J. T. S.
Vol. 6 No.1 (2012), pp.79-92
https://doi.org/10.56424/jts.v6i01.10458

Submanifold of a Globally Para framed Metric Manifold

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(Received: 3 March, 2011)

(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In this paper we have defined various kinds of $\underset{x}{H}$ −connexions and stated and proved many theorems related to them. Some useful results have been derived in the form of corollaries. We have also generalized Gauss Characteristic and Mainardi-Codazzi equations and obtained the equations in the hypersurface therein.

1. Introduction

Let us consider two differentiable manifolds V_m and $V_n (m > n)$ of class C^∞ with the structures $\{F, G\}$ and $\{f, g\}$ of dimensions m and n respectively.

Let b be the inclusion map defined by

$$b : V_n \to V_m,$$

such that

$$p \in V_n \Rightarrow bp \in V_m.$$

The inclusion map b induces a Jacobian map B, defined by

$$B : T_n^1 \to T_m^1,$$

where T_n^1 is the tangent space at p in V_n and T_m^1 is the tangent space at bp in V_m, such that X in V_n at p and BX in V_m at bp. Let g be the induced metric tensor in V_n, then

$$(G(BX, BY))ob = g(X, Y). \quad (1.1)\text{a}$$

Let $\underset{x}{N} : x = n+1, ..., m$ be a system of C^∞ mutually orthogonal unit normal vector fields in V_n at p. Then

$$(G(\underset{x}{N}, BX))ob = 0, \tag{1.1)b}$$

$$(G(\underset{x}{N}, \underset{y}{N}))ob = \overset{x}{\underset{y}{\delta}} = \left\{ \begin{array}{ll} 1 \text{ if } & x = y \\ 0 \text{ if } & x \neq y \end{array} \right\} \tag{1.1)c}$$

Let E be the Riemannian connexion in V_m and D the induced Riemannian connexion in V_n. The Gauss and the Weingarten equations are given by

$$E_{BX}BY = BD_XY +' \underset{x}{H}(X,Y)\,\underset{x}{N}, \tag{1.2)a}$$

$$E_{BX}\,\underset{x}{N} = -B\,\underset{x}{H}\,X + \overset{y}{\underset{x}{L}}(X)\,\underset{y}{N}, \tag{1.2)b}$$

where $'\underset{x}{H}$ are the symmetric bilinear function, called second fundamental magnitudes in V_n and

$$\text{g}(\underset{x}{H}\text{X}, \text{Y}) \overset{\text{def}}{=} {'\underset{x}{H}}(X,Y) = {'\underset{x}{H}}(Y,X) = g(\underset{x}{H}Y, X) = g(X, \underset{x}{H}Y), \tag{1.3}$$

where $\underset{x}{H}$ are known as Weingarten maps and $\overset{y}{\underset{x}{L}}$ are called third fundamental forms in V_n.

If the submanifold be totally geodesic, then

$$'\underset{x}{H}(X,Y) = 0. \tag{1.4}$$

The submanifold V_n is said to be hypersurface of V_m if $m = n+1$.

If V_n is hypersurface of V_m, the equations (1.1)b and (1.1)c assume the forms

$$(G(N, BX))ob = 0, \tag{1.5)a}$$

$$(G(N, N))ob = 1. \tag{1.5)b}$$

Equations (1.2)a and (1.2)b reduce to

$$E_{BX}BY = BD_XY +' H(X,Y)N, \tag{1.6}$$

$$E_{BX}N = -BHX. \tag{1.7}$$

The hypersurface Vn is said to be totally geodesic, if

$$'H(X,Y) = 0. \tag{1.8}$$

The map b is called conformal or (strictly conformal), if

$$(G(BX, BY))ob = hg(X,Y), \tag{1.9}$$

where h is C^∞ real-valued positive function, called the scale function.

The map $\underset{x}{H}$ are strictly conformal, if

$$\underset{x}{H}(\underset{x}{H}(X)) = hX, \qquad (1.10)a$$

or

$$g(\underset{x}{H}(X), \underset{x}{H}(Y)) = hg(X, Y). \qquad (1.10)b$$

2. General $\underset{x}{H}$ −Connexion

A connexion D in V_n is called general $\underset{x}{H}$ −connexion, if

$$(D_X \underset{x}{H})Y = 0, \qquad (2.1)a$$

or equivalently

$$D_X \underset{x}{H} Y = \underset{x}{H} D_X Y. \qquad (2.1)b$$

Theorem (2.1). Let E be an arbitrary connexion in V_m, then the connexion D defined by

$$D_X Y = P_4(hE_X Y + \underset{x}{H} E_X \underset{x}{H} Y) + P_5(\underset{x}{H} E_X Y + E_X \underset{x}{H} Y)$$

$$+P_6(E_{\underset{x}{H} X} \underset{x}{H} Y + \underset{x}{H} E_{\underset{x}{H} X} Y) + P_7(hE_{\underset{x}{H} X} Y + \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y), \qquad (2.2)$$

is an $\underset{x}{H}$ −connexion.

