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Some Results on Trans-Sasakian Manifolds admitting Semi-symmetric Non Metric Connection

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In the present paper we obtained several results on trans-sasakian manifold, admitting semi-symmetric non metric connection under the condition, $\phi(grad\,\alpha)$ $= (n-2)\,grad\,\beta$.

Keywords and Phrases : Trans-sasakian manifolds,semi-symmetric non metric connection.

1. Introduction

In 1924, Friedman and Schouter [1] introduced the notion of semi-symmetric linear connection on differentiable manifold. Then in 1932, Hyden [2] introduced the idea of metric connection with a torsion on a Riemannian manifold. A systematic study of semi symmetric metric connection on Riemannian manifold has been given by K. Yano [3] in 1970. In 1992, Agashe and Chafle [7] have defined the notion of semi-symmetric non-metric connection on Riemannian manifold. In 2002 S. K. Srivastva studied the semi symmetric non metric connection on manifold with generalized structures [12]. In 2007 C. S. Bagwadi [10] has obtained some results on trans-sasakian manifold with respect to semi-symmetric metric connection.

2. Preliminaries

An odd dimensional differentiable manifold M^n is said to be an almost contact metric manifold [10] if it admits a $a(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{3}$$

for all vector field $X, Y \in M^n$.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be a trans-sasakian structure of the type (α, β) [10] if

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(X, Y) - \eta(Y)\phi X] \tag{4}$$

for some smooth functions α and β on M^n.

Let M^n be a n-dimensional trans-sasakian manifold, then from equation (4) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{5}$$

and

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{6}$$

In n-dimensional trans-sasakian manifold, we have

$$R(\xi, X)\,\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X) \tag{7}$$

$$2\alpha\beta + \xi\alpha = 0 \tag{8}$$

and

$$S(X, \xi) = [(n-1)(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (n-2)X\beta - \phi(X)\alpha. \tag{9}$$

Further,in trans-sasakian manifold of type (α, β), we have

$$\phi(grad\,\alpha) = (n-2)\,grad\,\beta \tag{10}$$

Using (10), the equations (7) and (9) reduces to

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X) \tag{11}$$

and

$$S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\eta(X) \tag{12}$$

Let (M^n, g) be a n-dimensional Riemannian manifold with metric tensor g and ∇ be the Riemannian connection (Levi-civita connection) on M^n. A linear connection $\overline{\nabla}$ on (M^n, g) is said to be semi-symmetric non metric if the torsion tensor T of the connection for trans-sasakian manifold satisfying

$$TR(X,Y) = \eta(Y)X - \eta(X)Y \tag{13}$$

where η is 1-form on M^n with ξ as a associated vector field such that

$$\eta(X) = g(X,\xi)$$

and

$$(\overline{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y) \tag{14}$$

The relation between semi-symmetric non-metric connection $\overline{\nabla}$ and Riemannian connection ∇ of(M^n, g) has been obtained and given by [7]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{15}$$

Furthe, a relation between curvature tensor R and $\overline{R}$ of ∇ and $\overline{\nabla}$ are given by

$$\overline{R}(X,Y)Z = R(X,Y)Z - K(Y,Z)X + K(X,Z)Y \tag{16}$$

where $K(X,Y)$ is tensor field of type (0, 2) given by

$$K(X,Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) \tag{17}$$

For a trans-sasakian manifold M^n under the condition (10) with semi-symmetric metric connection,we have following useful results:

(i) $$K(\nabla_Y \xi, Z) = -\alpha^2[g(Y,Z) - \eta(Y)\eta(Z)] - 2\alpha\beta g(\phi Y, Z) + \beta^2[g(Y,Z) - \eta(Y)\eta(Z)] \tag{17.1}$$

(ii) $$K(Y, \nabla_Z \xi) = \alpha^2[g(Y,Z) - \eta(Y)\eta(Z)] + \beta^2[g(Y,Z) - \eta(Y)\eta(Z)]. \tag{17.2}$$

Theorem 1. For a trans-sasakian manifold M^nunder the condition (10) with semi symmetric non-metric connection,the following results are true

$$K(Y,Z) = \alpha g(Y,\phi Z) + \beta g(Y,Z) - (\beta+1)\eta(Y)\eta(Z), \tag{18.1}$$

$$(K(Y,\xi) = -\eta(Y), \tag{18.2}$$

$$K(\overline{\nabla}_Y \xi, Z) = (\beta^2 - \alpha^2 - \beta)g(Y,Z) - \alpha(2\beta+1)g(\phi Y, Z) + (\alpha^2 - \beta^2 - \beta - 1)\eta(Y)\eta(Z), \tag{18.3}$$

$$K(Y, \overline{\nabla}_Z \xi) = (\alpha^2 + \beta^2 + \beta)g(Y,Z) - \alpha g(\phi Y, Z) - (\alpha^2 + \beta^2 + \beta + 1)\eta(Y)\eta(Z)). \tag{18.4}$$

