J. T. S.
Vol. 6 No.1 (2012), pp.117-126
https://doi.org/10.56424/jts.v6i01.10455

Projectively Flat Finsler Metric with Special Curvature Properties

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(Received: 25 March, 2011)

(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

The (α, β) metric is a Finsler metric which is constructed from Riemannian metric α and differential 1-form β. In this paper, we find the mean Cartan torsion I and mean Landsberg curvature J for Kropina metric $L = \frac{\alpha^2}{\beta}$ and also find the neccessary and sufficient condition for $J + cLI = 0$.

Key Words : Finsler space, Projective change, Kropina metric, Projective flatness, Mean Cartan torsion , Landsberg curvature.
2000 AMS Subject Classification : 53B40, 53C60.

1. Introduction

A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces. There are several notions of curvatures in Finsler geometry. The flag curvature **K** is an analogue of the sectional curvature in Riemannian geometry. The distortion τ is a basic invariant which characterizes Riemannian metrics among Finsler metrics, namely, $\tau = 0$ if and only if the Finsler metric is Riemannian. The vertical derivative of τ on tangent spaces gives rise to the mean Cartan torsion **I**. The horizontal derivative of τ along geodesics is the so called *S-curvature* **S**. The vertical Hessian of $(1/2)$**S** on tangent spaces is called the *E-curvature*. Thus if the S-curvature is *isotropic*, so is the *E-curvature*. The horizontal derivative of $\mathbf{I}$ along geodesics is called the *mean Landsberg curvature* $\mathbf{J}$. Thus $\mathbf{J}/\mathbf{I}$ is regarded as the relative growth rate of the mean Cartan torsion along geodesics. We see how these quantities are generated from the distortion except for the flag curvature $\mathbf{K}$. X. Chen and Z. Shen [2] have studied special class of Finsler metrics- Randers metrics with special curvature properties. In this paper, we extend the above results to Kropina metric for finding mean Cartan torsion and mean Landsberg curvature.

2. Preliminary Notes

Let L be a Finsler metric on a manifold M. In a standard local coordinate system (x^i, y^i) in TM, $L = L(x, y)$ is a function of (x^i, y^i). Let

$$g_{ij}(x,y) := \frac{1}{2}\left[L^2\right]_{y^i y^j}(x,y),$$

and $(g^{ij}) = (g_{ij})^{-1}$. For a non-zero vector $\mathbf{y} = y^i(\frac{\partial}{\partial x^i}) \mid_x \in T_xM$, L induces an inner product on T_xM,

$$g_y(u,v) = g_{ij}(x,y)u^i v^j,$$

where $u = u^i(\frac{\partial}{\partial x^i}) \mid_x$, $v = v^i(\frac{\partial}{\partial x^i}) \mid_x \in T_xM$. $g = \{g_{\mathbf{y}}\}$ is called the fundamental metric of L. For a non-zero vector $\mathbf{y} \in T_xM$, define

$$h_y(u,v) = g_y(u,v) - L^{-2}(\mathbf{y})g_y(y,u)g_y(y,v), \quad \mathbf{u},\mathbf{v} \in T_xM$$

$h = \{h_{\mathbf{y}}\}$ is called the angular metric of L.

The geodesics of L are characterized locally by

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0$$

The Riemannian curvature is a family of endomorphisms $\mathbf{R}_{\mathbf{y}} = R^i{}_k dx^k \otimes (\frac{\partial}{\partial x^i}) : T_xM \longrightarrow T_xM$, defined by

$$(R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}).$$

L is said to be constant flag curvature $\mathbf{K} = \lambda$, if

$$g_{\mathbf{y}}\left(\mathbf{R}_{\mathbf{y(u)}}, \mathbf{v}\right) = \lambda L(y)^2\, h_{\mathbf{y}}(\mathbf{u},\mathbf{v}),$$

or equivalently

$$R^i{}_k = \lambda\left\{L^2\delta^i_k - LL_{y^k}y^i\right\}.$$

There are many interesting non-Riemannian quantities in Finsler geometry [13]. Let $dV_L = \sigma(x)dx^1...dx^n$ denote the volume form of L, where

$$\sigma(x) = \frac{Vol(\mathbb{B}^n(1))}{Vol\{(y^i) \in R^n \mid L(y^i(\frac{\partial}{\partial x^i}) \mid_x) < 1\}}$$