Proof. Let us put

$$D_X Y = P_1 E_X Y + P_2 E_{\underset{x}{H} X} Y + P_3 \underset{x}{H} E_X Y + P_4 \underset{x}{H} E_X \underset{x}{H} Y$$

$$+P_5 E_X \underset{x}{H} Y + P_6 E_{\underset{x}{H} X} \underset{x}{H} Y + P_7 \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + P_8 \underset{x}{H} E_{\underset{x}{H} X} Y. \qquad (2.3)$$

Applying $\underset{x}{H}$ on Y in equation (2.3) and using (1.10)a in the resulting equation, we get

$$D_X \underset{x}{H} Y = P_1 E_X \underset{x}{H} Y + P_2 E_{\underset{x}{H} X} \underset{x}{H} Y + P_3 \underset{x}{H} E_X \underset{x}{H} Y + hP_4 \underset{x}{H} E_X Y$$

$$+hP_5 E_X Y + hP_6 E_{\underset{x}{H} X} Y + hP_7 \underset{x}{H}(E_{\underset{x}{H} X} Y) + P_8 \underset{x}{H}(E_{\underset{x}{H} X} \underset{x}{H} Y). \qquad (2.4)$$

Applying $\underset{x}{H}$ in equation (2.3) throughout and using the equation (1.10)a, then subtracting the resulting equation from (2.4),we get

$$(D_X \underset{x}{H})Y = (P_1 - hP_4)(E_X \underset{x}{H})Y + (P_2 - hP_7)(E_{\underset{x}{H}X} \underset{x}{H})Y$$
$$+(P_3 - P_5)\,\underset{x}{H}((E_X \underset{x}{H})Y) + (P_8 - P_6)\,\underset{x}{H}((E_{\underset{x}{H}X} \underset{x}{H})Y). \qquad (2.5)$$

Now the connexion D is general $\underset{x}{H}$ −connexion, iff

$$P_1 = hP_4, \qquad P_2 = hP_7, \qquad P_3 = P_5, \qquad P_6 = P_8. \qquad (2.6)$$

Since E being arbitrary connexion, substituting from (2.7) in (2.3), we have (2.2).

Corollary (2.1). For the $\underset{x}{H}$ − connexion D in V_n, we have

$$D_{\underset{x}{H}X}\, \underset{x}{H}\, Y = \underset{x}{H}\, D_{\underset{x}{H}X} Y, \qquad (2.7)\text{a}$$

$$\underset{x}{H}\, D_{\underset{x}{H}X}\, \underset{x}{H}\, Y = hD_{\underset{x}{H}X} Y, \qquad (2.7)\text{b}$$

$$\underset{x}{H}\, D_X\, \underset{x}{H}\, Y = hD_X Y. \qquad (2.7)\text{c}$$

Proof. We know that

$$D_X\, \underset{x}{H}\, Y = (D_X \underset{x}{H})Y + \underset{x}{H}\, D_X Y. \qquad (2.8)$$

Replacing X by $\underset{x}{H} X$ in (2.1)b, we obtain (2.7)a. Applying $\underset{x}{H}$ on (2.7)a and (2.1)b, using (1.10)a in the resulting equations, we get (2.7)b and (2.7)c respectively.

3. Nearly $\underset{x}{H}$ −Connexion

A connexion D in V_n is called nearly $\underset{x}{H}$ −connexion, if

$$(D_X \underset{x}{H})Y + (D_Y \underset{x}{H})\, X = 0, \qquad (3.1)$$

or equivalently

$$D_X\, \underset{x}{H}\, Y + D_Y\, \underset{x}{H}\, X = \underset{x}{H}\, D_X Y + \underset{x}{H}\, D_Y X. \qquad (3.2)$$

Theorem (3.1). Let E be an arbitrary connexion in V_m, then the connexion D defined by

$$D_X Y = P_4(hE_X Y + \underset{x}{H}\, E_X\, \underset{x}{H}\, Y) + P_5(\underset{x}{H}\, E_X Y + E_X\, \underset{x}{H}\, Y)$$
$$+P_7(hE_{\underset{x}{H}X} Y + \underset{x}{H}\, E_{\underset{x}{H}X}\, \underset{x}{H}\, Y) + P_8(E_{\underset{x}{H}X}\, \underset{x}{H}\, Y + \underset{x}{H}\, E_{\underset{x}{H}X} Y), \qquad (3.3)$$

is a nearly $\underset{x}{H}$ −connexion.

Proof. Applying $\underset{x}{H}$ on Y in equation (2.3) and using (1.10)a, we get

$$D_X \underset{x}{H} Y = P_1 E_X \underset{x}{H} Y + P_2 E_{\underset{x}{H} X} \underset{x}{H} Y + P_3 \underset{x}{H} E_X \underset{x}{H} Y + hP_4 \underset{x}{H} E_X Y$$
$$+hP_5 E_X Y + hP_6 E_{\underset{x}{H} X} Y + hP_7 \underset{x}{H} E_{\underset{x}{H} X} Y + P_8 \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y. \quad (3.4)$$

Interchanging X and Y in equation (3.4) and adding the resulting equation with the equation (3.4), we get

$$\begin{aligned} D_X \underset{x}{H} Y + D_Y \underset{x}{H} X = {} & P_1(E_X \underset{x}{H} Y + E_Y \underset{x}{H} X) + P_2(E_{\underset{x}{H} X} \underset{x}{H} Y + E_{\underset{x}{H} Y} \underset{x}{H} X) \\ & + P_3(\underset{x}{H} E_X \underset{x}{H} Y + \underset{x}{H} E_Y \underset{x}{H} X) \\ & + hP_4(\underset{x}{H} E_X Y + \underset{x}{H} E_Y X) + hP_5(E_X Y + E_Y X) \\ & + hP_6(E_{\underset{x}{H}X} Y + E_{\underset{x}{H}Y} X) + hP_7(\underset{x}{H} E_{\underset{x}{H}X} Y + \underset{x}{H} E_{\underset{x}{H} Y} X) \\ & +P_8(\underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + \underset{x}{H} E_{\underset{x}{H} Y} \underset{x}{H} X). \end{aligned} \quad (3.5)$$