Proof. (18.1). We have

$$K(Y,Z) = (\overline{\nabla}_Y \eta)Z - \eta(Y)\eta(Z).$$

Using equations (2), (3), (6) and simplifying, we get

$$K(Y,Z) = \alpha g(Y,\phi Z) + \beta g(Y,Z) - (\beta+1)\eta(Y)\eta(Z).$$

Proof. (18.2). Putting $Z=\xi$ in equation (18.1), we have

$$K(Y,\xi) = \alpha g(Y,\phi\xi) + \beta g(Y,\xi) - (\beta+1)\eta(Y)\eta(\xi).$$

Using equation(1) and (3), we obtained

$$K(Y,\xi) = -\eta(y).$$

Proof. (18.3). Since

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + \eta(Y)X \\ K(\overline{\nabla}_Y\xi, Z) &= K(\nabla_Y\xi + \eta(\xi)Y, Z) \\ &= K(\nabla_Y\xi + Y, Z).\end{aligned}$$

Using equation (18.1) this gives

$$\begin{aligned}K(\overline{\nabla}_Y\xi, Z) &= \alpha g(\nabla_Y\xi + Y, \phi Z) + \beta g(\nabla_Y\xi + Y, Z) - (\beta+1)\eta(\nabla_Y\xi + Y)\eta(Z) \\ &= \alpha g(\nabla_Y\xi, \phi Z) + \alpha g(Y,\phi Z) + \beta g(\nabla_Y\xi, Z) + \beta g(Y,Z) \\ &\quad - (\beta+1)\eta(\nabla_Y\xi)\eta(Z) - (\beta+1)\eta(Y)\eta(Z)\end{aligned}$$

Using equation (5) and simplify we get

$$\begin{aligned}K(\overline{\nabla}_Y\xi, Z) &= (\beta^2 - \alpha^2 - \beta)g(Y,Z) - \alpha(2\beta+1)g(\phi Y, Z) \\ &\quad + (\alpha^2 - \beta^2 - \beta - 1)\eta(Y)\eta(Z)\end{aligned}$$

Proof. (18.4).

$$\begin{aligned}K(Y, \overline{\nabla}_Z\xi) &= K(Y, \nabla_Z\xi + \eta(\xi)Z) \\ &= K(Y, \nabla_Z\xi + Z).\end{aligned}$$

Using equations (5),(18.1) and (18.3), we get the required result.

Theorem 2. For a trans-sasakian manifold M^n under the condition (10) with semi symmetric non metric connection we have

$$\begin{aligned}(\overline{\nabla}_\xi K)(Y,Z) - (\overline{\nabla}_Y K)(\xi, Z) &= (2\beta - \alpha^2 + \beta^2)g(Y,Z) \\ -2\alpha(\beta+1)g(\phi Y, Z) &+ (\alpha^2 - \beta^2 - 3\beta)\eta(Y)\eta(Z)\end{aligned} \tag{19}$$

Proof. Since,

$$L_\xi g(Y,Z) = 2\beta[g(Y,Z) - \eta(Y)\eta(Z)] \tag{20}$$

for all X and Y, where L is Lie derivative.

From equation (20) and $k(Y,Z) = g(AY,Z)$, we have

$$(L_\xi K)(Y,Z) = 2\beta K(Y,Z) + \beta\eta(Y)\eta(Z). \tag{21}$$

Now,

$$\begin{aligned}(\overline{\nabla}_\xi K)(Y,Z) &= \xi K(Y,Z) - K(\overline{\nabla}_\xi Y,Z) - K(Y,\overline{\nabla}_\xi Z)\\ &= \xi K(Y,Z) - K(\nabla_\xi Y + \eta(Y)\xi, Z) - K(Y,\nabla_\xi Z + \eta(Z)\xi)\\ &= \xi K(Y,Z) - K([\xi,Y] + \nabla_Y\xi, Z) - \eta(Y)K(\xi,Z)\\ &\quad - K(Y,[\xi,Z] + \nabla_Z\xi) - \eta(Z)K(Y,\xi)\\ &= (L_\xi K)(Y,Z) - K(\nabla_Y\xi,Z) - K(Y,\nabla_Z\xi) + 2\eta(Y)\eta(Z).\end{aligned}$$

