i(x,y)y^i\right\}. \tag{2.1}$$

Let $d_i(x,y)$ be a vector field in the Finsler space $F^n = (M^n, L)$. Then the vector d_i is called as h-vector if it satisfies the following conditions:

(2.2) (i) $d_{i\|j} = 0$

and (ii) $LC^h_{ij} d_h = \rho h_{ij}$.

Here $\|_r$ denotes the v-covariant differentiation with respect to the Cartan's connection $C\Gamma$, $C^h_{ij}$is Cartan's C-tensor and h_{ij} is the angular metric tensor. The function ρ is given by:

(2.3) $$\rho = \frac{1}{(n-1)} LC^i d_i.$$

where $C^i$is the torsion vector field $C^i_{jk} g^{jk}$.

For an h-vector we have the following lemmas defined in [2] :

Lemma 2.1. If d_i is an h-vector then the function ρ and $l^*_i = d_i - \rho l_i$ are independent of y.

Lemma 2.2. The magnitude d of an h-vector d_i is independent of y.

Since d_i is an h-vector, thus from (2.2) and (2.3) we get:

(2.4) (i) $$\dot{\partial}_j d_i = \rho L^{-1} h_{ij}$$

and (ii) $$\dot{\partial}_j \delta = d_j.$$

The following notations have been used throughout the paper:

$$\dot{\partial}_j L = L_j,\ \dot{\partial}_i \dot{\partial}_j L = L_{ij} \text{ etc.}$$

and the quantities referring to F^{*n} are indicated using *.

3. Fundamental Tensors

If l_i, h_{ij}, g_{ij} and C_{ijk} denote the normalized supporting element, angular metric tensor, metric tensor and Cartan's C-tensor respectively of the space $F^n = (M^n, L)$ then the corresponding quantities of the space$F^{*n} = (M^n, L^*)$ equipped with metric L^* defined in (2.1) are determined using the following relations between two spaces F^n and F^{*n}:

(3.1) (a) $$L^*_i = e^\sigma (L_i + d_i),$$

(b) $$L^*_{ij} = e^\sigma (1+\rho) L_{ij},$$

(c) $$L^*_{ijk} = e^\sigma (1+\rho) L_{ijk}$$

and

(d) $L^*_{ijkh} = e^\sigma (1+\rho) L_{ijkh}$. Thus, from (3.1) we have:

(3.2) $$l^*_i = e^\sigma (l_i + d_i)$$

and $$h^*_{ij} = e^\sigma \lambda h_{ij}.$$

where $$\lambda = \tau(1+\rho)$$

and $\tau = L^* L^{-1}$.

To obtain the value of g^*_{ij} we substitute values from (3.2) in the relation $g^*_{ij} = h^*_{ij} + l^*_i l^*_j$ and get the following result:

(3.3) $$g^*_{ij} = e^\sigma[\lambda g_{ij} + (e^\sigma - \lambda) l_i l_j + e^\sigma (l_i d_j + l_j d_i + d_i d_j)].$$

The relation between the contravariant components of the fundamental tensors are derived as:

(3.4) $$g^{*ij} = \tfrac{e^{-\sigma}}{\lambda}[g^{ij} - \tfrac{(1+\rho)}{\lambda}(l^i d^j + d^i l^j) - \tfrac{(1+\rho)^2}{\lambda^2}\theta l^i l^j].$$

where $\theta = 1 - d^2 - e^{-\sigma}\lambda$,

$$d^2 = g_{ij} d^i d^j$$

and λ being defined in (3.2).

Theorem 3.1. The fundamental metric function g^*_{ij} of the conformally transformed Finsler space $F^{*n} = (M^n, L^*)$ and its contravariant component in terms of $F^n = (M^n, L)$ are given as:

$$g^*_{ij} = e^\sigma \left[\lambda g_{ij} + (e^\sigma - \lambda) l_i l_j + e^\sigma (l_i d_j + l_j d_i + d_i d_j)\right]$$

and

$$g^{*ij} = \frac{e^{-\sigma}}{\lambda}\left[g^{ij} - \frac{(1+\rho)}{\lambda}(l^i d^j + d^i l^j) - \frac{(1+\rho)^2}{\lambda^2}\theta l^i l^j\right].$$

From the lemma (2.1) we get:

(3.5) $$\dot\partial_i \lambda = \tfrac{e^\sigma}{L}(1+\rho) m_i.$$

where

(3.6) $$\dot\partial_i \lambda = \tfrac{\partial\lambda}{\partial y^i}$$

and $m_i = d_i - \frac{\delta}{L} l_i$.