For a non-zero vector $\mathbf{y} \in T_xM$, the distortion $\tau(\mathbf{y})$ is defined by

$$\tau(\mathbf{y}) = \ln\left[\frac{\sqrt{det(g_{ij}(x,y))}}{\sigma(x)}\right]$$

The distortion characterizes Riemannian metrics among Finsler metrics, namely, L is Riemannian if and only if $\tau = 0$.

The mean Cartan torsion $\mathbf{I_y} = I_i(x,y)dx^i : T_xM \longrightarrow R$ is defined as the vertical derivative of τ on T_xM,

$$I_i = \frac{\partial\tau}{\partial y^i} = \frac{1}{4}g^{jk}[L^2]_{y^iy^jy^k}. \tag{2.1}$$

The S-curvature $\mathbf{S}$ is defined as the horizontal derivative of τ along geodesics,

$$\mathbf{S}(\mathbf{y}) = \frac{d}{dt}\left[\tau\left(\dot{c}(t)\right)\right]\mid_{t=o},$$

where *c(t)* is the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y} \in T_xM$. A direction computation yields

$$\mathbf{S}(\mathbf{y}) = \frac{\partial G^i}{\partial y^i}(x,y) - \frac{y^i}{\sigma(x)}\frac{\partial\sigma}{\partial x^i}(x). \tag{2.2}$$

For a non-zero $\mathbf{y} \in T_xM$, the E-curvature $\mathbf{E_y} = E_{ij}(x,y)dx^i \otimes dx^j : T_xM \times T_xM \longrightarrow R$ is defined as the vertical Hessian of $(1/2)\mathbf{S}$ on T_xM,

$$E_{ij} = \frac{1}{2}\mathbf{S}_{y^iy^j} = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m\partial y^i\partial y^j}(x,y). \tag{2.3}$$

L is said to be *weakly Berwaldian* if $\mathbf{E} = 0$.

For a non-zero vector $\mathbf{y} \in T_xM$, the mean Landsberg curvature $\mathbf{J_y} = J_i(x,y)dx^i : T_xM \longrightarrow R$ is defined as the horizontal derivative of $\mathbf{I}$ along geodesics

$$J_i = y^j\frac{\partial I_i}{\partial x^i} - I_j\frac{\partial G^j}{\partial y^i} - 2G^j\frac{\partial I_i}{\partial y^j} = -\frac{1}{2}LL_{y^l}g^{jk}\frac{\partial^3 G^l}{\partial y^i\partial y^j\partial y^k}, \tag{2.4}$$

L is said to be *weakly Landsbergian* if $\mathbf{J} = 0$.

The above mentioned geometric quantities are computable for Randers metrics. Consider a Randers metric $L = \alpha + \beta$ be a Randers metric on a manifold M, where

$$\alpha(y) = \sqrt{a_{ij}y^i y^j}\ , \quad \beta(y) = b_i(x)y^i,$$

with $\|\beta\|_x := sup_{y \in T_x M}\beta(y)/\alpha(y) < 1$. An easy computation yields

$$g_{ij} = \frac{L}{\alpha}\left(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha}\right) + \left(\frac{y_i}{\alpha} + b_i\right)\left(\frac{y_j}{\alpha} + b_j\right), \tag{2.5}$$

