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On Conservative C-Bochner Curvature Tensor in trans-Sasakian Manifold admitting Semi-symmetric metric Connection

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

The paper deals with the study on conservative C-Bochner curvature tensor in a trans-Sasakian manifold admitting semi-symmetric metric connection.

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1. Introduction

In 1924, Friedman and Schouten [8] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932, Hayden [10] introduced the idea of metric connection with a torsion on a Riemannain manifold. A systematic study of semi-symmetric metric connection on a Riemannain manifold has been given by Yano [14] in 1970 and later studied by K.S.Amur and S.S.Pujar, [1] C.S.Bagewadi, [2] U.C.De et al, [7] Sharafuddin and Hussain [13] and others.

The authors U.C.De and Absos Ali Shaikh, [7] C.S.Bagewadi and Venkatesha, [4] have obtained results on the conservativeness of different curvature tensors like projective, pseudo projective, conformal in K- contact and trans-Sasakian manifolds. The authors C.S. Bagewadi, D.G. Prakasha and Venkatesha [3] have extended the above study to trans-Sasakian manifolds admitting semi-symmetric metric connection.

In this paper we study conservative C-Bochner curvature tensor in a trans-Sasakian manifold admitting semi-symmetric metric connection and obtain some results.

2. Preliminaries

Let M^n be an almost contact metric manifold [5] with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta.\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM^n$.

An almost contact metric structure (ϕ, ξ, η, g) on M^n is called a trans-Sasakian structure [11] if $(M^n \times R, J, G)$ belongs to the class w_4, [9] where J is the almost complex structure on $M^n \times R$ defined by $J(X, \lambda d/dt) = (\phi X - \lambda\xi, \eta(X)d/dt)$ for all vector fields X on M^n and smooth functions λ on $M^n \times R$ and G is the product metric on $M^n \times R$. This may be expressed by the condition [6]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

for some smooth functions α and β on M^n, and we say that the trans-Sasakian structure is of type (α, β).

Let M^n be a trans-Sasakian manifold. From (2.4) it is easy to see that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In an n-dimensional trans-Sasakian manifold, we have

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{2.7}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$S(X, \xi) = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (n-2)X\beta - (\phi X)\alpha. \tag{2.9}$$

Further in a trans-Sasakian manifold of type (α, β) we have

$$\phi(grad\alpha) = (n-2)grad\beta. \tag{2.10}$$

Using (2.10) the equations (2.7) and (2.9) reduces to

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{2.11}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\eta(X). \tag{2.12}$$

In this paper we study trans-Sasakian manifold under the condition (2.10).

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g and ∇ be the Levi-Civita connection on M^n. A linear connection $\widetilde{\nabla}$ on (M^n, g) is said to be semi-symmetric [14] if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies

$$T(X, Y) = \pi(Y)X - \pi(X)Y, \tag{2.13}$$

where π is a 1-form on M^n with the associated vector field ρ, i.e. $\pi(X) = g(X, \rho)$ for any differentiable vector field X on M^n.

A semi-symmetric connection $\widetilde{\nabla}$ is called semi-symmetric metric connection [10] if $\widetilde{\nabla}g = 0$.

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form π of (2.13) with the contact form η, i.e. by setting [13]

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \tag{2.14}$$

with ξ as associated vector field. i.e., $g(X, \xi) = \eta(X)$.

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ of M^n has been obtained by K.Yano, [14] which is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{2.15}$$

Further a relation between the curvature tensor R and $\tilde{R}$ of type $(1, 3)$ of the connections ∇ and $\tilde{\nabla}$ respectively are given by

$$\begin{aligned}\tilde{R}(X, Y)Z = R(X, Y)Z - \tilde{\alpha}(Y, Z)X + \tilde{\alpha}(X, Z)Y \\ -g(Y, Z)FX + g(X, Z)FY,\end{aligned} \tag{2.16}$$

where $\tilde{\alpha}$ is a tensor field of type $(0, 2)$ defined by

$$\begin{aligned}\tilde{\alpha}(Y, Z) = g(FY, Z) &= (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y, Z), \\ &= (\tilde{\nabla}_Y \eta)(Z) - \frac{1}{2}\eta(\xi)g(Y, Z),\end{aligned} \tag{2.17}$$

for any vector fields Y, Z and for a tensor field F of type $(1, 1)$.