Applying $\underset{x}{H}$ in equation (2.3) and using (1.10)a in the resulting equation, we get

$$\underset{x}{H} D_X Y = P_1 \underset{x}{H} E_X Y + P_2 \underset{x}{H} E_{\underset{x}{H} X} Y + hP_3 E_X Y + hP_4 E_X \underset{x}{H} Y$$
$$+P_5 \underset{x}{H} E_X \underset{x}{H} Y + P_6 \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + hP_7 E_{\underset{x}{H} X} \underset{x}{H} Y + hP_8 E_{\underset{x}{H} X} Y. \quad (3.6)$$

Interchanging X and Y in equation (3.6) and adding the resulting equation with the equation (3.6), we get

$$\begin{aligned} \underset{x}{H} D_X Y + \underset{x}{H} D_Y X = {} & P_1(\underset{x}{H} E_X Y + \underset{x}{H} E_Y X) + P_2(\underset{x}{H} E_{\underset{x}{H} X} Y + \underset{x}{H} E_{\underset{x}{H} Y} X) \\ & + hP_3(E_X Y + E_Y X) + hP_4(E_X \underset{x}{H} Y + E_Y \underset{x}{H} X) \\ & + P_5(\underset{x}{H} E_X \underset{x}{H} Y + \underset{x}{H} E_Y \underset{x}{H} X) \\ & + P_6(\underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + \underset{x}{H} E_{\underset{x}{H} Y} \underset{x}{H} X) \\ & +hP_7(E_{\underset{x}{H} X} \underset{x}{H} Y + E_{\underset{x}{H} Y} \underset{x}{H} X) + hP_8(E_{\underset{x}{H} X} Y + E_{\underset{x}{H} Y} X). \end{aligned} \quad (3.7)$$

Substituting the values from the equations (3.5) and (3.7) in (3.2), we get

$$P_1(E_X \underset{x}{H} Y+E_Y \underset{x}{H} X)+P_2(E_{\underset{x}{H} X} \underset{x}{H} Y+E_{\underset{x}{H} Y} \underset{x}{H} X)+P_3(\underset{x}{H} E_X \underset{x}{H} Y+\underset{x}{H} E_Y \underset{x}{H} X)$$
$$+hP_4(\underset{x}{H} E_X Y + \underset{x}{H} E_Y X) + hP_5(E_X Y + E_Y X) + hP_6(E_{\underset{x}{H} X} Y + E_{\underset{x}{H} Y} X)$$
$$+hP_7(\underset{x}{H} E_{\underset{x}{H} X} Y + \underset{x}{H} E_{\underset{x}{H} Y} X) + P_8(\underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + \underset{x}{H} E_{\underset{x}{H} Y} \underset{x}{H} X)$$
$$= P_1(\underset{x}{H} E_X Y + \underset{x}{H} E_Y X) + P_2(\underset{x}{H} E_{\underset{x}{H} X} Y + \underset{x}{H} E_{\underset{x}{H} Y} X) + hP_3(E_X Y + E_Y X)$$

$$+hP_4(E_X \underset{x}{H} Y + E_Y \underset{x}{H} X) + P_5(\underset{x}{H} E_X \underset{x}{H} Y + \underset{x}{H} E_Y \underset{x}{H} X)$$
$$+P_6(\underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + \underset{x}{H} E_{\underset{x}{H} Y} \underset{x}{H} X) + hP_7(E_{\underset{x}{H} X} \underset{x}{H} Y + E_{\underset{x}{H} Y} \underset{x}{H} X)$$
$$+hP_8(E_{\underset{x}{H} X} Y + E_{\underset{x}{H} Y} X), \tag{3.8}$$

which gives

$$P_1 = hP_4, \qquad P_2 = hP_7, \qquad P_3 = P_5, \qquad P_6 = P_8. \tag{3.9}$$

Substituting from the equation (3.9) in equation (2.3), we get the equation (3.3).

Corollary (3.1). For nearly $\underset{x}{H}$ −connexion D in V_n, we have

$$D_{\underset{x}{H} X} \underset{x}{H} Y + hD_Y X = \underset{x}{H} D_{\underset{x}{H} X} Y + \underset{x}{H} D_Y \underset{x}{H} X, \tag{3.10)a}$$

$$hD_{\underset{x}{H} X} Y + hD_{\underset{x}{H} Y} X = \underset{x}{H} D_{\underset{x}{H} X} \underset{x}{H} Y + \underset{x}{H} D_{\underset{x}{H} Y} \underset{x}{H} X, \tag{3.10)b}$$

$$\underset{x}{H} D_X \underset{x}{H} Y + \underset{x}{H} D_Y \underset{x}{H} X = h(D_X Y + D_Y X), \tag{3.10)c}$$

$$h(\underset{x}{H} D_{\underset{x}{H} X} Y + \underset{x}{H} D_{\underset{x}{H} Y} X) = h(D_{\underset{x}{H} X} \underset{x}{H} Y + D_{\underset{x}{H} Y} \underset{x}{H} X). \tag{3.10)d}$$

Proof. Applying $\underset{x}{H}$ on X in equation (3.2) and using (1.10)a, we get (3.10)a. Similarly, applying $\underset{x}{H}$ on Y in equation (3.10)a then using (1.10)a, we obtain (3.10)b. Operating $\underset{x}{H}$ throughout in equation (3.2) and (3.10)b respectively and using (1.10)a, we get the equations (3.10)c and (3.10)d.