where $y_i = a_{ij}y^j$. By an elementary argument in linear algebra, we obtain

$$det(g_{jk}) = \left(\frac{L}{\alpha}\right)^{n+1} det(a_{ij}). \tag{2.6}$$

Define $b_{i\backslash j}$ by

$$b_{i\backslash j}\theta^j = db_i - b_j\theta_i^j,$$

where $\theta^i = dx^i$ and $\theta_i^j = \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij} = \frac{1}{2}\left(b_{i\backslash j} + b_{j\backslash i}\right), \quad s_{ij} = \frac{1}{2}\left(b_{i\backslash j} - b_{j\backslash i}\right),$$
$$s^i_j = a^{ih}s_{hj}, \quad s_j = b_i s^i_j, \quad e_{ij} = r_{ij} + b_i s_j + b_j s_i.$$

Then the geodesics coefficients G^i are given by

$$G^i = \bar{G}^i + \frac{e_{00}}{2L}y^i - s_0 y^i + \alpha s^i_0, \tag{2.7}$$

where $\bar{G}^i$ denote the geodesics coefficients of α, $e_{00} = e_{ij}y^i y^j$, $s_0 = s_i y^i$ and $s^i_0 = s^i_j y^j$ [3].

3. Mean Cartan torsion I and mean Landsberg curvature J for Kropina metric

In this section, we determine the mean Cartan torsion I and mean Landsberg curvature for Kropina metric.

Theorem 3.1. For a Kropina metric $L = \frac{\alpha^2}{\beta}$, the mean Cartan torsion $\mathbf{I} = I_i dx^i$ and the mean Landsberg curvature $\mathbf{J} = J_i dx^i$ are given by

$$I_i = \frac{1}{2}(n+1)L^{-1}\alpha^{-2}\beta^{-2}\left\{\beta(2-\alpha^2)y_i - \alpha^4 b_i\right\}, \tag{3.1}$$

$$
\begin{aligned}
J_i \;=\;& \frac{1}{4}(n+1)L^{-2}\alpha^{-2}\beta^{-4}\left[\alpha^6\left\{\beta(4\beta s_i-4s_0-s_{i0})-2e_{00}b_i+2e_{00}-\beta e_{i0}\right\}\right.\\
&-\alpha^2\beta^2\left\{(5e_{00}y_i+2\beta e_{i0}+2e_{00}b_i)+(2\alpha s_{00}y_i+6s_0y_i+8y_i)\right\}\\
&+\alpha^4\beta\left\{\beta(e_{00}b_i+\beta e_{i0})+2\beta(2s_i-3s_0y_i)+4y_i(\beta+s_0)+2e_{00}y_i\right\}\\
&+2\beta^2e_{00}y_i(5\beta+2)-2\alpha^7\beta s_i\\
&+2\alpha^3\left\{\beta s_0y_i(3\alpha^2-4)+2s_0y_i(\alpha^2+2)-(5\alpha^2+6)\beta^2s_{i0}\right\}\\
&\left.+\,4\alpha\beta\left\{7\beta s_{00}y_i-2\alpha s_0y_i\right\}\right].
\end{aligned}
\tag{3.2}
$$

Proof. First substituting (2.6) into (2.1).

$$
\begin{aligned}
\Rightarrow I_i &= \frac{\partial}{\partial y^i}\left(\frac{1}{2}\log\left(\sqrt{\det(g_{ij}(x,y))}\right)-\log(\sigma(x))\right),\\
&= \frac{1}{2}\frac{\partial}{\partial y^i}\left(\log\left(\frac{L}{\alpha}\right)^{n+1}\right),
\end{aligned}
$$

differentiating the above equation we get (3.1). Now we are going to compute J_i.