From (2.16), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-2)\tilde{\alpha}(Y,Z) - \tilde{A}\cdot g(Y,Z), \tag{2.18}$$

where $\tilde{S}$ denotes the Ricci tensor with respect to $\tilde{\nabla}$, $\tilde{A} = Tr\cdot\tilde{\alpha}$.

Differentiating (2.18) covariantly with respect to X, we obtain

$$\begin{aligned}(\tilde{\nabla}_X\tilde{S})(Y,Z) &= (\nabla_X S)(Y,Z) - (n-2)(\nabla_X\tilde{\alpha})(Y,Z) - \eta(Y)S(X,Z)\\ &+(n-2)\eta(Y)\tilde{\alpha}(X,Z) + g(X,Y)S(\xi,Z) - (n-2)g(X,Y)\tilde{\alpha}(\xi,Z)\\ &-\eta(Z)S(X,Y) + (n-2)\eta(Z)\tilde{\alpha}(Y,X) + g(X,Z)S(Y,\xi)\\ &-(n-2)g(X,Z)\tilde{\alpha}(Y,\xi).\end{aligned} \tag{2.19}$$

From (2.19), it follows that

$$\tilde{\nabla}_X\tilde{r} = d\tilde{r}(X) = \nabla_X r - (n-2)(\nabla_X\tilde{A}). \tag{2.20}$$

3. Some Basic Results

Theorem 3.1. For a trans-Sasakian manifold M^n, $n > 1$ under the condition (2.10), we have

$$\begin{aligned}[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] &= \beta S(Y,Z) - (n-1)(\alpha^2-\beta^2)\beta g(Y,Z)\\ &-(n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z) + \alpha S(Y,\phi Z).\end{aligned} \tag{3.1}$$

Proof. For a symmetric endomorphism Q of the tangent space at a point of M^n, we express the Ricci tensor S as

$$S(X,Y) = g(QX,Y). \tag{3.2}$$

Further, it is known that [6]

$$(L_\xi g)(X,Y) = 2\beta[g(X,Y) - \eta(X)\eta(Y)], \tag{3.3}$$

for all X and Y, where L is the Lie derivation.

In a trans-Sasakian manifold, from (3.2) and (3.3) we have

$$(L_\xi S)(X,Y) = 2\beta S(X,Y) - 2\beta(n-1)(\alpha^2-\beta^2)\eta(X)\eta(Y), \tag{3.4}$$

$$(L_\xi\widetilde{\alpha})(X,Y) = 2\beta\widetilde{\alpha}(X,Y) + \beta\eta(X)\eta(Y). \tag{3.5}$$

Consider

$$\begin{aligned}(\nabla_\xi S)(Y,Z) &= \xi S(Y,Z) - S(\nabla_\xi Y,Z) - S(Y,\nabla_\xi Z)\\ &= (L_\xi S)(Y,Z) - S(\nabla_Y\xi,Z) - S(Y,\nabla_Z\xi).\end{aligned}$$

Using (2.5), (2.12) and (3.4) in the above equation, we obtain

$$(\nabla_\xi S)(Y,Z) = 0. \tag{3.6}$$

Consequently,

$$(\nabla_\xi r)(Y,Z) = dr(\xi) = 0. \tag{3.7}$$

Also, we know that

$$(\nabla_Y S)(\xi,Z) = YS(\xi,Z) - S(\nabla_Y\xi,Z) - S(\xi,\nabla_Y Z).$$

By virtue of (2.5) and (2.12), then using (2.6) above equation takes the form

$$\begin{aligned}(\nabla_Y S)(\xi,Z) &= (n-1)(\alpha^2-\beta^2)\beta g(Y,Z) - \beta S(Y,Z)\\ &\quad +(n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z) - \alpha S(Y,\phi Z).\end{aligned} \tag{3.8}$$

