J. T. S.
Vol. 4 (2010), pp.41-48
https://doi.org/10.56424/jts.v4i01.10425

On Cartan Spaces with Generalized $(\alpha, \beta)-$metric

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(Received: July 23, 2009)

Abstract

In 1933 E.Cartan [1] introduced a new space known as Cartan space. It is considered as dual of Finsler space. H.Rund [4] , F.Brickell [2] and others studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron ([6] , [7]). T.Igrashi ([10] , [11]) introduced the notion of $(\alpha, \beta$)-metric in Cartan spaces and obtained the metric tensor and the invariants ρ and σ which characterize the special classes of Cartan spaces with $(\alpha, \beta$)-metric. Later on H.G.Nagaraja [3] studied Cartan spaces with $(\alpha, \beta$)-metric in 2007. This paper deals with a study of Cartan spaces with Generalized $(\alpha, \beta$)-metric admitting h-metrical d-connection. The conditions for these spaces to be locally Minkowaski and conformally flat have been obtained.

Keywords and Phrases : Cartan spaces, Generalized(α, β)-metric, h-metrical d-connection, locally Minkowaski and conformally flat spaces.
2000 AMS Subject Classification : 53C60, 53B40.

1. Introduction

In 1978, M.Matsumoto and H.Shimada [5] introduced the concept of 1-form metric $L(\beta_\lambda)$, where $L(\beta_\lambda)$ is positively homogeneous function of degree one in n-arguments $\beta_\lambda(x, y)$, where $\beta_\lambda(x, y) = b_{(\lambda)i}(x)y^i$, $1 \leq \lambda \leq n$, are n-linearly independent 1-forms. In this paper we consider a Cartan metric

$$K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}), \quad 1 \leq \lambda \leq n, \tag{1.1}$$

where (1.1) is a p-homogeneous function with respect to $\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}$ and $\alpha(x, p) = \left(a^{ij}p_ip_j\right)^{\frac{1}{2}}$ together with $\beta^{(r)}(x, p) = b^{(r)i}(x)p_i, r = 1, ..., \lambda$, which are $\lambda-$ linearly independent 1-forms. For $\lambda = 1$ this metric is nothing but (α, β)-metric.

Let M be a real smooth manifold and (T M, π, M) its cotangent bundle. Let $C^n = (M, K(x,p))$, where $K : T^*M \longrightarrow R$ is a scalar function which is differentiable on T*M=TM-{0} and is homogeneous on fibres of T*M. The hessian of K^2 i.e., $g^{ij}(x,p) = \frac{1}{2}\partial^{\cdot i}\partial^{\cdot j}K^2$, where $\partial^{\cdot i} = \frac{\partial}{\partial p^i}$, is positively homogeneous on T^*M. Here C^n is called the Cartan space and the functions $K(x,p)$ and $g^{ij}(x,p)$ are called, respectively, the fundamental function and the metric tensor of the Cartan space C^n. The reciprocal $g_{ij}(x,p)$ of $g^{ij}(x,p)$ is given by $g_{ij}(x,p)g^{ik}(x,p) = \delta^k_j$, where $g_{ij}(x,p)$ and $g^{ij}(x,p)$ are both symmetric and homogeneous of order 0 in p_j.

A Cartan space $C^n = (M, K(x,p))$ is said to be with generalized $(\alpha, \beta\ )$-metric if $K(x,p)$ is a function of the variables $\alpha(x,p) = \left(a^{ij}p_ip_j\right)^{\frac{1}{2}}, \beta^{(r)}(x,p) = b^{(r)i}(x)p_i, r = 1, ..., \lambda$, where $a^{ij}(x)$ is a Riemannian metric and $b^{(r)i}(x)$ is a vector field depending only on x. Clearly, K must satisfy the conditions imposed to the fundamental functions of a Caratn space.

2. Generalized (α, β)-metric

Definition (2.1). A Cartan metric $K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$ is called Generalized (α, β)-metric.

In this paper we consider the Cartan spaces with generalized (α, β)-metric admitting h-metrical d-connection and their conformal change. To find the angular metric tensor g^{ij} of $C^n = (M, K(x,p))$ we use the following results:

(2.1) $$\partial^{\cdot i}\alpha = \frac{p^i}{\alpha}, \partial^{\cdot i}\beta^{(r)} = b^{(r)i}, \partial^{\cdot i}K = l^i, \partial^{\cdot j}l^i = \frac{1}{K}h^{ij},$$

where
$$\partial^{\cdot i} = \frac{\partial}{\partial p^i}, \quad h^{ij} = g^{ij} - l^il^j = K\,\frac{\partial^2 K}{\partial p_i \partial p_j} \text{ and } p^i = a^{ij}p_j.$$