4. N$\underset{x}{H}$ −Connexion

A connexion D in V_n, is called $N \underset{x}{H}$ −connexion, if

$$(D_X \underset{x}{H})Y + \underset{x}{H}(D_{\underset{x}{H} X} \underset{x}{H})\, Y = 0, \tag{4.1}$$

or equivalently

$$D_X \underset{x}{H} Y - \underset{x}{H} D_X Y + \underset{x}{H} D_{\underset{x}{H} X} \underset{x}{H} Y - hD_{\underset{x}{H} X} Y = 0. \tag{4.2}$$

Theorem (4.1). Let the connexion D and E be related by the equation (2.3), then the connexion D is $N \underset{x}{H}$ −connexion, iff E is also $N \underset{x}{H}$ −connexion.

Proof. Applying $\underset{x}{H}$ on Y in equation (2.3) and using (1.10)a, we get

$$D_X \underset{x}{H} Y = P_1 E_X \underset{x}{H} Y + P_2 E_{\underset{x}{H} X} \underset{x}{H} Y + P_3 \underset{x}{H} E_X \underset{x}{H} Y + hP_4 \underset{x}{H} E_X Y$$
$$+hP_5 E_X Y + hP_6 E_{\underset{x}{H} X} Y + hP_7 \underset{x}{H} E_{\underset{x}{H} X} Y + P_8 \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y. \tag{4.3}$$

Applying $\underset{x}{H}$ throughout in equation (2.3) and using (1.10)a, then subtracting the resulting equation from the equation (4.3), we get

$$\begin{aligned}(D_X \underset{x}{H})Y = {} & (P_1 - hP_4)(E_X \underset{x}{H} Y - \underset{x}{H} E_X Y) + (P_2 - hP_7)(E_{\underset{x}{H} X} \underset{x}{H} Y \\ & - \underset{x}{H} E_{\underset{x}{H} X} Y) + (P_3 - P_5)(\underset{x}{H} E_X \underset{x}{H} Y - hE_X Y) \\ & + (P_6 - P_8)(hE_{\underset{x}{H} X} Y - \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y).\end{aligned} \tag{4.4}$$

Applying $\underset{x}{H}$ on X in equation (4.4) and using the equation (1.10)a in the resulting equation, we get

$$\begin{aligned}(D_{\underset{x}{H} X} \underset{x}{H})Y = {} & (P_1 - hP_4)(E_{\underset{x}{H} X} \underset{x}{H} Y - \underset{x}{H} E_{\underset{x}{H} X} Y) + (P_2 - hP_7)(E_{hX} \underset{x}{H} Y \\ & - \underset{x}{H} E_{hX} Y) + (P_3 - P_5)(\underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y - hE_{\underset{x}{H} X} Y) \\ & + (P_6 - P_8)(hE_{hX} Y - \underset{x}{H} E_{hX} \underset{x}{H} Y).\end{aligned} \tag{4.5}$$

Applying $\underset{x}{H}$ throughout in equation (4.5) and using the equation (1.10)a, then adding the resulting equation with the equation (4.4), we get

$$\begin{aligned}(D_X \underset{x}{H})Y + \underset{x}{H}(D_{\underset{x}{H} X} \underset{x}{H})Y = {} & (P_1 - hP_4)(E_X \underset{x}{H} Y - \underset{x}{H} E_X Y + \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y - hE_{\underset{x}{H} X} Y) \\ & + (P_2 - hP_7)(E_{\underset{x}{H} X} \underset{x}{H} Y - \underset{x}{H} E_{\underset{x}{H} X} Y + \underset{x}{H} E_{hX} \underset{x}{H} Y - hE_{hX} Y) \\ & + (P_3 - P_5)(\underset{x}{H} E_X \underset{x}{H} Y - hE_X Y + hE_{\underset{x}{H} X} \underset{x}{H} Y - h \underset{x}{H} E_{\underset{x}{H} X} Y) \\ & + (P_6 - P_8)(hE_{\underset{x}{H} X} Y - \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + h \underset{x}{H} E_{hX} Y - hE_{hX} \underset{x}{H} Y)\end{aligned} \tag{4.6}$$

If E is $N \underset{x}{H}$ −connexion, then

$$(D_X \underset{x}{H})Y + \underset{x}{H}(D_{\underset{x}{H} X} \underset{x}{H})Y = 0.$$

Hence D is $N \underset{x}{H}$ −connexion.

Conversely, suppose D is $N \underset{x}{H}$ −connexion, then from the equation (4.6), it follows that E is also $N \underset{x}{H}$ −connexion.

Corollary (4.1). For $N \underset{x}{H}$ −connexion in V_n, we have

$$\underset{x}{H} D_X \underset{x}{H} Y - hD_X Y + hD_{\underset{x}{H} X} \underset{x}{H} Y - h \underset{x}{H} D_{\underset{x}{H} X} Y = 0, \tag{4.7)a}$$

$$D_{\underset{x}{H} X} \underset{x}{H} Y - \underset{x}{H} D_{\underset{x}{H} X} Y + \underset{x}{H} D_{hX} \underset{x}{H} Y - hD_{hX} Y = 0, \tag{4.7)b}$$

$$hD_{\underset{x}{H} X} Y - \underset{x}{H} D_{\underset{x}{H} X} \underset{x}{H} Y + h \underset{x}{H} D_{hX} Y - hD_{hX} \underset{x}{H} Y = 0. \tag{4.7)c}$$

Proof. Operating $\underset{x}{H}$ in equation (4.2) and using the equation (1.10)a, we get the equation (4.7)a. Operating $\underset{x}{H}$ on X in equation (4.2) and using the equation (1.10)a, we obtain (4.7)b. Similarly operating $\underset{x}{H}$ in equation (4.7)b and using the equation (1.10)a in the resulting equations, we get (4.7)c.