We have

$$\begin{aligned}(\nabla_\xi\widetilde{\alpha})(Y,Z) &= \xi\widetilde{\alpha}(Y,Z) - \widetilde{\alpha}(\nabla_\xi Y,Z) - \widetilde{\alpha}(Y,\nabla_\xi Z)\\ &= (L_\xi\widetilde{\alpha})(Y,Z) - \widetilde{\alpha}(\nabla_Y\xi,Z) - \widetilde{\alpha}(Y,\nabla_Z\xi).\end{aligned}$$

Using (2.17) and (3.5) in above equation, we obtain

$$(\nabla_\xi\widetilde{\alpha})(Y,Z) = 0. \tag{3.9}$$

Consequently,

$$(\nabla_\xi\widetilde{A})(Y,Z) = d\widetilde{A}(\xi) = 0. \tag{3.10}$$

We know that

$$(\nabla_Y\widetilde{\alpha})(\xi,Z) = Y\widetilde{\alpha}(\xi,Z) - \widetilde{\alpha}(\nabla_Y\xi,Z) - \widetilde{\alpha}(\xi,\nabla_Y\xi).$$

Using (2.17) in the above equation and simplifying we get

$$\begin{aligned}(\nabla_Y\widetilde{\alpha})(\xi,Z) = {}& \alpha g(\phi Y,Z) + 2\alpha\beta g(\phi Y,Z) + [\alpha^2 - \beta(\beta+1)](g(Y,Z)\\ & -\eta(Y)\eta(Z)).\end{aligned} \tag{3.11}$$

4. Trans-Sasakian Manifold Admitting a Semi-symmetric metric Connection with $Div\,\widetilde{B} = 0$

The C-Bochner curvature tensor with respect to semi-symmetric metric connection is given by

$$\begin{aligned}\tilde{B}(X,Y)Z &= \tilde{R}(X,Y)Z + \frac{1}{n+3}[g(X,Z)\tilde{Q}Y - \tilde{S}(Y,Z)X - g(Y,Z)\tilde{Q}X \\ &+ \tilde{S}(X,Z)Y + g(\phi X,Z)\tilde{Q}\phi Y - \tilde{S}(\phi Y,Z)\phi X - g(\phi Y,Z)\tilde{Q}\phi X \\ &+ \tilde{S}(\phi X,Z)\phi Y + 2\tilde{S}(\phi X,Y)\phi Z + 2g(\phi X,Y)\tilde{Q}\phi Z + \eta(Y)\eta(Z)\tilde{Q}X \\ &- \eta(Y)\tilde{S}(X,Z)\xi + \eta(X)\tilde{S}(Y,Z)\xi - \eta(X)\eta(Z)\tilde{Q}Y] \\ &- \frac{\tilde{D}+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\ &+ \frac{\tilde{D}}{n+3}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ &-\eta(X)g(Y,Z)\xi] - \frac{\tilde{D}-4}{n+3}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{4.1}$$

where, $\tilde{D} = \frac{(n-1+\tilde{r})}{(n+1)}$

Differentiating (4.1) covariantly followed by contraction and simplification we get $div\tilde{B}$. By virtue of conservativeness of $\tilde{B}$ i.e. $div\tilde{B} = 0$ and using (2.17), (2.18), (2.19) and (2.20)

