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Ricci Solitons in (ϵ)-Trans-Sasakian Manifolds

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

We study Ricci solitons in (ϵ)-trans-Sasakian manifolds. It is shown that a symmetric parallel second order covariant tensor in a (ϵ)-trans-Sasakian manifold is a constant multiple of the metric tensor. Using this it is shown that if $L_V g + 2S$ is parallel where V is a given vector field, then (g, V) is Ricci soliton. Further, by virtue of this result, Ricci solitons for n-dimensional (ϵ)-trans-Sasakian Manifolds are obtained. Next, Ricci solitons for 3-dimensional (ϵ)-trans-Sasakian Manifolds of type (α, β) are discussed.

Key Words : Ricci soliton, (ϵ)-trans-Sasakian manifold, Einstein.
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1. Introduction

A Ricci soliton is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where V is a vector field on M and λ is a constant. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero and positive respectively. Compact Ricci solitons are the fixed point of the Ricci flow $\frac{\partial g}{\partial t} = -2Ric$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds.

In 1923, L.P. Eisenhart [9] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925, Levy [12] obtained the necessary and sufficient conditions for the existence of such tensors. In 1989 and 1990, R. Sharma [20, 21] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an n-dimensional $(n > 2)$ space of constant curvature is a constant multiple of the metric tensor. It is also proved that in a Sasakian manifold there is no nonzero parallel 2-form.

In 2008, R. Sharma [22] studied Ricci solitons in K-contact manifolds, where the structure field ξ is killing and he proved that a complete K-contact gradient soliton is compact Einstein and Sasakian. In 2010, Constantin Calin and Mircea Crasmareanu [7] extended the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds. They studied the case of f-Kenmotsu manifolds satisfying a special condition called regular and a symmetric parallel tensor field of second order is a constant multiple of the Riemannian metric. Using this result, they obtained the results on Ricci solitons. Again in 2011, Amadendu Ghosh and Ramesh Sharma [1] studied on K-contact metrics as Ricci solitons.

The present paper is organized as follows: the second section is devoted to preliminaries. In the third section we prove that a symmetric parallel second order covariant tensor in an (ϵ)-trans-Sasakian manifold is a constant multiple of the associated metric tensor. A Ricci soliton in an n-dimensional η-Einstein (ϵ)-trans-Sasakian manifold is shrinking or expanding according as λ is positive or negative. Similarly also for Ricci soliton in 3-dimensional (ϵ)-trans-Sasakian manifold is either shrinking or expanding according as λ is positive or negative.

2. Preliminaries

Let M be an almost contact metric manifold of dimension n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \tag{2.1}$$

Almost contact metric manifold M is called (ϵ)-almost contact metric manifold if

$$g(\xi, \xi) = \epsilon, \quad \eta(X) = \epsilon g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \quad \forall X, Y \in TM \tag{2.3}$$

for all vector fields X, Y on M, where $\epsilon = g(\xi, \xi) = \pm 1$.

An (ϵ)-almost contact metric manifold is called an (ϵ)-trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \epsilon\eta(Y)X) + \beta(g(\phi X,Y)\xi - \epsilon\eta(Y)\phi X), \tag{2.4}$$

holds for some smooth functions α and β on M and $\epsilon = \pm 1$.

The notations used in Lemmas (2.1) to (2.4) are from [15] and [23].

Lemma 2.1. An (ϵ)-almost contact metric manifold M is an (ϵ)-trans-Sasakian manifold if and only if

$$\nabla_X \xi = \epsilon[-\alpha\phi X + \beta(X - \eta(X)\xi)]. \tag{2.5}$$

Proof. By taking $Y = \xi$ in (2.4) and making use of (2.1), we have (2.5).