5. Semi $\underset{x}{H}$ −Connexion

A connexion D in V_n is called a semi $\underset{x}{H}$ −connexion, if

$$(\text{div } \underset{x}{H})Y = 0, \tag{5.1)a}$$

or

$$(\text{div } \underset{x}{H}) \underset{x}{H} Y = 0. \tag{5.1)b}$$

Theorem (5.1). Let E be an arbitrary connexion in V_m, then the connexion D defined by

$$D_X Y = h(P_4 + P_6 - P_8)E_X Y + (hP_7 + P_5 - P_3)E_{\underset{x}{H} X} Y + P_3 \underset{x}{H} E_X Y$$
$$+P_4 \underset{x}{H} E_X \underset{x}{H} Y + P_5 E_X \underset{x}{H} Y + P_6 E_{\underset{x}{H} X} \underset{x}{H} Y + P_7 \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y + P_8 \underset{x}{H} E_{\underset{x}{H} X} Y, \tag{5.2}$$

is semi $\underset{x}{H}$ −connexion.

Proof. Applying $\underset{x}{H}$ on Y in equation (2.3) and using (1.10)a in the resulting equation, we get

$$D_X \underset{x}{H} Y = P_1 E_X \underset{x}{H} Y + P_2 E_{\underset{x}{H} X} \underset{x}{H} Y + P_3 \underset{x}{H} E_X \underset{x}{H} Y + hP_4 \underset{x}{H} E_X Y$$
$$+hP_5 E_X Y + hP_6 E_{\underset{x}{H} X} Y + hP_7 \underset{x}{H} E_{\underset{x}{H} X} Y + P_8 \underset{x}{H} E_{\underset{x}{H} X} \underset{x}{H} Y. \tag{5.3}$$

Applying $\underset{x}{H}$ throughout in equation (2.3) and using (1.10)a then subtracting the resulting equation from (5.3), we get

$$(D_X \underset{x}{H})Y = (P_1 - hP_4)(E_X \underset{x}{H})Y + (P_2 - hP_7)(E_{\underset{x}{H} X} \underset{x}{H})Y$$
$$+(P_3 - P_5) \underset{x}{H}((E_X \underset{x}{H})Y) + (P_8 - P_6) \underset{x}{H}((E_{\underset{x}{H} X} \underset{x}{H})Y). \tag{5.4}$$

Contracting the equation (5.4) with respect to X, we get

$$(\text{div } \underset{x}{H})Y = (P_1 - hP_4 - hP_6 + hP_8)(Div \underset{x}{H})Y + (P_2 - hP_7 + P_3 - P_5)(Div \underset{x}{H}) \underset{x}{H} Y \tag{5.5}$$

where div refers to the connexion D and Div refers to the connexion E.

The equation (5.6) is equivalent to

$$(\text{div } \underset{x}{H})\, \underset{x}{H}\, Y = (P_1 - hP_4 - hP_6 + hP_8)(Div\, \underset{x}{H})\, \underset{x}{H}\, Y + (P_2 - hP_7 + P_3 - P_5)(Div\, \underset{x}{H})Y. \tag{5.6}$$

Hence from the equations (5.5) and (5.6) we observe that the necessary and sufficient condition that D be semi $\underset{x}{H}$ −connexion are

$$P_1 = hP_4 + hP_6 - hP_8, \qquad P_2 = hP_7 + P_5 - P_3. \tag{5.7}$$

Since E being an arbitrary connexion (Div $\underset{x}{H}$)$Y \neq 0$, substituting from (5.7) in equation (2.3), we get the equation (5.2).

6. Almost $\underset{x}{H}$ −Connexion

A connexion D in V_n is called an almost $\underset{x}{H}$ −connexion, if

$$(D_X{}'\, \underset{x}{H})(Y,Z) + (D_Y{}'\, \underset{x}{H})(Z,X) + (D_Z{}'\, \underset{x}{H})(X,Y) = 0, \tag{6.1)a$$

or equivalently

$$g((D_X\, \underset{x}{H})Y,Z) + g((D_Y\, \underset{x}{H})Z,X) + g((D_Z\, \underset{x}{H})X,Y) = 0. \tag{6.1)b$$

Theorem (6.1). Let the connexion D and E be related by the equation (2.3), then the connexion D is an almost $\underset{x}{H}$ −connexion, if

$$\begin{aligned} \underset{x}{H}(D_X Y,Z) = {} & P_1\{{}'\underset{x}{H}(E_X Y,Z) + g(E_X\, \underset{x}{H}\, Y,Z)\} + P_2\{{}'\underset{x}{H}(E_{\underset{x}{H}\,X} Y,Z) \\ & + g(E_{\underset{x}{H}\,X}\, \underset{x}{H}\, Y,Z)\} + P_3\{hg(E_X Y,Z) +{}'\, \underset{x}{H}(E_X\, \underset{x}{H}\, Y,Z)\} \\ & + P_6\{{}'\underset{x}{H}(E_{\underset{x}{H}\,X}\, \underset{x}{H}\, Y,Z) + hg(E_{\underset{x}{H}\,X} Y,Z)\}. \end{aligned} \tag{6.2}$$