we obtain,

$$\begin{aligned}&\frac{n+2}{n+3}\,[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - (n-2)(\nabla_X \tilde{\alpha})(Y,Z) - \eta(Y)S(X,Z)] \\ &- (n-2)\eta(Y)\alpha g(\phi X,Z) + (n-2)\eta(Y)\beta g(\phi X,\phi Z) \\ &+ 2(n-2)\eta(Z)\alpha g(\phi X,Y) + g(X,Z)S(Y,\xi) \\ &+ (n-2)\eta(Y)g(X,Z) + (n-2)(\nabla_Y \tilde{\alpha})(X,Z) + \eta(X)S(Y,Z) \\ &+ (n-2)\eta(X)\alpha g(\phi Y,Z) - (n-2)\eta(X)\beta g(\phi Y,\phi Z) - g(Y,Z)S(X,\xi) \\ &- (n-2)\eta(X)g(Y,Z)] + \frac{1}{n+3}\left[\frac{1}{2}g(X,Z)\nabla_Y r - \frac{n-2}{2}g(X,Z)(\nabla_Y \tilde{A})\right. \\ &- \frac{1}{2}g(Y,Z)\nabla_X r + \frac{n-2}{2}g(Y,Z)(\nabla_X \tilde{A}) - g(\phi X,Z)(\nabla_{\phi Y} r) \\ &+ (n-2)g(\phi X,Z)(\nabla_{\phi Y}\tilde{A}) - (\nabla_{\phi X} S)(\phi Y,Z) + (n-2)(\nabla_{\phi X}\tilde{\alpha})(\phi Y,Z) \\ &+ (n-2)\eta(Z)\alpha g(\phi^2 Y,\phi X) - g(\phi X,Z)S(\phi Y,\xi) - g(\phi Y,Z)(\nabla_{\phi X} r) \\ &+(n-2)g(\phi Y,Z)(\nabla_{\phi X}\tilde{A}) + (\nabla_{\phi Y} S)(\phi X,Z) \\ &- (n-2)(\nabla_{\phi Y}\tilde{\alpha})(\phi X,Z) - (n-2)\eta(Z)\alpha g(\phi^2 X,Y) + g(\phi Y,Z)S(\phi X,\xi) \\ &+ 2(\nabla_{\phi Z} S)(\phi X,Y) - 2(n-2)(\nabla_{\phi Z}\tilde{\alpha})(\phi X,Y) + 2g(\phi Z,\phi X)S(\xi,Y) \\ &+2\,(n-2)\eta(Y)g(\phi Z,\phi X) - 2\eta(Y)S(\phi Z,\phi X) - 2(n-2)\eta(Y)g(\phi^2 X,\phi Z)\end{aligned}$$

$$+ 2(n-2)\eta(Y)\beta g(\phi^2 X, \phi^2 Z) + 2g(\phi Z, Y)S(\phi X, \xi) - 2g(\phi X, Y)(\nabla_{\phi Z} r)$$
$$+ (n-2)2g(\phi X, Y)(\nabla_{\phi Z}\tilde{A}) - (\nabla_\xi S)(X, Z)\eta(Y) + (n-2)(\nabla_\xi \tilde{\alpha})(X, Z)\eta(Y)$$
$$+ \frac{1}{2}\eta(Y)\eta(Z)\nabla_X r - \frac{n-2}{2}\eta(Y)\eta(Z)(\nabla_X \tilde{A}) + \eta(X)(\nabla_\xi S)(Y, Z)$$
$$- (n-2)(\nabla_\xi \tilde{\alpha})(Y, Z)\eta(X) - \frac{1}{2}\eta(X)\eta(Z)\nabla_Y r + \frac{n-2}{2}\eta(X)\eta(Z)(\nabla_Y \tilde{A})\Big]$$
$$+ \frac{1}{(n+1)(n+3)}[g(\phi X, Z)(\nabla_{\phi Y} r) - (n-2)g(\phi X, Z)(\nabla_{\phi Y}\tilde{A})$$
$$- g(\phi Y, Z)(\nabla_{\phi X} r) + (n-2)g(\phi Y, Z)(\nabla_{\phi X}\tilde{A}) + 2g(\phi X, Y)(\nabla_{\phi Z} r)$$
$$- (n-2)2g(\phi X, Y)(\nabla_{\phi Z}\tilde{A})] + \frac{(\tilde{r}+n-1)}{(n+1)(n+3)}[\alpha g(Y, \phi X)\eta(Z)$$
$$+ \alpha g(Z, \phi X)\eta(Y) - \beta g(Z, X)\eta(Y) - g(Z, X)\eta(Y) - \alpha g(X, \phi Y)\eta(Z)$$
$$- \alpha g(Z, \phi Y)\eta(X) + \beta g(Z, Y)\eta(X) + g(Z, Y)\eta(X)]$$
$$+ \frac{1}{(n+1)(n+3)}[g(X, Z)\eta(Y)(\nabla_\xi r) - (n-2)g(X, Z)\eta(Y)(\nabla_\xi \tilde{A})$$
$$- \eta(Y)\eta(Z)(\nabla_X r) + (n-2)\eta(Y)\eta(Z)(\nabla_X \tilde{A}) + \eta(X)\eta(Z)(\nabla_Y r)$$
$$- (n-2)\eta(X)\eta(Z)(\nabla_Y \tilde{A}) - g(Y, Z)\eta(X)(\nabla_\xi r) + (n-2)g(Y, Z)\eta(X)(\nabla_\xi \tilde{A})]$$
$$- \frac{1}{(n+1)(n+3)}[g(X, Z)(\nabla_Y r) - (n-2)g(X, Z)(\nabla_Y \tilde{A})$$
$$- g(Y, Z)(\nabla_X r) + (n-2)g(Y, Z)(\nabla_X \tilde{A})] = 0.$$