Let

$$H^i=\frac{e_{00}}{2L}y^i-s_0y^i+\alpha s_0^i.$$

We can rewrite (2.4) as follows

$$J^i=y^jI_{i\backslash j}-I_jH^j_{.i}-2H^jI_{i.j}\,, \tag{3.3}$$

where $H^j_{.i}=\frac{\partial H^j}{\partial y^i}$, $I_{i.j}=\frac{\partial I_i}{\partial y^j}$ and $I_{i\backslash j}$ are defined by

$$dI_i-I_j\frac{\partial\bar{G}^j}{\partial y^i\partial y^k}dx^k=I_{i\backslash j}dx^j+I_{i.j}\left(dy^j+\frac{\partial\bar{G}^j}{\partial y^k}dx^k\right).$$

By a direct computation, we obtain

$$
\begin{aligned}
I_{i.j} \;=\;& -\frac{(n+1)}{2}L^{-2}\alpha^{-2}\beta^{-4}\left\{\beta^2(4-2\alpha^2)y_iy_j+\beta(\alpha^2-2)b_jy_i\right.\\
&\left.-\,2\alpha^4\beta b_iy_j+\alpha^6b_ib_j\right\}-(n+1)L^{-1}\alpha^{-4}\beta^{-2}y_j\left\{2\beta y_i-\alpha^2\beta y_i-\alpha^4b_i\right\}\\
&+\frac{(n+1)}{2}L^{-1}\alpha^{-2}\beta^{-3}\left\{(\alpha^2-2)b_jy_i+\beta^2(2-\alpha^2)a_{ij}\right.\\
&\left.-\,2\beta^2y_iy_j+2\alpha^4b_ib_j-4\beta\alpha^2b_iy_j\right\},\\
H^j_{.i} \;=\;& \frac{e_{00}}{2L}\delta^j_i+\frac{e_{i0}}{2L}y^j-\frac{e_{00}}{2L^2\beta^2}y^j\left(2\beta y_i-\alpha^2b_i\right)\\
&-s_iy^j-s_0\delta^j_i+\alpha s^j_i+\alpha^{-1}s^j_0y_i\,,
\end{aligned}
$$

where $b_{i\backslash 0} = b_{i\backslash j}y^j$ and $b_{0\backslash 0} = b_{i\backslash j}y^i y^j$. Observe that

$$b_{i\backslash j} = r_{ij} + s_{ij} = e_{ij} - b_i s_j - b_j s_i + s_{ij}.$$

We have

$$b_{i\backslash 0} = e_{i0} - b_i s_0 - s_i\beta + s_{i0}, \qquad b_{0\backslash 0} = e_{00} - 2s_0\beta.$$

By these identities, we obtain

$$\begin{aligned}
y^j I_{i\backslash j} = & \frac{1}{2}(n+1)L^{-2}\beta^{-2}b_{0\backslash 0}\left\{\beta(2-\alpha^2)y_i - \alpha^4 b_i\right\} \\
& + \frac{1}{2}(n+1)L^{-1}\alpha^{-2}\beta^{-2}\left\{\alpha^2 b_{0\backslash 0}y_i - 2b_{0\backslash 0 y_i}\right\} \\
& + \frac{1}{2}(n+1)L^{-1}\beta^{-3}\left\{2\alpha^2 b_{0\backslash 0} - \alpha^2\beta b_{i\backslash 0}\right\} \\
= & \frac{1}{2}(n+1)L^{-2}\beta^{-2}\left(e_{00} - 2s_0\beta\right)\left\{\beta(2-\alpha^2)y_i - \alpha^4 b_i\right\} \\
& + \frac{1}{2}(n+1)L^{-1}\alpha^{-2}\beta^{-2}\left\{(\alpha^2-2)(e_{00}-2s_0\beta)y_i\right\} \\
& + \frac{1}{2}(n+1)L^{-1}\beta^{-3}\{\alpha^2(2e_{00} - \beta e_{i0}) + \alpha^2\beta s_0(b_i - 4) + \alpha^2\beta(s_i\beta - s_{i0})\}.
\end{aligned}$$

Plugging for $H^j I_{i.j}$, $I_j H^j_{.i}$, $y^j I_{i\backslash j}$ into (3.3) yields (3.2).