From (2.5), it follows that

$$(\nabla_X \eta)Y = \beta[g(X,Y) - \epsilon\eta(X)\eta(Y)] - \alpha g(\phi X, Y). \tag{2.6}$$

Lemma 2.2. In an (ϵ)-trans-Sasakian manifold M, the Riemannian curvature tensor R satisfies

$$\begin{aligned} R(X,Y)\xi =&(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+ \epsilon[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y], \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi,Y)X =&(\alpha^2 - \beta^2)[\epsilon g(X,Y)\xi - \eta(X)Y] + 2\alpha\beta[\epsilon g(\phi X,Y)\xi \\ &+ \eta(X)\phi Y] + \epsilon g(\phi X,Y)(grad\,\alpha) + \epsilon(X\alpha)\phi Y \\ &- \epsilon g(\phi X,\phi Y)(grad\,\beta) + \epsilon(X\beta)[Y - \eta(Y)\xi]. \end{aligned} \tag{2.8}$$

Proof. We know that $R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$. Using (2.5) the above equation becomes

$$\begin{aligned} R(X,Y)\xi =&\nabla_X(\epsilon[-\alpha\phi Y + \beta(Y - \eta(Y)\xi)]) - \nabla_Y(\epsilon[-\alpha\phi X + \beta(X \\ &- \eta(X)\xi)]) - \epsilon\{-\alpha\phi[X,Y] + \beta([X,Y] - \eta([X,Y])\xi)\}. \end{aligned} \tag{2.9}$$

Using (2.4), the above relation yields (2.7).

From (2.7) and $g(R(\xi,X)Y,Z) = g(R(Y,Z)\xi,X)$, we obtain (2.8).

Lemma 2.3. In an (ϵ)-trans-Sasakian manifold M, we have

$$\begin{aligned}\eta(R(X,Y)Z) = &\epsilon(\alpha^2-\beta^2)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\\ &+2\epsilon\alpha\beta[g(\phi Y,Z)\eta(X)-g(\phi X,Z)\eta(Y)]\\ &+[(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)]\\ &+[(X\beta)g(\phi^2 Y,Z)-(Y\beta)g(\phi^2 X,Z)].\end{aligned} \tag{2.10}$$

Consequently

$$\eta(R(X,Y)\xi)=0. \tag{2.11}$$

Proof. Now we have

$$\begin{aligned}\eta(R(X,Y)Z) &= \epsilon g(R(X,Y)Z,\xi)\\ &= -\epsilon g(R(X,Y)\xi,Z).\end{aligned}$$

Using (2.7), in the above equation, we have (2.10) that is

$$\begin{aligned}\eta(R(X,Y)Z) &= \epsilon(\alpha^2-\beta^2)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+2\epsilon\alpha\beta[g(\phi Y,Z)\eta(X)\\ &- g(\phi X,Z)\eta(Y)]+[(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)]\\ &+ [(X\beta)g(\phi^2 Y,Z)-(Y\beta)g(\phi^2 X,Z)].\end{aligned}$$

Replacing $Z=\xi$ in the above equation then we have (2.11).

Lemma 2.4. In an (ϵ)-trans-Sasakian manifold M, the following relations holds true

$$S(X,\xi)=[(n-1)(\alpha^2-\beta^2)-\epsilon(\xi\beta)]\eta(X)-\epsilon((\phi X)\alpha)-(n-2)\epsilon(X\beta) \tag{2.12}$$

and

$$\epsilon(\xi\alpha)+2\alpha\beta=0. \tag{2.13}$$

Proof. Taking $Y=Z=e_i$ in (2.11) and we obtain (2.12).

Taking $X=\xi$ in (2.7), we have

$$R(\xi,X)\xi=[(\alpha^2-\beta^2)-\epsilon(\xi\beta)][-Y+\eta(Y)\xi]-[2\alpha\beta+\epsilon(\xi\alpha)]\phi Y. \tag{2.14}$$

Taking $Y=\xi$ in (2.8), we obtain

$$R(\xi,X)\xi=[(\alpha^2-\beta^2)-\epsilon(\xi\beta)][-Y+\eta(Y)\xi]+[2\alpha\beta+\epsilon(\xi\alpha)]\phi Y. \tag{2.15}$$

Comparing (2.14) and (2.15), we obtain (2.13).