Using equation (17.1), (17.2), (18.1) and (21) we get

$$(\overline{\nabla}_\xi K)(Y,Z) = -(\beta-2)\eta(Y)\eta(Z). \tag{22}$$

Again,

$$\begin{aligned}(\overline{\nabla}_Y K)(\xi,Z) &= YK(\xi,Z) - K(\overline{\nabla}_Y\xi,Z) - K(\xi,\overline{\nabla}_Y Z)\\ &= -Y\eta(Z) - K(\overline{\nabla}_Y\xi,Z) - K(\xi,\nabla_Y Z + \eta(Z)Y)\\ &= -Y\eta(Z) - K(\overline{\nabla}_Y\xi,Z) + \eta(\nabla_Y Z) + \eta(Y)\eta(Z)\\ &= -[Y\eta(Z) - \eta(\nabla_Y Z)] - K(\overline{\nabla}_Y\xi,Z) + \eta(Y)\eta(Z)\\ &= -(\nabla_Y\eta)(Z) - K(\overline{\nabla}_Y\xi,Z) + \eta(Y)\eta(Z).\end{aligned}$$

Using equations (6), (18.3) and simplify, we have

$$\begin{aligned}(\overline{\nabla}_Y K)(\xi,Z) &= 2\alpha g(\phi Y,Z) - 2\beta g(Y,Z) + \alpha^2[g(Y,Z) - \eta(Y)\eta(Z)]\\ &+ 2\alpha\beta g(\phi Y,Z) - \beta^2[g(Y,Z) - \eta(Y)\eta(Z)] + 2(\beta+1)\eta(Y)\eta(Z). \end{aligned}\tag{23}$$

Subtracting equation (23) from (22), we get

$$\begin{aligned}(\overline{\nabla}_\xi K)(Y,Z) - (\overline{\nabla}_Y K)(\xi,Z) &= (2\beta - \alpha^2 + \beta^2)g(Y,Z)\\ &- 2\alpha(\beta+1)g(\phi Y,Z) + (\alpha^2 - \beta^2 - 3\beta)\eta(Y)\eta(Z).\end{aligned}$$

Theorem 3. For a trans-Sasakian manifold $M^n (n > 1)$ under the condition (10), we have

$$\begin{aligned}(\overline{\nabla}_\xi S)(Y,Z) - (\overline{\nabla}_Y S)(\xi,Z) &= (n-1)\alpha(\alpha^2 - \beta^2)g(\phi Y,Z)\\ &- \beta(n-1)(\alpha^2-\beta^2)g(Y,Z) - \alpha S(\phi Y,Z) + (\beta+1)S(Y,Z)\\ &- (n-1)(\alpha^2-\beta^2)\eta(Y)\eta(Z) \end{aligned}\tag{24}$$

Proof. For a symmetric endomorphism Q of the tangent space at a point of M, we express the Ricci tensor as

$$S(Y,Z)=g(QY,Z). \tag{25}$$

In trans-Sasakian manifold, from equations (20) and (25), we have

$$(L_\xi S)(Y,Z)=2\beta S(Y,Z)-2\beta(n-1)(\alpha^2-\beta^2)\eta(Y)\eta(Z). \tag{26}$$

Now,

$$\begin{aligned}(\overline{\nabla}_\xi S)(Y,Z)&=\xi S(Y,Z)-S(\overline{\nabla}_\xi Y,Z)-S(Y,\overline{\nabla}_\xi Z)\\ &=\xi S(Y,Z)-S(\nabla_\xi Y+\eta(Y)\xi,Z)-S(Y,\nabla_\xi Z+\eta(Z)\xi)\\ &=\xi S(Y,Z)-S([\xi,Y]+\nabla_Y\xi,Z)-\eta(Y)S(\xi,Z)\\ &\quad -S(Y,[\xi,Z]+\nabla_Z\xi)-\eta(Z)S(Y,\xi)\\ &=(L_\xi S)(Y,Z)-S(\nabla_Y\xi,Z)-S(Y,\nabla_Z\xi)-\eta(Y)S(\xi,Z)\\ &\quad -\eta(Z)S(Y,\xi).\end{aligned}$$