Now putting $X = \xi$ in the above equation and using (2.12), (3.6), (3.8), (3.9) and (3.11)
and on simplification we obtain

$$\frac{n+2}{n+3}[(n-2)\eta(Y)\eta(Z) - (n-1)(\alpha^2 - \beta^2)\beta g(Y, Z) + \beta S(Y, Z)$$
$$-(n-1)(\alpha^2 - \beta^2)\alpha g(Y, \phi Z) + \alpha S(Y, \phi Z) + 2(n-2)\alpha g(\phi Y, Z)$$
$$+2(n-2)\alpha\beta g(\phi Y, Z) + (n-2)[\alpha^2 - \beta(\beta+1)](g(Y, Z) - \eta(Y)\eta(Z)) \quad (4.2)$$
$$+S(Y, Z) - (n-2)\beta g(\phi Y, \phi Z) - (n-1)(\alpha^2 - \beta^2)g(Y, Z) - (n-2)g(Y, Z)]$$
$$+\frac{(\tilde{r}+n-1)}{(n+1)(n+3)}[(\beta+1)(g(Y, Z) - \eta(Y)\eta(Z)) - \alpha g(Z, \phi Y)] = 0.$$

Replacing Z by ϕZ in (4.2) and on simplification we get

$$\frac{n+2}{n+3}[-(n-1)(\alpha^2 - \beta^2)\beta g(Y, \phi Z) + \beta S(Y, \phi Z)$$
$$-(n-1)(\alpha^2 - \beta^2)\alpha g(Y, \phi^2 Z) + \alpha S(Y, \phi^2 Z) + 2(n-2)\alpha g(\phi Y, \phi Z)$$

$$+2(n-2)\alpha\beta g(\phi Y,\phi Z)+(n-2)[\alpha^2-\beta(\beta+1)](g(Y,\phi Z))+S(Y,\phi Z)$$
$$-(n-2)\beta g(\phi Y,\phi^2 Z)-(n-1)(\alpha^2-\beta^2)g(Y,\phi Z)-(n-2)g(Y,\phi Z)]$$
$$+\frac{(\tilde{r}+n-1)}{(n+1)(n+3)}[(\beta+1)(g(Y,\phi Z)-\alpha g(\phi Y,\phi Z))]=0. \quad (4.3)$$