From the definition of m_i, we have the following results:

(3.7) (a) $m_i . l^i = 0$,

(b) $m_i . d^i = d^2 - \frac{\delta^2}{L^2} = m_i . m^i = m^2$,

(c) $h_{ij} . m^i = h_{ij} d^i = m_j$

and

(d) $C^h_{ij}m_h = \frac{\rho}{L}h_{ij}$.

We know that:

(3.8) $$2C_{ijk} = \dot{\partial}_k h_{ij} + L^{-1}(h_{ik}l_j + h_{jk}l_i).$$

Differentiating (3.2) with respect to y_k and using the relation (3.8) we have:

(3.9) $$C^*_{ijk} = e^{\sigma}[\lambda C_{ijk} + e^{\sigma}\tfrac{(1+\rho)}{2L}cycl(i,j,k)(h_{ij}m_k)].$$

where m_k is defined in (3.6) and cycle (i, j, k) stands for cyclic interchange of i, j and k.

A non-Riemannian Finsler space is called C-reducible if the torsion tensor C_{ijk} is of the form [4]:

$$C_{ijk} = h_{ij}M_k + h_{jk}M_i + h_{ik}M_j$$

From (3.2) and (3.9) it follows that:

Proposition 3.1. If the Finsler space F^n is C-reducible then the Finsler space F^{*n} is also C- reducible.

Multiplying (3.4) with (3.9) and using (2.2) and (3.7) we get:

(3.10) $$C^{*h}_{ij} = C^h_{ij} + \tfrac{e^{\sigma}}{2L^*}(h_{ij}m^h + h^h_i m_j + h^h_j m_i) - \tfrac{e^{\sigma}}{L^*}A_{ij}l^h.$$

where $C^h_{ij} = g^{hk}C_{ijk}$

and $A_{ij} = \left[e^{-\sigma}\rho + \frac{1}{2}\tau^{-1}m^2\right]h_{ij} + \tau^{-1}m_i m_j.$

Theorem 3.2. The Cartan's tensor of the conformally transformed Finsler space $F^{*n} = (M^n, L^*)$ expressed in terms of $F^n = (M^n, L)$ takes the following forms:

$$C^*_{ijk} = e^{\sigma}[\lambda C_{ijk} + e^{\sigma}\frac{(1+\rho)}{2L}cycl(i,j,k)(h_{ij}m_k)]$$

and

$$C^{*h}_{ij} = C^h_{ij} + \frac{e^{\sigma}}{2L^*}(h_{ij}m^h + h^h_i m_j + h^h_j m_i) - \frac{e^{\sigma}}{L^*}A_{ij}l^h.$$

If L is a Riemmanian metric, then

(I) $C_{ijk} = 0.$

Hence the Cartan's tensor is expressed as:

$$C^*_{ijk} = h^*_{ij}M^*_k + h^*_{ik}M^*_j + h^*_{jk}M^*_i$$

where $M^*_i = \frac{e^\sigma}{2L^*} m_i$.

Hence we have:

Theorem 3.3. The conformally transformed Finsler space F^{*n} is C-reducible if the metric L is Riemannian and does not vanish.

4. Cartan's Connection of F^{*n}

The Cartn's connection $C\Gamma$ is denoted by $C\Gamma = (F^i_{jk},\ N^i_k,\ C^i_{jk})$ where $N^i_k = F^i_{0k} = y^j F^i_{jk}$ is the non-linear connection and $C^h_{ij} = g^{hk} C_{ijk}$.

In virtue of $L_{ij\|k} = 0$, we have:

$$\partial_k L_{ij} = L_{ijr} N^r_k + L_{rj} F^r_{ik} + L_{ir} F^r_{jk}. \tag{4.1}$$

Differentiation of (3.1)(b) leads to:

$$\partial_k L^*_{ij} = e^\sigma [(1+\rho)\partial_k L_{ij} + (\rho_k + (1+\rho)\sigma_k) L_{ij}]. \tag{4.2}$$

where $\rho_k = \partial_k \rho$ and $\sigma_k = \partial_k \sigma$.

We shall now suppose

$$D^i_{jk} = F^{*i}_{jk} - F^i_{jk}. \tag{4.3}$$

Clearly the difference D^i_{jk} is a tensor of type (1, 2).