Proof. Applying $\underset{x}{H}$ on Y in equation (2.3) and using the equation (1.10)a then applying g in the resulting equation and using the equation (1.3), we get

$$\begin{aligned} (D_X{}'\, \underset{x}{H})(Y,Z) +{}'\, \underset{x}{H}(D_X Y,Z) = {} & P_1 g(E_X\, \underset{x}{H}\, Y,Z) + P_2 g(E_{\underset{x}{H}\,X}\, \underset{x}{H}\, Y,Z) \\ & + P_3\, {}'\underset{x}{H}(E_X\, \underset{x}{H}\, Y,Z) + hP_4\, {}'\underset{x}{H}(E_X Y,Z) \\ & + hP_5 g(E_X Y,Z) + hP_6 g(E_{\underset{x}{H}\,X} Y,Z) \\ & + hP_7\, {}'\underset{x}{H}(E_{\underset{x}{H}\,X} Y,Z) + P_8\, {}'\underset{x}{H}(E_{\underset{x}{H}\,X}\, \underset{x}{H}\, Y,Z). \end{aligned} \tag{6.3}$$

Similarly, two more relations of the type (6.3) can be obtained by interchanging X, Y, Z cyclically in (6.3) then adding the resulting equations in (6.3) and using (6.1)a, we get

$$'\underset{x}{H}(D_X Y, Z) +' \underset{x}{H}(D_Y Z, X) +' \underset{x}{H}(D_Z X, Y)$$

$$\begin{aligned}
&= P_1[g(E_X \underset{x}{H} Y, Z) + g(E_Y \underset{x}{H} Z, X) + g(E_Z \underset{x}{H} X, Y)] \\
&+ P_2[g(E_{\underset{x}{H} X} \underset{x}{H} Y, Z) + g(E_{\underset{x}{H} Y} \underset{x}{H} Z, X) + g(E_{\underset{x}{H} Z} \underset{x}{H} X, Y) \\
&+ P_3['\underset{x}{H}(E_X \underset{x}{H} Y, Z) +' \underset{x}{H}(E_Z \underset{x}{H} X, Y)] \\
&+ hP_4['\underset{x}{H}(E_X Y, Z) +' \underset{x}{H}(E_Y Z, X) + '\underset{x}{H}(E_Z X, Y)] \\
&+ hP_5[g(E_X Y, Z) + g(E_Y Z, X) + g(E_Z X, Y)] + hP_6[g(E_{\underset{x}{H} X} Y, Z) \\
&+ g(E_{\underset{x}{H} Y} Z, X) + g(E_{\underset{x}{H} Z} X, Y)] \\
&+ hP_7['\underset{x}{H}(E_{\underset{x}{H} X} Y, Z) +' \underset{x}{H}(E_{\underset{x}{H} Y} Z, X) +' \underset{x}{H}(E_{\underset{x}{H} Z} X, Y)].
\end{aligned} \tag{6.4}$$

Applying $\underset{x}{H}$ on equation (2.3) throughout and using the equation (1.10)a then applying g in the resulting equation and using the equation (1.3), we get

$$\begin{aligned}
'\underset{x}{H}(D_X Y, Z) &= P_1 '\underset{x}{H}(E_X Y, Z) + P_2 '\underset{x}{H}(E_{\underset{x}{H} X} Y, Z) + hP_3 g(E_X Y, Z) \\
&+hP_4 g(E_X \underset{x}{H} Y, Z) + P_5 '\underset{x}{H}(E_X \underset{x}{H} Y, Z) + P_6 '\underset{x}{H}(E_{\underset{x}{H} X} \underset{x}{H} Y, Z) \\
&+hP_7 g(E_{\underset{x}{H} X} \underset{x}{H} Y, Z) + hP_8 g(E_{\underset{x}{H} X} Y, Z).
\end{aligned} \tag{6.5}$$

Similarly, two more relations of the type (6.5) can be obtained by interchanging X, Y, Z cyclically in (6.5), adding the resulting equations in (6.5), we get

$$'\underset{x}{H}(D_X Y, Z) + '\underset{x}{H}(D_Y Z, X) + '\underset{x}{H}(D_Z X, Y)$$

$$\begin{aligned}
&= P_1['\underset{x}{H}(E_X Y, Z) +' \underset{x}{H}(E_Y Z, X) +' \underset{x}{H}(E_Z X, Y)] \\
&+ P_2['\underset{x}{H}(E_{\underset{x}{H} X} Y, Z) + '\underset{x}{H}(E_{\underset{x}{H} Y} Z, X) + '\underset{x}{H}(E_{\underset{x}{H} Z} X, Y)] \\
&+ hP_3[g(E_X Y, Z) + g(E_Y Z, X) + g(E_Z X, Y)] \\
&+ hP_4[g(E_X \underset{x}{H} Y, Z) + g(E_Y \underset{x}{H} Z, X) + g(E_Z \underset{x}{H} X, Y)] \\
&+ P_5['\underset{x}{H}(E_X \underset{x}{H} Y, Z) +' \underset{x}{H}(E_Y \underset{x}{H} Z, X) +' \underset{x}{H}(E_Z \underset{x}{H} X, Y)]
\end{aligned}$$

$$+ P_6[{}'\underset{x}{H}(E_{\underset{x}{H}X}\underset{x}{H}Y, Z) + {}'\underset{x}{H}(E_{\underset{x}{H}Y}\underset{x}{H}Z, X) + {}'\underset{x}{H}(E_{\underset{x}{H}Z}\underset{x}{H}X, Y)]$$
$$+ hP_7[g(E_{\underset{x}{H}X}\underset{x}{H}Y, Z) + g(E_{\underset{x}{H}Y}\underset{x}{H}Z, X) + g(E_{\underset{x}{H}Z}\underset{x}{H}X, Y) \tag{6.6}$$
$$+ hP_8[g(E_{\underset{x}{H}X}Y, Z) + g(E_{\underset{x}{H}Y}Z, X) + g(E_{\underset{x}{H}Z}X, Y)].$$

Comparing the equations (6.4) and (6.6), we get

$$P_1 = hP_4, \qquad P_2 = hP_7, \qquad P_3 = P_5, \qquad P_6 = P_8 \tag{6.7}$$

Substituting these values from equation (6.7) in equation (6.5), we get the equation (6.2).