Remark 1. If $I_i = 0$ i.e., distortion τ is independent of y^i or τ is constant.

$$\begin{aligned}
\Rightarrow I_i = \frac{1}{2}(n+1)L^{-1}\alpha^{-2}\beta^{-2}\left\{\beta(2-\alpha^2)y_i - \alpha^4 b_i\right\} &= 0 \\
\Rightarrow b_i L^2 - \left(1 - \frac{2}{\alpha^2}\right)y_i L &= 0,
\end{aligned}$$

a quadratic equation in L. Solving for L, we get the metric in the form $L = \left(\frac{2}{\alpha^2} - 1\right)\frac{y_i}{b_i}$.

Theorem 3.2. If the Kropina metric $L = \alpha^2/\beta$ is projectively flat, then mean Landsberg curvature reduces to

$$\begin{aligned}
J_i = & \frac{(n+1)}{4}L^{-2}\alpha^{-2}\beta^{-4}\left[\alpha^6\{-2e_{00}b_i + 2e_{00} - \beta e_{i0}\}\right. \\
- & \alpha^2\beta^2\{5e_{00}y_i + 2\beta e_{i0} + 2e_{00}b_i + 8y_i\} \\
+ & \alpha^4\beta\{\beta(e_{00}b_i + \beta e_{i0}) + 4\beta y_i + 2e_{00}y_i\} \\
+ & \left. 2\beta^2 e_{00}y_i(5\beta + 2)\right].
\end{aligned} \tag{3.4}$$

4. Kropina Metric with $\mathbf{J} + cL\mathbf{I} = 0$

As we know, the mean Landsberg curvature J can be expressed in terms of α and β. But the formula (3.2) is very complicated. So the equation $J+cLI = 0$ is complicated too. In this section, we are going to find a simpler necessary and sufficient condition for $J + cLI = 0$.

Theorem 4.3. Let $L = \alpha^2/\beta$ be a Kropina metric on a manifold M. For a scalar function $c = c(x)$ on M, the following are equivalent

(a) $\mathbf{J} + cL\mathbf{I} = 0$

(b) $e_{00} = \xi\alpha^4\beta - 2c(\alpha^2 - \beta^2)$, where $\xi = 60/\omega$ and β is closed.

Proof. Let $f_{ij} = e_{ij} - 2c(a_{ij} - b_ib_j)$ and $f_{i0} = f_{ij}y^j$, $f_{00} = f_{ij}y^iy^j$. We have

$$\begin{aligned}
&\alpha^6 (2e_{00} - 2e_{00}b_i - \beta e_{i0}) - \alpha^2\beta^2 (5e_{00}y_i + 2\beta e_{i0} + 2e_{00}b_i) \\
&+\alpha^4\beta^2 (e_{00}b_i + \beta e_{i0}) + 2\alpha^4\beta e_{00}y_i + 2\beta^2 e_{00}y_i (5\beta + 2) \\
=\ &\alpha^6 (2f_{00} - 2f_{00}b_i - \beta f_{i0}) - \alpha^2\beta^2 (5f_{00}y_i + 2\beta f_{i0} + 2f_{00}b_i) \\
&+\alpha^4\beta^2 (f_{00}b_i + \beta f_{i0}) + 2\alpha^4\beta f_{00}y_i + 2\beta^2 f_{00}y_i (5\beta + 2) \\
&+2c(\alpha^2 - \beta^2) \left[2\alpha^2 + \left(\alpha^2\beta^2 + 2\beta^2 - 2\right) \left(\alpha^2 b_i + \beta y_i\right)\right. \\
&\left. + \beta y_i \left(10\beta^2 - 5\alpha^2\beta + 2\alpha^4 + 4\beta + 1\right)\right].
\end{aligned}$$