The successive differentiation of (1.1) with respect to p_i and p_j gives

(2.2) $$l^i = K_\alpha \frac{p^i}{\alpha} + \sum_{r=1}^{\lambda} K_{\beta^{(r)}} b^{(r)i}$$

(2.3) $$h^{ij} = \frac{KK_{\alpha\alpha}p^ip^j}{\alpha^2} + \sum_{r=1}^{\lambda} \frac{KK_{\alpha\beta^{(r)}}}{\alpha}\left(b^{(r)i}p^j + b^{(r)j}p^i\right) + \frac{KK_\alpha}{\alpha}a^{ij}$$

$$-\frac{KK_\alpha}{\alpha^3}p^i p^j + \sum_{r=1}^{\lambda}\sum_{s=1}^{\lambda} KK_{\beta^{(r)}\beta^{(s)}} b^{(r)i} b^{(s)j},$$

where

$$K_\alpha = \frac{\partial K}{\partial \alpha}, \qquad K_{\beta^{(r)}} = \frac{\partial K}{\partial \beta^{(r)}}, \qquad K_{\alpha\alpha} = \frac{\partial^2 K}{\partial \alpha^2}, \qquad K_{\alpha\beta^{(r)}} = \frac{\partial^2 K}{\partial \alpha \partial \beta^{(r)}},$$

$$K_{\beta^{(r)}\beta^{(s)}} = \frac{\partial^2 K}{\partial \beta^{(r)}\ \partial \beta^{(s)}}.$$

From (2.2) and (2.3) , we get the metric tensor of C^n, given by

$$g^{ij} = \rho a^{ij} + \sum_{r=1}^{\lambda}\sum_{s=1}^{\lambda} \rho^{rs} b^{(r)i} b^{(s)j} + \sum_{r=1}^{\lambda} \rho^r \left(b^{(r)i}p^j + b^{(r)j}p^i\right) + \sigma p^i p^j, \tag{2.4}$$

where ρ^{rs}, ρ^r and σ are functions of α and $\beta^{(r)}$, given by

$$\rho = \frac{KK_\alpha}{\alpha}, \quad \rho^{rs} = KK_{\beta^{(r)}\beta^{(s)}} + K_{\beta^{(r)}}K_{\beta^{(s)}}, \rho^r = \frac{KK_{\alpha\beta^{(r)}} + K_\alpha K_{\beta^{(r)}}}{\alpha} \tag{2.5}$$

and

$$\sigma = \frac{KK_{\alpha\alpha} - \alpha^{-1}KK_\alpha + K_\alpha^2}{\alpha^2}.$$

The homogeneity of K in α and $\beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}$ gives the identity

$$\sum_{r=1}^{\lambda} \rho^{rs}\beta^{(r)} + \rho^s \alpha^2 = KK_{\beta^{(s)}}. \tag{2.6}$$

Let γ^i_{jk} denote the christoffel symbol constructed from a_{ij} and $F_\gamma = \{\gamma^i_{jk}, \gamma^i_{0j} = p^k\gamma^i_{kj}, 0\}$ be the linear Finsler connection of the space C^n, induced from the Riemannian connection $\gamma = (\gamma^i_{jk}(x))$ of the associated Riemannian space (M^n, α). We denote ':' the covariant differentiation with respect to F_γ.Then $a_{ij:k} = 0$, $a^{ij}_{:k} = 0, p^i_{:k} = 0$. Since $p_i = a_{ij}p^j$, it follows that $p_{i:k} = 0$. Also, $\alpha^2 = a^{ij}(x)p_i p_j$ gives $\alpha_{:k} = 0$. Now, if we assume that $b^{(r)i}_{:k} = 0$ for $r = 1, ..., \lambda$, then $\beta^{(r)} = b^{(r)i}p_i$ gives $\beta^{(r)}_{:k} = 0$ for $r = 1,...,\lambda$. Consequently, (1.1) gives

$$K_{:k} = K_\alpha \alpha_{:k} + \sum_{r=1}^{\lambda} L_{\beta^{(r)}}\beta^{(r)}_{:k} = 0.$$

Since K_α is a function of $\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}$, we have $(K_\alpha)_{:k} = 0$. Similarly, $K_{\alpha\beta^{(r)}:k} = 0, K_{\beta^{(r)}\beta^{(s)}:k} = 0$, which in view of (2.5) give $\rho_{:k} = 0$, $\rho^{rs}_{:k} = 0, \rho^r_{:k} = 0$ and $\sigma_{:k} = 0$. Therefore, from (2.4) it follows that $g^{ij}_{:k} = 0$.