Lemma 2.5. In an (ϵ)-trans-Sasakian manifold M of type (α,β), if

$$\phi(grad\,\alpha)=(n-2)(grad\,\beta), \tag{2.16}$$

then we have

$$(\xi\beta) = 0. \tag{2.17}$$

Thus the directional derivative of β with respect to characteristic vector field ξ is zero.

Proof. We know that

$$\begin{aligned} X\beta = g(X, grad\,\beta) =&\ g(X, \frac{\phi(grad\,\alpha)}{(n-2)}) \\ =&\ -\frac{1}{(n-2)}g(\phi X, grad\,\alpha), \end{aligned} \tag{2.18}$$

which implies

$$(n-2)X\beta + (\phi X)\alpha = 0. \tag{2.19}$$

On putting $X = \xi$ in (2.19), we obtain (2.17).

2.1. **Example** [14] We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3; z \neq 0\}$, where (x, y, z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = \frac{x}{z}\frac{\partial}{\partial x}, \quad E_2 = \frac{y}{z}\frac{\partial}{\partial y}, \quad E_3 = \epsilon\frac{\partial}{\partial z}. \tag{2.20}$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$ and $g(E_1, E_1) = g(E_2, E_2) = 1,\ g(E_3, E_3) = \epsilon$, where $\epsilon = \pm 1$ and g is given by

$$g = \frac{z^2}{x^2}dx \otimes dx + \frac{z^2}{y^2}dy \otimes dy + \epsilon dz \otimes dz.$$

The (ϕ, ξ, η) is given by $\eta = \epsilon dz,\ \ \xi = E_3 = \frac{\partial}{\partial z}\ \phi E_1 = E_2,\ \phi E_2 = -E_1$ and $\phi E_3 = 0$. The linearity property of ϕ and g yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$, $g(\phi U, \phi W) = g(U, W) - \epsilon\eta(U)\eta(W)$, for any vector fields U, W on M. Hence for $E_3 = \xi$, (ϕ, ξ, η, g) defines an (ϵ)-almost contact metric structure on M. By definition of Lie bracket, we have $[E_1, E_2] = 0$, $[E_1, E_3] = -\frac{\epsilon}{z}E_1$ $[E_2, E_3] = \frac{\epsilon}{z}E_2$. Let ∇ be Levi-Civita connection with respect to the above metric g given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) =& X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{2.21}$$

Using (2.21), we have

$$2g(\nabla_{E_1}E_3, E_1) = 2g(\frac{\epsilon}{z}E_1, E_1) + 2g(\epsilon E_2, E_1) = 2g(\frac{\epsilon}{z}E_1 + \epsilon E_2, E_1), \tag{2.22}$$

since $g(E_1, E_2) = 0$. Thus $\nabla_{E_1}E_3 = \frac{\epsilon}{z}E_1 + \epsilon E_2$.

Again by (2.21), we get

$$2g(\nabla_{E_2}E_3, E_2) = 2g(\frac{\epsilon}{z}E_2, E_2) - 2g(\epsilon E_2, E_1) = 2g(\frac{\epsilon}{z}E_2 - \epsilon E_1, E_2), \qquad (2.23)$$

since $g(E_1, E_2) = 0$. Therefore we have $\nabla_{E_2}E_3 = \frac{\epsilon}{z}E_2 - \epsilon E_1$.