Substituting above equation into (3.2), we see that $\mathbf{J} + cL\mathbf{I} = 0$ if and only if

$$\begin{aligned}
&\beta^2 (4\beta s_i - 4s_0 - s_{i0}) - \alpha^2\beta^2 (2\alpha s_{00}y_i + 2s_0y_i + 8y_i) \\
&+4\alpha^4\beta y_i (\beta + s_0) - 2\alpha^7\beta s_i + 2\alpha^3\beta s_0y_i \left(3\alpha^2 - 4\right) \\
&+4\alpha^3 s_0y_i \left(\alpha^2 + 2\right) + 4\alpha\beta (7\beta s_{00}y_i - 2\alpha s_0y_i) + \alpha^6 (-2f_{00}b_i + 2f_{00} - \beta f_{i0}) \\
&-2\alpha^2\beta^2 f_{00}b_i + \alpha^4\beta^2 (f_{00}b_i + \beta f_{i0}) + 2\alpha^4\beta f_{00}y_i + 2\beta^2 f_{00}y_i (5\beta + 2) \\
&+(\alpha^2 - \beta^2) \left[4c\alpha^2 + 2c\alpha^2 b_i \left(2\beta^2 - \alpha^2 - 2\right)\right] - 2c\alpha^4\beta^4 b_i \\
&+2c\beta y_i(\alpha^2 - \beta^2) \left(12\beta^2 + \alpha^4 - 5\alpha^2\beta + 4\beta - 1\right) \\
&+2c\alpha^2\beta y_i \left(2\alpha^2 - \beta^4\right) = 0, \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
2\alpha^2\beta^2 (2s_i - 3s_0y_i) - 2\alpha\beta^2 s_{i0} \left(5\alpha^2 + 6\right) - \beta^2 (5f_{00}y_i + 2\beta f_{i0}) &= 0, \\
\Rightarrow 2\alpha^2 (2s_i - 3s_0y_i) - 2\alpha s_{i0} \left(5\alpha^2 + 6\right) - (5f_{00}y_i + 2\beta f_{i0}) &= 0. \qquad (4.2)
\end{aligned}$$

Differentiating (4.2) with respect to y^j, y^k and y^l, we obtain

$$\begin{aligned}
0 = &- 12\beta (b_lb_jf_{ik} + b_kb_lf_{ij} + b_kb_jf_{il}) \\
&- 12b_jb_kb_lf_{i0} - 10\beta [f_{k0} (b_la_{ij} + b_ja_{il}) + f_{l0} (b_ka_{ij} + b_ja_{ik})
\end{aligned}$$

$$
\begin{aligned}
&+ f_{j0}\left(b_l a_{ik} + b - k a_{il}\right)\big] - 5\beta\left[f_{jk}\left(2y_i b_l + \beta a_{il}\right)\right. \\
&+ f_{kl}\left(2b_j y_i + \beta a_{ij}\right) + f_{jl}\left(2b_k y_i + \beta a_{ik}\right)\big] \\
&-10 f_{00}\left(b_k b_l a_{ij} + b_j b_l a_{ik} + b_j b_k a_{il}\right) \\
&-10 y_i\left(b_k b_l f_{j0} + b_j b_l f_{k0} + b_j b_k f_{l0}\right). \qquad (4.3)
\end{aligned}
$$

Contracting (3.4) with a^{kl} yields

$$
\begin{aligned}
&\omega f_{ij} + 15\beta\left(4\alpha^2 + \lambda\beta\right) a_{ij} + 20\alpha\left(3n + 10\right) s_{ij} \\
&+12(n+4)\left[s_i y_j + s_j y_i\right] + 24 s_i y_j = 0, \qquad (4.4)
\end{aligned}
$$

where $\omega = 94\beta^2 + 20\alpha^2 + 24\beta$, $\lambda = a^{kl} f_{kl}$. It follows from (4.4) that

$$
\begin{aligned}
f_{ij} &= \frac{-15\beta(4\alpha^2 + \lambda\beta)a_{ij}}{\omega} - 12\frac{(2n+5)}{\omega}\left[s_i y_j + s_j y_i\right], \qquad (4.5) \\
20\alpha(3n+10)s_{ij} &= 12n(s_i y_j + s_j y_i) - 12(s_i y_j - s_j y_i). \qquad (4.6)
\end{aligned}
$$