On account of (4.1) and (4.3) we may write (4.2) as:

$$(1+\rho)\left[L_{ijr} D^r_{ok} + L_{ir} D^r_{jk} + L_{rj} D^r_{ik}\right] = (\rho_k + (1+\rho)\sigma_k) L_{ij}. \tag{4.4}$$

Now, differentiation of (3.1)a. yields:

$$\partial_j L^*_i = e^\sigma [\partial_j L_i + \partial_j d_i + (L_i + d_i)\sigma_j]. \tag{4.5}$$

In virtue of $L_{i\|j} = 0$ and $d_{ir} = \rho L_{ir}$, we get:

$$(1+\rho) L_{ir} N^{*r}_j + (L_r + d_r) F^{*r}_{ij} = (1+\rho) L_{ir} N^r_j + (L_r + d_r) F^r_{ij} + d_{i\|j} + (L_i + d_i)\sigma_j.$$

By means of (3.1) and (4.3) the above equation reduces to:

$$(1+\rho) L_{ir} D^r_{oj} + (L_r + d_r) D^r_{ij} = d_{i\|j} + (L_i + d_i)\sigma_j. \tag{4.6}$$

Now to find the difference tensor we introduce the following lemma defined in [4]:

Lemma 4.1. The system of algebraic equations

(i) $L_{ir}A^r = B_i$

and (ii) $(l_r + d_r)A^r = B$

has a unique solution A^r for given B_i and B such that $B_i.l^i = 0$. The solution is given by:

$$A^i = LB^i + \tau^{-1}(B - LB_\beta)l^i$$

where subscript β denotes contraction with d_i.

Now, the equation (4.6) is equivalent to two equations:

$$(1+\rho)(L_{ir}D^r_{oj} + L_{jr}D^r_{oi}) + 2(l_r + d_r)D^r_{ij} = 2E_{ij} \tag{4.7}$$

and

$$(1+\rho)(L_{ir}D^r_{oj} - L_{jr}D^r_{oi}) = 2F_{ij}. \tag{4.8}$$

where

$$2E_{ij} = [d_{i\|j} + (L_i + d_i)\sigma_j] + [d_{j\|i} + (L_j + d_j)\sigma_i] \tag{4.9}$$

and

$$2F_{ij} = [d_{i\|j} + (L_i + d_i)\sigma_j] - [d_{j\|i} + (L_j + d_j)\sigma_i].$$

On the other hand equation (4.4) is equivalent to:

$$(1+\rho)(2L_{jr}D^r_{ik} + L_{ijr}D^r_{ok} + L_{jkr}D^r_{oi} - L_{ikr}D^r_{oj}) = (1+\rho)U_{ijk} + V_{ijk}. \tag{4.10}$$

where $U_{ijk} = \sigma_k L_{ij} + \sigma_i L_{jk} - \sigma_j L_{ik}$

and $V_{ijk} = \rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ik}$.

Contraction of (4.7) and (4.8) with y^j yields:

$$(1+\rho)L_{ir}D^r_{oo} + 2(l_r + d_r)D^r_{io} = 2E_{io} \tag{4.11}$$

and

$$(1+\rho)L_{ir}D^r_{oo} = 2F_{io}. \tag{4.12}$$

Similarly, contracting (4.10) with y^j and interchanging j and k we get:

$$(1+\rho)(L_{ir}D^r_{oj} + L_{jr}D^r_{oi} + L_{ijr}D^r_{oo}) = (\rho_o + (1+\rho)\sigma_o)L_{ij}. \tag{4.13}$$

Contracting (4.11) with y^j again we get:

(4.14) $$(l_r + d_r)D^r_{oo} = E_{oo}.$$

Applying lemma (4.1)to the set of equations (4.12) and (4.14) we get the following results:

(4.15) $$D^i_{oo} = 2L(1+\rho)^{-1}F^i_o + \tau^{-1}[E_{oo} - 2L(1+\rho)^{-1}F_{\beta o}]\, l^i$$

where $F^r_o = g^{ir}F_{io}$.

We may now add (4.8) and (4.13) to obtain the following:

(4.16) $$L_{ir}D^r_{oj} = G_{ij}.$$

where

(4.17) $$G_{ij} = [2(1+\rho)]^{-1}[2F_{ij} + (\rho_o + (1+\rho)\sigma_o)L_{ij} - (1+\rho)L_{ijr}D^r_{oo}].$$

Then equation (4.11) is written in the following form:

(4.18) $$(l_r + d_r)D^r_{oj} = G_j.$$

where

(4.19) $$G_j = E_{jo} - \tfrac{(1+\rho)}{2}L_{jr}D^r_{oo} = E_{jo} - F_{jo}.$$

Substituting the value of D^r_{oo} from (4.15) in (4.17) we obtain the value of G_{ij} as:

(4.20) $$G_{ij} = (1+\rho)^{-1}[F_{ij} - LL_{ijr}F^r_o + L_{ij}Q_o(2L^*)^{-1}].$$

where

$$Q_o = (\rho_o + (1+\rho)\sigma_o)L^* + (1+\rho)E_{oo} - 2LF_{\beta o}.$$

Using (3.8) in the above equation we now have:

(4.21) $$G_{ij} = (1+\rho)^{-1}[K_{ij} + L^{-1}(l_iF_{jo} + l_jF_{io}) + Gh_{ij}]$$

where

$$K_{ij} = F_{ij} - 2C_{ijr}F^r_o$$

and

$$G = (2LL^*)^{-1}[(1+\rho)E_{oo} - 2LF_{\beta o} + L^*(\rho_o + (1+\rho)\sigma_o)].$$

Now, applying lemma 4.1 to the system of equations (4.16) and (4.18) we obtain:

(4.22) $$D^i_{oj} = LG^i_j + \tau^{-1}(G_j - LG_{\beta j})l^i.$$

where $G^i_j = g^{ir}G_{rj}$.