Again replace Z by Y and Y by Z in (4.3), we get

$$\frac{n+2}{n+3}[-(n-1)(\alpha^2-\beta^2)\beta g(Z,\phi Y)+\beta S(Z,\phi Y)$$
$$-(n-1)(\alpha^2-\beta^2)\alpha g(Z,\phi^2 Y)+\alpha S(Z,\phi^2 Y)+2(n-2)\alpha g(\phi Z,\phi Y)$$
$$+2(n-2)\alpha\beta g(\phi Z,\phi Y)+(n-2)[\alpha^2-\beta(\beta+1)](g(Z,\phi Y))+S(Z,\phi Y)$$
$$-(n-2)\beta g(\phi Z,\phi^2 Y)-(n-1)(\alpha^2-\beta^2)g(Z,\phi Y)-(n-2)g(Z,\phi Y)]$$
$$+\frac{(\tilde{r}+n-1)}{(n+1)(n+3)}[(\beta+1)(g(Z,\phi Y)-\alpha g(\phi Z,\phi Y))]=0. \quad (4.4)$$

Now adding equations (4.3) and (4.4) and on simplification we have

$$\begin{aligned} S(Y,Z) &= \Big[(n-1)(\alpha^2-\beta^2)+2(n-2)(1+\beta) \\ &- \left\{\frac{(n-2)\{-2(n-1)\beta-n+2\}-2n(\tilde{A}-1)+2(\tilde{r}-1)}{2(n+1)(n+2)}\right\}\Big]g(Y,Z) \\ &+ \Big[-2(n-2)(1+\beta)+\Big\{\frac{(n-2)\{-2(n-1)\beta-n+2\}}{2(n+1)(n+2)} \\ &- \frac{2n(\tilde{A}-1)+2(\tilde{r}-1)}{2(n+1)(n+2)}\Big\}\Big]\eta(Y)\eta(Z). \end{aligned}$$

This equation is of the form

$$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z).$$

From this it follows that

$$\begin{aligned} r_1 &= n(n-1)(\alpha^2-\beta^2)+2(n-2)(1+\beta)(n-1) \\ &-\left\{\frac{(n-2)\{-2(n-1)\beta-n+2\}-2n(\tilde{A}-1)+2(\tilde{r}-1)}{2(n+1)(n+2)}\right\}(n-1) \end{aligned} \quad (4.5)$$

where $\tilde{r}=r-(n-1)(n-2)\beta-\left\{\frac{n(n-2)}{2}\right\}+(n-2)-\tilde{A}n$.

Theorem 4.2. If M^n be a trans-Sasakian manifold with the condition (2.10) admit semi-symmetric metric connection and let $\tilde{B}$ denote C-Bochner curvature tensor with respect to this connection and $div\tilde{B}=0$. Then the manifold is η-Einstein with respect to Levi-Civita connection.

5. Example

We consider 3-dimensional manifold $M = \{(x, y, z) \in R^3; z \neq 0\}$, where (x, y, z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad E_2 = z\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}. \tag{5.1}$$

Let g be the Riemannian metric defined by $g = \frac{1}{z^2}[(1 - y^2z^2)dx \otimes dx + dy \otimes dy + z^2 dz \otimes dz]$. Then (ϕ, ξ, η) is given by

$$\eta = dz - ydx, \quad \xi = E_3 = \frac{\partial}{\partial z},$$
$$\phi E_1 = E_2, \qquad \phi E_2 = -E_1, \qquad \phi E_3 = 0.$$

By definition of Lie bracket we have

$$[E_1, E_2] = yE_2 - z^2E_3, \quad [E_1, E_3] = -\frac{1}{z}E_1, \quad [E_2, E_3] = -\frac{1}{z}E_2. \tag{5.2}$$