Contracting (4.6) with $b^i = b_r a^{ri}$ yields

$$s_j = 0.$$

Substituting $s_j = 0$ into (4.6) we obtain that

$$s_{ij} = 0,$$

and

$$f_{ij} = \frac{-15\beta(4\alpha^2 + \lambda\beta)}{\omega} a_{ij}, \quad \textit{where } \omega = 94\beta^2 + 24\beta + 20\alpha^2. \qquad (4.7)$$

Now equation (3.3) simplifies to

$$\lambda\left[\mu b_i + \varsigma y_i + p\right] = 0, \qquad (4.8)$$

where

$$
\begin{aligned}
\mu \;=\; &\big[15\alpha^4\beta^2\left(3\alpha^4 - 2\alpha^2\beta^2 + 2\beta^2\right) \\
&+\frac{1}{\lambda}\big\{60\alpha^6\beta\left(3\alpha^4 - 2\alpha^2\beta^2 + 2\beta^2\right) \\
&-8c\alpha^4\left(6\beta - 5\alpha^2\right)\left(\alpha^2 + 2\right) + 4\alpha^2\beta^4\left(94 - 27\alpha^2\right) \\
&+ 4c\alpha^2\beta^3\left(2 - \alpha^2\right)\left(12 + 47\beta^3 + 12\beta^2\right) - 4c\alpha^6\beta^2\left(17 + 10\beta^2\right)\big\}\big],
\end{aligned}
$$

$$
\begin{aligned}
\varsigma = &\big[30\alpha^2\beta^3\left(\alpha^4 - 5\beta^2 - 2\beta\right) + \frac{1}{\lambda}\{120\alpha^4\beta^2\left(\alpha^4 - 5\beta^2 - 2\beta\right) \\
&- 4c\beta^5\left(1 + 133\alpha^2 - 57\alpha^4\right) + 4c\beta^4\left(12 + 160\alpha^2 - 297\alpha^4\right) \\
&+ 4c\alpha^2\beta^3\left(44\alpha^2 + 37\alpha^4 - 11\right) + 8c\alpha^2\beta^2\left(32\alpha^2 - 19\alpha^4 - 6\right) \\
&- 16\left(166 + 3\alpha^2\right)c\beta^6 + 4c\beta^7\left(564 - 47\alpha^2\right) + 40c\alpha^2\beta\left(\alpha^2 + 2\right) - 940c\alpha^6\beta^6\}\big],
\end{aligned}
$$

and

$$p=\frac{1}{\lambda}\left[120\alpha^{1}0\beta+4c\omega\left(\alpha^{2}-\beta^{2}\right)\alpha^{2}\right]-30\alpha^{8}\beta^{2}.$$

Taking $y_i=b_i$ in (4.8), we obtain

$$\lambda\left(\mu+\varsigma+q\right)b_i=0. \tag{4.9}$$

Assume that $\beta\neq 0$. It follows from (4.9) that $\lambda=0$. From (4.7), we conclude that $f_{ij}=-\xi\alpha^2\beta a_{ij}$ where $\xi=60/\omega$. Substituting this in (), we obtain

$$\begin{aligned} e_{ij} &= \xi\alpha^2\beta a_{ij}-2c\left(a_{ij}-b_ib_j\right),\\ e_{00} &= \xi\alpha^4\beta-2c\left(\alpha^2-\beta^2\right), \end{aligned}$$

where $\xi=\frac{60}{\omega}$, with $\omega=94\beta^2+20\alpha^2+24\beta$.