Using (2.21), we have

$$\begin{array}{lll} \nabla_{E_1}E_1 = -\frac{\epsilon}{z}E_3, & \nabla_{E_2}E_2 = -\frac{\epsilon}{z}E_3, & \nabla_{E_3}E_3 = 0, \\ \nabla_{E_1}E_2 = 0, & \nabla_{E_2}E_1 = 0, & \nabla_{E_1}E_3 = \frac{\epsilon}{z}E_1 + \epsilon E_2, \\ \nabla_{E_3}E_1 = 0, & \nabla_{E_2}E_3 = \frac{\epsilon}{z}E_2 - \epsilon E_1, & \nabla_{E_3}E_2 = 0. \end{array} \qquad (2.24)$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, that is $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$ where $a_i, b_i (i = 1, 2, 3)$ are scalars.

Now, for $\xi = E_3$, above results that is (2.24) satisfy (2.5) that is

$$\nabla_X\xi = \epsilon[-\alpha\phi X + \beta(X - \eta(X)\xi)],$$

with $\alpha = -1$ and $\beta = \frac{1}{z}$. Consequently M is a 3-dimensional (ϵ)-trans-Sasakian manifold.

3. Parallel symmetric second order tensors and Ricci solitons in (ϵ)-trans-Sasakian manifolds

Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [20]

$$\nabla^2h(X, Y; Z, W) - \nabla^2h(X, Y; W, Z) = 0, \qquad (3.1)$$

we obtain the relation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0, \qquad (3.2)$$

Replacing $Z = W = \xi$ in (3.2) and using (2.7) and by the symmetry of h, we have

$$\begin{aligned} &2(\alpha^2 - \beta^2)[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] + 4\alpha\beta[\eta(Y)h(\phi X, \xi) \\ &-\eta(X)h(\phi Y, \xi)] + 2\epsilon[(Y\alpha)h(\phi X, \xi) - (X\alpha)h(\phi Y, \xi) \\ &+(Y\beta)h(\phi^2X, \xi) - (X\beta)h(\phi^2Y, \xi)] = 0. \end{aligned} \qquad (3.3)$$

Put $X = \xi$ in (3.3) and by virtue of (2.1), we have

$$\begin{aligned} &2(\alpha^2 - \beta^2)[\eta(Y)h(\xi, \xi) - h(Y, \xi)] - 2(\epsilon(\xi\alpha) + 2\alpha\beta)h(\phi Y, \xi) \\ &-2\epsilon(\xi\beta)h(\phi^2Y, \xi) = 0. \end{aligned} \qquad (3.4)$$

By using (2.13) and (2.17) in (3.4) we have

$$2(\alpha^2 - \beta^2)[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.5}$$

And suppose $2(\alpha^2 - \beta^2) \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.6}$$

Differentiating (3.6) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) \ + h(Y,\nabla_X \xi) &= [(\nabla_X \eta)(Y) + \eta(\nabla_X Y)]h(\xi,\xi) \\ &+ \eta(Y)[(\nabla_X h)(Y,\xi) + 2h(\nabla_X \xi,\xi)]. \end{aligned}\tag{3.7}$$

By using (2.6) and (3.6) in the above equation, we have

$$-\epsilon\alpha h(Y,\phi X) + \beta h(Y,X) = -\alpha g(\phi X,Y)h(\xi,\xi) + \beta g(Y,X)h(\xi,\xi).$$

Put $X = \phi X$ in the above equation and on simplification, we have

$$h(X,Y) = \epsilon g(X,Y)h(\xi,\xi), \tag{3.8}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (3.6). Now, by considering the above conditions we can state the following theorem:

Theorem 3.1. A symmetric parallel second order covariant tensor in an (ϵ)-trans-Sasakian manifold is a constant multiple of the associated metric tensor.

Corollary 3.1. A locally Ricci symmetric $(\nabla S = 0)$ (ϵ)-trans-Sasakian manifold is an Einstein manifold.

3.1. **Remark.** The following statements for (ϵ)-trans-Sasakian manifold are equivalent:

(1) Einstein,
(2) locally Ricci symmetric,
(3) Ricci semi-symmetric that is $R \cdot S = 0$.