7. Generalization of Gauss Characteristic and Mainardi-Codazzi Equations

Let K and $\overline{K}$ be the curvature tensors at p and bp in V_n and V_m respectively, then

$$\overline{K}(BX, BY, BZ) = E_{BX}E_{BY}BZ - E_{BY}E_{BX}BZ - E_{[BX,BY]}BZ. \tag{7.1}$$

In consequence of (1.2)a and (1.2)b, we have

$$E_{BX}E_{BY}BZ = BD_XD_YZ +' \underset{x}{H}(X, D_YZ)\underset{x}{N} -' \underset{x}{H}(Y, Z)B\underset{x}{H}X$$
$$+' \underset{x}{H}(Y, Z)\overset{y}{\underset{x}{L}}(X)\underset{y}{N} + \underset{x}{N}X('\underset{x}{H}(Y, Z)). \tag{7.2}$$

Similarly

$$E_{BY}E_{BX}BZ = BD_YD_XZ +' \underset{x}{H}(Y, D_XZ)\underset{x}{N} -' \underset{x}{H}(X, Z)B\underset{x}{H}Y$$
$$+' \underset{x}{H}(X, Z)\overset{y}{\underset{x}{L}}(Y)\underset{y}{N} + \underset{x}{N}Y('\underset{x}{H}(X, Z)), \tag{7.3}$$

and

$$E_{[BX,BY]}BZ = BD_{[X,Y]}Z +' \underset{x}{H}([X, Y], Z)\underset{x}{N}. \tag{7.4}$$

Substituting from (7.2), (7.3) and (7.4) in (7.1), we get

$$\overline{K}(BX, BY, BZ) = BK(X, Y, Z) -' \underset{x}{H}(Y, Z)B\underset{x}{H}(X) + '\underset{x}{H}(X, Z)B\underset{x}{H}Y$$
$$+ '\underset{x}{H}(X, D_YZ)\underset{x}{N} - '\underset{x}{H}(Y, D_XZ)\underset{x}{N} + '\underset{x}{H}(Y, Z)\overset{y}{\underset{x}{L}}(X)\underset{y}{N}$$
$$-' \underset{x}{H}(X, Z)\overset{y}{\underset{x}{L}}(Y)\underset{y}{N} + \underset{x}{N}X('\underset{x}{H}(Y, Z)) \tag{7.5}$$
$$- \underset{x}{N}Y('\underset{x}{H}(X, Z)) - '\underset{x}{H}([X, Y], Z)\underset{x}{N}.$$

Let us put

$$'\overline{K}(BX, BY, BZ, BU) = (G(\overline{K}(BX, BY, BZ), BU))ob, \tag{7.6}$$

and

$$'\overline{K}(BX, BY, BZ, \underset{x}{N}) = (G(\overline{K}(BX, BY, BZ), \underset{x}{N}))ob. \tag{7.7}$$

Then from the equations (7.5), (7.6), (1.1)a and (1.1)b, we have

$$'\overline{K}(BX, BY, BZ, BU) = {}'K(X, Y, Z, U) - {}'\underset{x}{H}(Y, Z)'\underset{x}{H}(X, U) + {}'\underset{x}{H}(X, Z)'\underset{x}{H}(Y, U). \tag{7.8}$$

Similarly, from the equations (7.5) and (7.7), we have

$$\begin{aligned}'\overline{K}(BX, BY, BZ, \underset{x}{N}) = {} & G(BK(X, Y, Z), \underset{x}{N})ob - {}'\underset{x}{H}(Y, Z)G(B\underset{x}{H}X, \underset{x}{N})ob \\ & + {}'\underset{x}{H}(X, Z)G(B\underset{x}{H}Y, \underset{x}{N})ob + {}'\underset{x}{H}(X, D_Y Z)G(\underset{x}{N}, \underset{x}{N})ob \\ & - {}'\underset{x}{H}(Y, D_X Z)G(\underset{x}{N}, \underset{x}{N})ob + {}'\underset{y}{H}(Y, Z)G(\overset{x}{\underset{y}{L}}(X)\underset{x}{N}, \underset{x}{N})ob \\ & - {}'\underset{y}{H}(X, Z)G\overset{x}{\underset{y}{L}}(Y)\underset{x}{N}, \underset{x}{N})ob + X('\underset{x}{H}(Y, Z))G(\underset{x}{N}, \underset{x}{N})ob \\ & - Y'\underset{x}{H}(X, Z)G(\underset{x}{N}, \underset{x}{N})ob - {}'\underset{x}{H}([X, Y], Z)G(\underset{x}{N}, \underset{x}{N})ob.\end{aligned} \tag{7.9}$$