Finally from (4.7) and (4.10) we have the following set of equations:

(II) $L_{ir}D^r_{jk} = H_{ijk}$

and $(l_r + d_r)D^r_{jk} = H_{jk}$

where

(4.23) $$H_{ijk} = \tfrac{1}{2}\{cycl(i,j,k)(\sigma_i L_{jk}) + (1+\rho)^{-1} cycl(i,j,k)(\rho_i L_{jk}) - A^i_{jk}\},$$

$$A^i_{jk} = L_{ijr}D^r_{ok} + L_{ikr}D^r_{oj} - L_{kjr}D^r_{oi}$$

and

$$H_{jk} = E_{jk} - \tfrac{(1+\rho)}{2}(L_{jr}D^r_{ok} + L_{kr}D^r_{oj}).$$

Hence, applying lemma (4.1) to (II) we get:

(4.24) $$D^i_{jk} = LH^i_{jk} + \tau^{-1}(H_{jk} - LH_{\beta jk})l^i$$

where $H^i_{jk} = g^{hi}H_{hjk}$

and H_{ijk} and H_{jk} are defined in (4.23).

Hence, we now establish the following theorem:

Theorem 4.1. The connection parameters of the Cartan's connection of the conformally transformed Finsler space F^{*n} are completely determined by the set of equations (4.4) and (4.6) in terms of the one of F^n utilizing (4.24).

5. The v-Curvature Tensor of F^{*n}

The v-curvature tensor S^*_{hijk} of $F^{*n} = (M^n, L^*)$ is defined as:

(5.1) $$S^*_{hijk} = C^*_{hkm}C^{*m}_{ij} - C^*_{hjm}C^{*m}_{ik}.$$

From (3.7), (3.9) and (3.10) we first obtain:

(5.2) $$C^*_{hkm}C^{*m}_{ij} = e^{\sigma}\{\lambda C_{hkm}C^m_{ij} + e^{2\sigma}\phi h_{ij}h_{hk} + e^{\sigma}\frac{(1+\rho)}{2L} cycl\,(i,j,k,h)(C_{ijk}m_h) + e^{2\sigma}\frac{(1+\rho)}{4LL^*}J_{hijk}\}$$

where

(5.3) $$\phi = \tfrac{(1+\rho)}{L}\left[\tfrac{e^{-\sigma}\rho}{L} + \tfrac{1}{4L^*}m^2\right],$$

$$J_{hijk} = 2h_{ij}m_h m_k + 2h_{hk}m_i m_j + h_{ik}m_j m_h + h_{ih}m_j m_k + h_{jk}m_i m_h + h_{hj}m_i m_k$$

and λ being defined in (3.2).

Thus, from (5.1) we get:

$$S^*_{hijk} = e^{\sigma}[\lambda S_{hijk} + e^{2\sigma}(h_{ij}d_{hk} + h_{hk}d_{ij} - h_{ik}d_{hj} - h_{hj}d_{ik})] \tag{5.4}$$

where $\quad d_{ij} = \frac{\phi}{2}h_{ij} + \frac{(1+\rho)}{4LL^*}m_i m_j.$

In virtue of (I) and (5.1) we get:

$$S_{hijk} = 0. \tag{III}$$

Hence, the v-curvature tensor takes the following form:

$$L^{*2}S^*_{hijk} = h_{ij}P_{hk} + h_{hk}P_{ij} - h_{ik}P_{hj} - h_{hj}P_{ik}. \tag{5.5}$$

where

$$P_{hk} = e^{3\sigma}\frac{\lambda}{4}[2(e^{-\sigma}\rho\tau + \frac{1}{4}m^2)h_{hk} + m_h m_k].$$

We may now state the following theorem:

Theorem 5.1. If condition (III) holds good for the conformally transformed Finsle space then the v-curvature tensor S^*_{hijk} takes the following form:

$$L^{*2}S^*_{hijk} = h_{ij}P_{hk} + h_{hk}P_{ij} - h_{ik}P_{hj} - h_{hj}P_{ik}.$$

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