The implication (1) $\longrightarrow$ (2) $\longrightarrow$ (3) is trivial. Now we prove the implication (3) $\longrightarrow$ (1) and $R \cdot S = 0$ means exactly (3.2) with replaced h by S, that is

$$(R(X,Y)\cdot S)(U,V) \ = \ -S(R(X,Y)U,V) - S(U,R(X,Y)V). \tag{3.9}$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in equation (3.9), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{3.10}$$

By using (2.8), (2.12), (2.13) and (2.17), we obtain

$$\begin{aligned}
&(\alpha^2-\beta^2)[\epsilon g(U,Y)S(\xi,V)-\eta(U)S(Y,V)]+2\alpha\beta[\epsilon g(\phi U,Y)S(\xi,V)\\
&+\eta(U)S(\phi Y,V)]+\epsilon g(\phi U,Y)S(grad\alpha,V)+\epsilon(U\alpha)S(\phi Y,V)\\
&-\epsilon g(\phi U,\phi Y)S(grad\beta,V)+\epsilon(U\beta)[S(Y,V)-\eta(Y)S(\xi,V)]\\
&+(\alpha^2-\beta^2)[\epsilon g(V,Y)S(U,\xi)-\eta(V)S(U,Y)]+2\alpha\beta[\epsilon g(\phi V,Y)S(U,\xi)\\
&+\eta(V)S(U,\phi Y)]+\epsilon g(\phi V,Y)S(U,grad\alpha)+\epsilon(V\alpha)S(U,\phi Y)\\
&-\epsilon g(\phi V,\phi Y)S(U,grad\beta)+\epsilon(V\beta)[S(U,Y)-\eta(Y)S(U,\xi)]=0.
\end{aligned}$$

Again by putting $U=\xi$ in above equation and by using (2.1), (2.16), (2.12) and (2.17), on simplification we obtain

$$S(Y,V)=\epsilon(n-1)(\alpha^2-\beta^2)g(Y,V). \tag{3.11}$$

In conclusion:

Proposition 3.2. A Ricci semi-symmetric (ϵ)-trans-Sasakian manifold is an Einstein manifold.

A Ricci soliton in an (ϵ)-trans-Sasakian manifold defined by (1.1). In the theorem 3.1 we proved that if an (ϵ)-trans-Sasakian manifold admits a symmetric parallel $(0,2)$ tensor, then the tensor is a constant multiple of the metric tensor. Thus $\mathcal{L}_V g+2S$ is parallel. Hence $\mathcal{L}_V g+2S$ is a constant multiple of the metric tensor g that is $(\mathcal{L}_V g+2S)(X,Y)=\epsilon g(X,Y)h(\xi,\xi)$, where $h(\xi,\xi)$ is a nonzero constant. We close this section with applications of our Theorem 3.1 to Ricci solitons:

Corollary 3.2. Suppose that on a (ϵ)-trans-Sasakian manifold the $(0,2)$-type field $\mathcal{L}_V g+2S$ is parallel where V is a given vector field or point-wise collinear with ξ. Then (g,V) yield a Ricci soliton. In particular, if the given (ϵ)-trans-Sasakian manifold is Ricci semi-symmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.

Proof. Follows from theorem 3.1 and corollary 3.1

Corollary 3.3. If a Ricci soliton (g,ξ,λ) in an n-dimensional (ϵ)-trans-Sasakian manifold cannot be steady.

Proof. From Linear Algebra either the vector field $V\in Span\,\xi$ or $V\perp\xi$. However the second case seems to be complex to analyse in practice. For this reason we investigate for the case $V=\xi$.