Further, F_γ has vanishing (h) h-torsion tensor T, deflection tensor D and (h) hv-torsion tensor C. Therefore, by the definition of Rund connection, we have

Proposition (2.1). If $b^{(r)i}_{:k} = 0, r = 1, ..., \lambda,$ is satisfied in a Cartan space C^n with generalized $(\alpha, \beta)-$ metric then the linear Cartan connection F_γ is nothing but the Rund's connection $R\Gamma$ of C^n.

It is remarked that the h-covariant derivative with respect to $R\Gamma$ coincides with that with respect to the Cartan connection $C\Gamma$.

Using the Christoffel symbols $\Gamma^i_{jk}(p) = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ constructed from $g_{ij}(x, p)$, we can define canonical N-connection

(2.7)
$$N_{ij} = \Gamma^k_{ij}\rho_k - \frac{1}{2}\Gamma^k_{hr}\rho_k\rho^r\partial^{\cdot h}g_{ij}.$$

We consider the canonical d-connection

(2.8)
$$D\Gamma = \left(N_{jk}, H^i_{jk}, C^{jk}_i\right),$$

where

(2.9)
$$H^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}).$$

The d-tensor field C^{jk}_i of type (2, 1) is given by

(2.10)
$$C^{jk}_i = -\frac{1}{2}g_{ir}\partial^{\cdot r}g^{jk} = g_{ir}C^{rjk}.$$

Let $_{|k}$ denote the h-covariant derivative with respect to $D\Gamma$, then we have

Definition (2.2). A d-connection $D\Gamma$ of a Cartan space C^n with generalized $(\alpha, \beta)-$metric is called h-metrical d-connection if it satisfies the following conditions:

(i) h-deflection tensor $D_{ij} = (p_{i|j}) = 0$,

(ii) $a^{ij}_{|k} = 0$,

(iii) $g^{ij}_{|k} = 0$.

3. Cartan Spaces with generalized (α, β)- metric admitting h-metrical d-connection

From the condition (i) of definition (2.2), we get $D_{ij} = p_{i|j} = 0$, therefore, the equation $K^2 = g^{ij}p_ip_j$ and condition (iii) of definition (2.2) give $K_{|k} = 0$.

Again, by the condition (i) and (ii), on the basis of the equation $p^i = a^{ij}(x)p_j$ and $\alpha^2 = a^{ij}(x)\ p_i p_j$, we get

$$\alpha_{|k} = 0, \qquad p^i_{|k} = 0. \tag{3.1}$$

Since $K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$, $1 \leq \lambda \leq n$, on the basis of (3.1), we get

$$K_{|k} = \sum_{r=1}^{\lambda} K_{\beta^{(r)}} \beta^{(r)}_{|k} = 0.$$

Therefore, $\beta^{(r)}_{|k} = 0$ for $r = 1, ..., \lambda$ and $K_{\beta^{(r)}}$ are linearly independent. Since, $\beta^{(r)}\ (x,p) = b^{(r)i}\ (x)p_i, r = 1, ..., \lambda$, on the basis of condition (i) of definition(2.3), we get

$$\beta^{(r)}_{|k} = b^{(r)i}_{|k}\ (x)p_i = 0,\ r = 1, ..., \lambda. \tag{3.2}$$

Since, $K_{|k} = 0$, $\alpha_{|k} = 0$, $\beta^{(r)i}_{|k} = 0$ for $r = 1, ..., \lambda$ and $K_\alpha, K_{\alpha\alpha}, K_{\alpha\beta^{(r)}}, K_{\beta^{(r)}\beta^{(s)}}$ are functions of $\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}$, therefore, $\rho_{|k} = 0$, $\rho^r_{|k} = 0$, $\rho^{rs}_{|k} = 0, \sigma_{|k} = 0$. Hence, h-covariant derivative of (2.4), on the basis of the conditions(ii) and (iii) of definition (2.2) gives

$$g^{ij}_{|k} = 0 = \sum_{r=1}^{\lambda} b^{(r)i}_{|k} \left(\sum_{s=1}^{\lambda} \rho^{rs} b^{(r)j} + \rho^s p^j \right) + \sum_{s=1}^{\lambda} b^{(s)i}_{|k} \left(\sum_{r=1}^{\lambda} \rho^{rs} b^{(r)i} + \rho^s p^i \right).$$

Contracting by p_j and using the identity (2.6) and equation (3.2), we get

$$\sum_{r=1}^{\lambda} K_{\beta^{(r)}} b^{(r)i}_{|k} = 0.$$

Since $K_{\beta^{(r)}}$ are linearly independent, we have

$$b^{(r)i}_{|k} = 0, r = 1, ..., \lambda. \tag{3.3}$$

Now from $a^{ij}_{|k} = 0$, we get $H^i_{jk} = \gamma^i_{jk}$. Hence, we have

$$b^{(r)i}_{:k} = 0, \qquad r = 1, ..., \lambda. \tag{3.4}$$

Also, the curvature tensor D^i_{hjk} of $D\Gamma$ coincides with the curvature tensor R^i_{hjk} of Riemannian connection $R\Gamma = \left(\gamma^i_{jk}, \gamma^i_{jk} p_i, 0\right)$. If $R^i_{hjk} = 0$, then $D^i_{hjk} = 0$. Thus, we have the following proposition:

Proposition (3.1). A Cartan space C^n with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection is locally flat if and only if the associated Riemannian space is locally flat.