Now in consequence of (7.9), (1.1)b and (1.1)c, we have

$$\begin{aligned}'\overline{K}(BX, BY, BZ, \underset{x}{N}) = {} & {}'\underset{x}{H}(X, D_Y Z) - {}'\underset{x}{H}(Y, D_X Z) + {}'\underset{y}{H}(Y, Z)\overset{x}{\underset{y}{L}}(X) - {}'\underset{y}{H}(X, \\ & Z)\overset{x}{\underset{y}{L}}(Y) + X('\underset{x}{H}(Y, Z)) - Y('\underset{x}{H}(X, Z)) - {}'\underset{x}{H}([X, Y], Z),\end{aligned} \tag{7.10}$$

but

$$X('\underset{x}{H}(Y, Z)) = (D_X{}'\underset{x}{H})(Y, Z) + {}'\underset{x}{H}(D_X Y, Z) + {}'\underset{x}{H}(Y, D_X Z, \tag{7.11)a}$$

and

$$D_X Y - D_Y X = [X, Y]. \tag{7.11)b}$$

Then

$$\begin{aligned}X('\underset{x}{H}(Y, Z)) - Y('\underset{x}{H}(X, Z)) = {} & (D_X{}'\underset{x}{H})(Y, Z) - (D_Y{}'\underset{x}{H})(X, Z) \\ & + {}'\underset{x}{H}([X, Y], Z) + {}'\underset{x}{H}(Y, D_X Z) - {}'\underset{x}{H}(X, D_Y Z).\end{aligned} \tag{7.11)c}$$

Substituting the value from (7.11)c in (7.10), we get

$$\begin{aligned} {}'\overline{K}(BX, BY, BZ, \underset{x}{N}) = &+{}'\underset{y}{H}(Y,Z)\,\overset{x}{\underset{y}{L}}(X) - {}'\underset{y}{H}(X,Z)\,\overset{x}{\underset{y}{L}}(Y) \\ &+ (D_X\,{}'\underset{x}{H})(Y,Z) - (D_Y\,{}'\underset{x}{H})(X,Z). \end{aligned} \tag{7.12}$$

The equations (7.8) are the generalization of Gauss Characteristic equations and the equations (7.12) are the generalization of Mainardi-Codazzi equations.

Let

$$\overline{K}(BX, BY, \underset{x}{N}) = E_{BX}E_{BY}\,\underset{x}{N} - E_{BY}E_{BX}\,\underset{x}{N} - E_{[BX,BY]}\,\underset{x}{N}\,. \tag{7.13}$$

In consequence of (1.2)a and (1.2)b, we have

$$\begin{aligned} E_{BX}E_{BY}\,\underset{x}{N} &= E_{BX}(-B\,\underset{x}{H}\,Y + \overset{y}{\underset{x}{L}}(Y)\,\underset{y}{N}) \\ &= -BD_X\,\underset{x}{H}\,Y -{}'\,\underset{y}{H}(X, \underset{x}{H}\,Y)\,\underset{y}{N} - \overset{y}{\underset{x}{L}}(Y)B\,\underset{y}{H}\,X + \overset{y}{\underset{x}{L}}(Y)\,\overset{z}{\underset{y}{L}}(X)\,\underset{y}{N} \\ &\quad + X(\overset{y}{\underset{x}{L}}(Y))\,\underset{y}{N}, \end{aligned} \tag{7.14}$$

and

$$E_{[BX,BY]}\,\underset{x}{N} = -B\,\underset{x}{H}([X,Y]) + \overset{y}{\underset{x}{L}}([X,Y])\,\underset{y}{N}\,. \tag{7.15}$$

Substituting from (7.14) and (7.15) in (7.13), we get

$$\begin{aligned} \overline{K}(BX, BY, \underset{x}{N}) = &B\{D_Y\,\underset{x}{H}\,X - D_X\,\underset{x}{H}\,Y - \overset{y}{\underset{x}{L}}(Y)\,\underset{x}{H}\,X + \overset{y}{\underset{x}{L}}(X)\,\underset{x}{H}\,Y \\ &+ \underset{x}{H}([X,Y]\} -{}'\,\underset{y}{H}(X, \underset{x}{H}\,Y)\,\underset{y}{N} +{}'\,\underset{y}{H}(Y, \underset{x}{H}\,X)\,\underset{y}{N} \\ &+ \overset{y}{\underset{x}{L}}(Y)\,\overset{z}{\underset{y}{L}}(X)\,\underset{y}{N} - \overset{y}{\underset{x}{L}}(X)\,\overset{z}{\underset{y}{L}}(Y)\,\underset{y}{N} + (D_X\,\overset{y}{\underset{x}{L}})(Y)\,\underset{y}{N} \\ &- (D_y\,\overset{y}{\underset{x}{L}})(X)\,\underset{y}{N}\,. \end{aligned} \tag{7.16}$$

Let us put

$${}'\overline{K}(BX, BY, \underset{x}{N}, \underset{y}{N}) \overset{def}{=} G(\overline{K}(BX, BY, \underset{x}{N}), \underset{y}{N})ob. \tag{7.17}$$

Then

$$'\overline{K}(BX, BY, \underset{x}{N}, \underset{y}{N}) = {'\underset{y}{H}}(\underset{x}{H}X, Y) - {'\underset{y}{H}}(\underset{x}{H}Y, X) + (D_X \overset{y}{\underset{x}{L}})(Y) - (D_Y \overset{y}{\underset{x}{L}})(X) + \overset{y}{\underset{x}{L}}(Y)\,\overset{z}{\underset{y}{L}}(X) - \overset{y}{\underset{x}{L}}(X)\,\overset{z}{\underset{y}{L}}(Y). \tag{7.18}$$

This equation is identically satisfied for $x = y$.

If V_n be a hypersurface then the generalization of Gauss characteristic and Mainardi-Codazzi equations reduce to

$$'\overline{K}(BX, BY, BZ, BU) = {'K}(X, Y, Z, U) - {'H}(Y, Z)'H(X, U) + {'H}(X, Z)'H(Y, U) \tag{7.19}$$

$$'\overline{K}(BX, BY, BZ, N) = (D_X{'H})(Y, Z) - (D_Y{'H})(X, Z). \tag{7.20}$$

Also when V_n is a hypersurface, these equations are identically satisfied.

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