A simple computation of $\mathcal{L}_\xi g + 2S$ gives

$$(\mathcal{L}_\xi g)(X, Y) = 2\beta[\epsilon g(X, Y) - \eta(X)\eta(Y)]. \tag{3.12}$$

From equation (1.1), we have $h(X, Y) = -2\lambda g(X, Y)$ and then putting $X = Y = \xi$, we have

$$h(\xi, \xi) = -2\lambda\epsilon, \tag{3.13}$$

where

$$h(\xi, \xi) = (\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi), \tag{3.14}$$

by using (3.12) and (2.12) in the above equation, we have

$$h(\xi, \xi) = 2(n-1)[(\alpha^2 - \beta^2)]. \tag{3.15}$$

Equating (3.13) and (3.15), we have

$$\lambda = -(n-1)\epsilon(\alpha^2 - \beta^2). \tag{3.16}$$

Since α and β are some nonzero functions, we have $\lambda \neq 0$, that is a Ricci soliton in an n-dimensional (ϵ)-trans-Sasakian manifold cannot be steady. Hence the proof.

Proposition 3.3. If an n-dimensional (ϵ)-trans-Sasakian manifold is η-Einstein then the Ricci soliton (g, ξ, λ) in an (ϵ)-trans-Sasakian manifold with varying scalar curvature cannot be steady but it is shrinking or expanding according as λ is positive or negative, that is

(1) shrinking($\lambda < 0$) for $\epsilon = 1$ and $\alpha^2 > \beta^2$
(2) expanding($\lambda > 0$) for $\epsilon = -1$ and $\alpha^2 > \beta^2$
(3) expanding($\lambda > 0$) for $\epsilon = 1$ and $\alpha^2 < \beta^2$
(4) shrinking($\lambda < 0$) for $\epsilon = -1$ and $\alpha^2 < \beta^2$.

Proof. The proof consists of three parts:

In first step we prove that the metric tensor is η-Einstein: that is the metric g is called η-Einstein if there exists two real functions a and b such that the Ricci tensor of g is given by

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y). \tag{3.17}$$

Let $e_i = 1, 2, \dots n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.17) and taking summation over i, then we get

$$r = na + b\epsilon. \tag{3.18}$$

Again putting $X = Y = \xi$ in (3.17) then by using (2.12), we have

$$\epsilon a + b = (n-1)(\alpha^2 - \beta^2). \tag{3.19}$$

Then from (3.18) and (3.19), we have

$$a = \frac{r}{(n-1)} - \epsilon(\alpha^2 - \beta^2), \qquad b = -\frac{r\epsilon}{(n-1)} + n(\alpha^2 - \beta^2). \tag{3.20}$$

Substituting the value of a and b in (3.17), we have

$$\begin{aligned} S(X,Y) = & \left[\frac{r}{(n-1)} - \epsilon(\alpha^2 - \beta^2)\right] g(X,Y) \\ & + \left[n(\alpha^2 - \beta^2) - \frac{r\epsilon}{(n-1)}\right] \eta(X)\eta(Y). \end{aligned} \tag{3.21}$$

Equation (3.21) is an η-Einstein (ϵ)-trans-Sasakian manifold.

In the second step we prove that the scalar curvature r is varying: A Ricci solitons in an (ϵ)-trans-Sasakian manifolds with $V = \xi$ in (1.1) and it reduced to

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{3.22}$$

The above equation can be written as

$$h(X,Y) + 2\lambda g(X,Y) = 0, \tag{3.23}$$

where h is a symmetric parallel covariant tensor of type $(0,2)$ and is given by

$$h(X,Y) \;=\; (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.24}$$

By using (3.12) and (3.21) in (3.24), we have

$$\begin{aligned} h(X,Y) \;=\; & \left[\frac{2r}{(n-1)} - 2\epsilon(\alpha^2 - \beta^2) + 2\epsilon\beta\right] g(X,Y) \\ + \; & \left[-\frac{2r\epsilon}{(n-1)} + 2n(\alpha^2 - \beta^2) - 2\beta\right] \eta(X)\eta(Y). \end{aligned} \tag{3.25}$$