Conversely, we suppose that $e_{00}=\xi\alpha^4\beta-2c(\alpha^2-\beta^2)$. Then

$$\begin{aligned} e_{i0} &= \xi\alpha^2\beta y_i-2c(y_i-b_i\beta),\\ e_{00} &= \xi\alpha^4\beta-2c(\alpha^2-\beta^2). \end{aligned}$$

Considering e_{ij} terms of (3.2) we get

$$\begin{aligned} -2e_{00}b_i+2e_{00}-\beta e_{i0} &= b_i\left[4c\left(\alpha^2-\beta^2\right)-2c\beta^2-2\xi\alpha^4\beta\right]\\ &\quad +y_i\left(2c\beta-\xi\alpha^2\beta\right)+2\left[\xi\alpha^4\beta-2c\left(\alpha^2-\beta^2\right)\right], \qquad (4.10)\\ 5e_{00}y_i+2\beta e_{i0}+2e_{00}b_i &= b_i\left[4c\beta^2+2\xi\alpha^4\beta-4c\left(\alpha^2-\beta^2\right)\right]\\ &\quad +y_i\left[5\xi\alpha^4\beta-10c\left(\alpha^2-\beta^2\right)+2\xi\alpha^2\beta^2-4c\beta\right], \qquad (4.11)\\ \beta e_{00}b_i+\beta^2e_{i0}+2e_{00}y_i &= b_i\left[\xi\alpha^4\beta^2-2c\beta\left(\alpha^2-\beta^2\right)+2c\beta^3\right]\\ &\quad +y_i\left[\xi\alpha^2\beta^3-2c\beta^2+2\xi\alpha^4\beta-4c\left(\alpha^2-\beta^2\right)\right]. \qquad (4.12) \end{aligned}$$

Substituting them into (3.2) yields

$$\begin{aligned} J_i \;=\; & \frac{(n+1)}{2}L^{-2}\beta^{-4}\left[-c\left[2b_i\left\{2\alpha^4\beta^2+2\beta^4-\alpha^6-\alpha^2\beta^2-\alpha^2\beta^4\right\}\right.\right.\\ & -2\alpha^{-2}\beta y_i\left\{\alpha^4\beta^2-\alpha^6+\left(\alpha^2-\beta^2\right)\left(5\alpha^2\beta-10\beta^2-4\beta\right)\right\}\\ & \left.+2\alpha^4\left(\alpha^2-\beta^2\right)\right]+\xi\alpha^4\beta\frac{b_i}{2}\left\{\alpha^2\beta^2-2\alpha^4-2\beta^2\right\}\\ & -\alpha^2\beta^2\frac{y_i}{2}\left\{\xi\alpha^6+5\xi\alpha^4\beta-8\xi\alpha^2\beta^2-\xi\alpha^4\beta^2-2\xi\alpha^4-4\xi\alpha^2\beta-4\alpha^2-8\right\}\\ & +\xi\alpha^8\beta+\alpha^2\beta s_i\left\{2\alpha^2\beta-\alpha^3-\beta\right\}-s_0y_i\left\{2\alpha^4\beta+3\alpha^2\beta^2-2\alpha^2\beta\right.\\ & \left.-\alpha\beta\left(3\alpha^2-4\right)-2\alpha\left(\alpha^2+2\right)+4\beta\right\}\\ & \left.+\alpha\beta\frac{s_{i0}}{2}\left\{28\beta-\alpha^3-12\alpha^2\beta-12\beta\right\}\right]. \qquad (4.13) \end{aligned}$$

Further, suppose that β is closed, hence $s_{ij} = 0$. From (3.1) and (4.13), we obtain

$$J_i = -\frac{1}{2}(n+1)cL^{-2}\beta{-}4\left\{\alpha^2\beta(2-\alpha^2)y_i - \alpha^6 b_i\right\} = -cLI_i.$$

This proves the theorem.

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