Let ∇ be the Levi-Civita connection with respect to the above metric g given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then

$$\nabla_{E_1} E_3 = -\frac{1}{z}E_1 + \frac{1}{2}z^2E_2, \qquad \nabla_{E_3} E_3 = 0, \qquad \nabla_{E_2} E_3 = -\frac{1}{z}E_2 - \frac{1}{2}z^2E_1,$$
$$\nabla_{E_2} E_2 = yE_1 + \frac{1}{z}E_3, \qquad \nabla_{E_1} E_2 = -\frac{1}{2}z^2E_3, \qquad \nabla_{E_2} E_1 = \frac{1}{2}z^2E_3 - yE_2,$$
$$\nabla_{E_1} E_1 = \frac{1}{z}E_3, \qquad \nabla_{E_3} E_2 = -\frac{1}{2}z^2E_1, \qquad \nabla_{E_3} E_1 = \frac{1}{2}z^2E_2. \tag{5.3}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, a_i, b_i are scalars. Clearly ϕ, ξ, η, g and X, Y satisfy equations (2.1), (2.2), (2.3) and (2.5) with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = -\frac{1}{z} \neq 0$. Thus M is a trans-Sasakian manifold.

The Ricci tensor $S(X, Y)$ is given by

$$\begin{aligned} S(X, Y) &= \sum_{i=1}^{3} g(R(X, E_i)E_i, Y) \\ &= g(R(X, E_1)E_1, Y) + g(R(X, E_2)E_2, Y) \\ &+ g(R(X, E_3)E_3, Y). \end{aligned} \tag{5.4}$$

The non vanishing components of the curvature tensors by virtue of (5.3) are given by

$$\begin{aligned}
R(E_2,E_1)E_1 &= -\left(\frac{3}{4}z^4+\frac{1}{z^2}+y^2\right)E_2 - yz^2E_3,\\
R(E_3,E_1)E_1 &= -yz^2E_2-\left(\frac{2}{z^2}-\frac{1}{4}z^4\right)E_3,\\
R(E_1,E_2)E_2 &= -\left(\frac{3}{4}z^4+\frac{1}{z^2}+y^2\right)E_1-\frac{y}{z}E_3,\\
R(E_3,E_2)E_2 &= -\frac{y}{z}E_1-\left(\frac{2}{z^2}-\frac{1}{4}z^4\right)E_3,\\
R(E_1,E_3)E_3 &= \left(\frac{1}{4}z^4-\frac{2}{z^2}\right)E_1,\\
R(E_2,E_3)E_3 &= -\left(\frac{2}{z^2}-\frac{1}{4}z^4\right)E_2.
\end{aligned}$$

Using these in (5.4), we have

$$\begin{aligned}
S(X,Y) = &-\left(\frac{1}{2}z^4+\frac{2}{z^2}+y^2\right)g(X,Y)+\left(z^4-\frac{1}{z^2}+y^2\right)\eta(X)\eta(Y)\\
&-[g(X,E_2)\eta(Y)+g(Y,E_2)\eta(X)]yz^2-[g(X,E_1)\eta(Y)+g(Y,E_1)\eta(Y)]\frac{y}{z}.
\end{aligned}$$

Now using $X=E_1,\ Y=E_2\ and\ Z=E_3\ in\ div\tilde{B}$ we have

$$\begin{aligned}
(div\tilde{B})(E_1,E_2)E_3 = &\frac{5}{6}\left[4y^2z^2+z^6+\frac{z^2}{4}+\frac{z}{2}+2\alpha-\frac{7}{2}\right]\\
&+\frac{1}{6}\left[2y^2z^2-z^6-z^2-\frac{y}{z^2}-\alpha+2\right]+\frac{(r-\frac{2}{z}+\frac{3}{2}-3\tilde{A})\alpha}{12}
\end{aligned}$$

i.e. $(div\tilde{B})(E_1,E_2)E_3\neq 0$, since the component $(div\tilde{B})(E_1,E_2)E_3$ of $(div\tilde{B})$ $(X,Y)Z$, where $X=\sum_{i=1}^{3}a_iE_i$, $Y=\sum_{j=1}^{3}b_jE_j$ and $Z=\sum_{k=1}^{3}c_kE_k$ is not zero.

Hence it is shown that if M is a generalized η-Einstein with respect to Levi-Civita connection then it is not C-Bochner conservative with respect to semi-symmetric metric connection. Hence the converse of Theorem 4.2 is not true.

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