Differentiating the above equation with respect to Z, we have

$$\begin{aligned} (\nabla_Z h)(X,Y) = & \left[\frac{2(\nabla_Z r)}{(n-1)} - 2\epsilon[2\alpha(Z\alpha) - 2\beta(Z\beta)] + 2\epsilon(Z\beta)\right] g(X,Y) \\ + & \left[-\frac{2\epsilon(\nabla_Z r)}{(n-1)} + 2n[2\alpha(Z\alpha) - 2\beta(Z\beta)] - 2(Z\beta)\right] \eta(X)\eta(Y) \\ + & \left[-\frac{2r\epsilon}{(n-1)} + 2n(\alpha^2 - \beta^2) - 2\beta\right] [-\alpha g(\phi Z, X)\eta(Y) + \beta g(X,Z)\eta(Y) \\ & -2\epsilon\beta\eta(X)\eta(Y)\eta(Z) - \alpha g(\phi Z, Y)\eta(X) + \beta g(Z,Y)\eta(X)]. \end{aligned} \tag{3.26}$$

By substituting $Z = \xi$ and $X = Y \in (Span\xi)^{\perp}$ in (3.26) and a tensor h is parallel. By using (2.17) in (3.26), we have

$$\nabla_{\xi} r = -4(n-1)\alpha^2\beta, \tag{3.27}$$

Thus (3.27) implies that the scalar curvature r is not constant.

In the third step we prove that the Ricci soliton in an (ϵ)-trans-Sasakian manifold is shrinking or expanding according as λ is positive or negative: From equation (3.23), we have

$$h(X,Y) = -2\lambda g(X,Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = -2\lambda\epsilon. \tag{3.28}$$

Now,

$$\begin{aligned} h(\xi,\xi) &= \left[\frac{2r}{(n-1)} - 2\epsilon(\alpha^2-\beta^2) + 2\epsilon\beta\right] g(\xi,\xi) \\ &+ \left[-\frac{2r\epsilon}{(n-1)} + 2n(\alpha^2-\beta^2) - 2\beta\right] \eta(\xi)\eta(\xi). \end{aligned}$$

The above equation reduced as,

$$h(\xi,\xi) = 2(n-1)[(\alpha^2-\beta^2)]. \tag{3.29}$$

Equating (3.28) and (3.29) and by using (2.17), we have

$$\lambda = -(n-1)\epsilon(\alpha^2-\beta^2). \tag{3.30}$$

From (3.30) we can see that the Ricci soliton in an η-Einstein (ϵ)-trans-Sasakian manifold is shrinking or expanding according as λ is positive or negative. This completes the proof.

Now, we restrict our study to 3-dimensional (ϵ)-trans-Sasakian manifolds:

Proposition 3.4. If a Ricci soliton (g,ξ,λ) of 3-dimensional (ϵ)-trans-Sasakian manifold with varying scalar curvature the is shrinking or expanding according as λ is positive or negative, that is

(1) expanding$(\lambda > 0)$ for $\epsilon = 1$ and $\alpha^2 < \beta^2$
(2) shrinking$(\lambda < 0)$ for $\epsilon = -1$ and $\alpha^2 < \beta^2$
(3) shrinking$(\lambda < 0)$ for $\epsilon = 1$ and $\alpha^2 > \beta^2$
(4) expanding$(\lambda > 0)$ for $\epsilon = -1$ and $\alpha^2 > \beta^2$.

Proof. The proof consists of three parts:

In first step we find 3-dimensional η-Einstein (ϵ)-trans-Sasakian manifolds: A general expression of Ricci tensor S is known by us for the 3-dimensional η-Einstein (ϵ)-trans-Sasakian manifolds by considering 3-dimensional Riemannian manifold that is,

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &- \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.31}$$

put $Z = \xi$ in the above equation and by using (2.7) and (2.12) we have

$$\begin{aligned} &(\alpha^2-\beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+\epsilon[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y] = \epsilon[\eta(Y)QX - \eta(X)QY] \\ &+2(\alpha^2-\beta^2)[\eta(Y)X - \eta(X)Y] + \epsilon[((\phi X)\alpha)Y + (X\beta)Y - ((\phi Y)\alpha)X \\ &-(Y\beta)X] - \frac{r\epsilon}{2}[\eta(Y)X - \eta(X)Y]. \end{aligned}$$