Using equations (5) and (26) we get

$$(\overline{\nabla}_\xi S)(Y,Z)=-2(n-1)(\alpha^2-\beta^2)\eta(Y)\eta(Z) \tag{27}$$

Again,

$$\begin{aligned}(\overline{\nabla}_Y S)(\xi,Z)&=YS(\xi,Z)-S(\overline{\nabla}_Y\xi,Z)-S(\xi,\overline{\nabla}_Y Z)\\ &=YS(\xi,Z)-S(\nabla_Y\xi+\eta(\xi)Y,Z)-S(\xi,\nabla_Y Z+\eta(Z)Y)\\ &=YS(\xi,Z)-S(\nabla_Y\xi,Z)-S(Y,Z)-S(\xi,\nabla_Y Z)-\eta(Z)S(\xi,Y).\end{aligned}$$

Using equations (5) and (12)

$$\begin{aligned}(\overline{\nabla}_Y S)(\xi,Z)&=(n-1)(\alpha^2-\beta^2)(\nabla_Y\eta)(Z)+\alpha S(\phi Y,Z)-(\beta+1)S(Y,Z)\\ &\quad+\beta\eta(Y)S(\xi,Z)-(n-1)(\alpha^2-\beta^2)\eta(Y)\eta(Z).\end{aligned}$$

Using (6) and (12), again we get

$$\begin{aligned}(\overline{\nabla}_Y S)(\xi,Z)=&-(n-1)(\alpha^2-\beta^2)\alpha g(\phi Y,Z)\\ &+\beta(n-1)(\alpha^2-\beta^2)g(Y,Z)+\alpha S(\phi Y,Z)-(\beta+1)S(Y,Z)\\ &-(n-1)(\alpha^2-\beta^2)\eta(Y)\eta(Z).\end{aligned} \tag{28}$$

Subtracting equation(28) from (27) we get

$$\begin{aligned}(\overline{\nabla}_\xi S)(Y,Z)-(\overline{\nabla}_Y S)(\xi,Z)&=(n-1)\alpha(\alpha^2-\beta^2)g(\phi Y,Z)\\ -\beta(n-1)(\alpha^2-\beta^2)g(Y,Z)&-\alpha S(\phi Y,Z)+(\beta+1)S(Y,Z)\\ &-(n-1)(\alpha^2-\beta^2)\eta(Y)\eta(Z).\end{aligned}$$

Definition. A trans-Sasakian manifold M^n is said to be $\eta-Einstein$ if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for any vector field X, Y where a and b are smooth function on M^n.

Theorem 4. Suppose a trans-Sasakian manifold M^n $(n > 2)$ under the condition (10) admits a semi-symmetric non metric connection whose curvature tensor with respect to this connection is conservative, then the manifold is $\eta-Einstein$.

Proof. Using equations (16), (19), (24) and simplify we get

$$S(Y,Z) = f_1(\alpha,\beta)g(Y,Z) + f_2(\alpha,\beta)\eta(Y)\eta(Z)$$

which is a $\eta - Einstein$ manifold.

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References

[1] Friedmann, A. and Schouten, J. A. : Math zeitschr, 21 (1924), 211-233.
[2] Hyden, H. A. : Proc. London Math. Soc., 34 (1932), 27-50.
[3] Yano, K. : Revue Roumine Math. pures App., 15 (1970), 1579-1586..
[4] Imai, T. : Tensor N. S., 24 (1972), 293-296.
[5] Amur, K. and Pujar, S. S. : Tensor N.S., 32 (1978), 35-38.
[6] Oubina, J. A. : Publ. Math. Debrecen, 32 no. 3-4 (1985), 187-193.
[7] Agashe, Nirmala S. and Chafle, Mangala R. : Ind. J. Pure Appl. Math., 23 (6) (1992), 399-409.
[8] De, U. C. and Shaikh, A. A. Bull. Cal. Math. Soc., 89 (1997), 349-354.
[9] Bagewadi, C. S. and Gatti, N. B. : Bull. Cal. Math. Soc., 97 (2005), 245-252.
[10] Bagewadi, C. S., Prakash, D. G. and Venkatesha : An. St. Univ. ovivius constanta, 15 (2) (2007), 5-18.
[11] Bagewadi, C. S. and Venkatesha : Turk. J. Math., 30 (2007), 1-11.
[12] Srivastava, S. K. : Semi-symmetric non metric connections on manifolds with generalised structure, Journal of Ravishankar University, 15-16 (2002), 49-58.