Again put $Y = \xi$ in the above equation and by using (2.1), (2.16) and (2.17) we get

$$QX = \left[\frac{r}{2} - \epsilon(\alpha^2-\beta^2)\right]X + \left[3\epsilon(\alpha^2-\beta^2) - \frac{r}{2}\right]\eta(X)\xi \tag{3.32}$$

and

$$S(X,Y) = \left[\frac{r}{2} - \epsilon(\alpha^2-\beta^2)\right]g(X,Y) + \left[3(\alpha^2-\beta^2) - \frac{r\epsilon}{2}\right]\eta(X)\eta(Y). \tag{3.33}$$

Equation (3.33) is an 3-dimensional η-Einstein (ϵ)-trans-Sasakian manifold.

In the second step we prove that the scalar curvature r is varying: A Ricci solitons in a 3-dimensional (ϵ)-trans-Sasakian manifolds with $V = \xi$ in (1.1) and it reduced to

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{3.34}$$

The above equation can be written as

$$h(X,Y) + 2\lambda g(X,Y) = 0, \tag{3.35}$$

where h is a symmetric parallel covariant tensor of type $(0,2)$ and is given by

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.36}$$

By using (3.12) and (3.33) in (3.36), we have

$$\begin{aligned} h(X,Y) &= [r - 2\epsilon(\alpha^2-\beta^2) + 2\epsilon\beta]g(X,Y) \\ &+ [6(\alpha^2-\beta^2) - \epsilon r - 2\beta]\eta(X)\eta(Y). \end{aligned} \tag{3.37}$$

Differentiating the above equation covariantly with respect to Z, we have

$$(\nabla_Z h)(X,Y) = [\nabla_Z r - 4\epsilon(\alpha(Z\alpha) - \beta(Z\beta)) + 2\epsilon(Z\beta)]g(X,Y) \\ +[12[\alpha(Z\alpha) - \beta(Z\beta)] - \epsilon(\nabla_Z r) - 2(Z\beta)]\eta(X)\eta(Y) \\ +[6(\alpha^2 - \beta^2) - \epsilon r - 2\beta]\{-\alpha g(\phi Z, X)\eta(Y) + \beta g(X,Z)\eta(Y) \\ -2\epsilon\beta\eta(X)\eta(Y)\eta(Z) - \alpha g(\phi Z, Y)\eta(X) + \beta g(Z,Y)\eta(X)\}. \quad (3.38)$$

Substituting $Z = \xi$, $X = Y \in (Span\xi)^{\perp}$ in (3.38) and a tensor h is parallel. By using (2.17), we have

$$\nabla_\xi r = -8\alpha^2\beta. \quad (3.39)$$

Thus, (3.39) implies that the scalar curvature r is not constant.

In the third step we prove that the Ricci soliton in 3-dimensional (ϵ)-trans-Sasakian manifold is shrinking or expanding according as λ is positive or negative: From equation (3.35), we have

$$h(X,Y) = -2\lambda g(X,Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = -2\lambda\epsilon. \quad (3.40)$$

Now,

$$h(X,Y) = [r - 2\epsilon(\alpha^2 - \beta^2) + 2\epsilon\beta]g(X,Y) + [6(\alpha^2 - \beta^2) - \epsilon r - 2\beta]\eta(X)\eta(Y).$$

If $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = 4(\alpha^2 - \beta^2). \quad (3.41)$$

Equating (3.40) and (3.41), we have

$$\lambda = -2\epsilon(\alpha^2 - \beta^2). \quad (3.42)$$

From (3.42) we can see that the Ricci soliton in 3-dimensional (ϵ)-trans-Sasakian manifold is shrinking or expanding according as λ is positive or negative. This completes the proof.

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