If the connection $D\Gamma$ is h-metrical, then $g^{ij}_{\shortmid h} = 0$, $\alpha_{\shortmid h} = 0, a^{ij}_{\shortmid h} = 0, b^{(r)k}_{\shortmid h} = 0, p^k_{\shortmid h} = 0$. From (2.1), (2.4) and (2.5) it follows that $C^{ijk} = -\frac{1}{2}\partial^{\cdot k} g^{ij}$ can be determined in terms of $a^{ij}, p^i, b^{(r)i}, K$ and its partial derivatives of first, second and third orders with respect to α and $\beta^{(r)}, (r = 1, ..., \lambda)$. Since the h-covariant derivative of all these quantities vanishes, we have $C^{ijk}_{\shortmid h} = 0$. Hence, in view of (2.10) and condition (iii) of definition (2.2), it follows that

$$C^{ij}_{k\shortmid h} = 0. \tag{3.5}$$

Definition (3.1). A Cartan space C^n is a Berwald space if and only if $C^{ij}_{k\shortmid h} = 0$.

Hence, from (3.5), we have the following proposition:

Proposition (3.2). A Cartan space C^n with generalized $(\alpha, \beta)-$metric admitting h-metrical d-connection is a Berwald space.

As we know [9] a locally Minkowaski space is a Berwald space in which the curvature tensor vanishes. Hence, from the propositions (3.1) and (3.2), we have the following theorem:

Theorem (3.1). A Cartan space C^n with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection is locally Minkowaski if and only if the associated Riemannian space is locally flat.

4. Conformal change of Cartan space

Let $C^n = (M, K(x, p))$ be an n-dimensional Cartan space with generalized $(\alpha, \beta)-$metric $K = K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$, $1 \leq \lambda \leq n$, by a conformal change $\eta : K \longrightarrow \overline{K}$ such that $\overline{K}\left(\overline{\alpha}, \overline{\beta}^{(1)}, \overline{\beta}^{(2)}, ..., \overline{\beta}^{(\lambda)}\right) = e^{\eta} K(\alpha, \beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)})$, $1 \leq \lambda \leq n$, we have another Cartan space $\overline{C}^n = \left(M, \overline{K}\left(\overline{\alpha}, \overline{\beta}^{(1)}, ..., \overline{\beta}^{(\lambda)}\right)\right)$, where $\overline{\alpha} = e^{\eta}\alpha$ and $\left(\overline{\beta}^{(1)}, \overline{\beta}^{(2)}, ..., \overline{\beta}^{(\lambda)}\right) = e^{\eta}\left(\beta^{(1)}, \beta^{(2)}, ..., \beta^{(\lambda)}\right)$. Putting $\alpha(x, p) = \left(a^{ij} p_i p_j\right)^{\frac{1}{2}}$ and $\beta^{(r)}(x, p) = b^{(r)i}(x) p_i$, $r = 1, ..., \lambda$, we get $\overline{a}^{ij} = e^{2\eta} a^{ij}$ and $\overline{b}^{(r)i} = e^{\eta} b^{(r)i}$. Then the Christoffel symbol $\overline{\gamma}^i_{jk}$ constructed from $\overline{a}^{ij}$ is written as

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + B^i_{jk}, \tag{4.1}$$

where

$$B^i_{jk} = \delta^i_j \eta_k + \delta^i_k \eta_j - \eta^i a_{jk}, \qquad \eta^i = \eta_j a^{ij}.$$

Taking covariant derivative of $\overline{b}^{(r)i}$ with respect to $\overline{\gamma}^i_{jk}$, we get

$$\overline{b}^{(r)i}_{:k} = e^{\eta} \sum_{r=1}^{\lambda} \left(b^{(r)i}_{:k} + 2\eta_k b^{(r)i} + b^{(r)j}\eta_j \delta^i_k - \eta^i a_{jk} b^{(r)j} \right).$$

Transvecting by $\overline{b}^{(r)k}$ and putting

$$M^i = \frac{1}{B^2} \sum_{r=1}^{\lambda} \left(b^{(r)k} b^{(r)i}_{:k} - \frac{1}{n+4} b^{(r)j}_{:j} b^{(r)i} \right), \tag{4.2}$$

we have

$$\eta^i = \overline{M}^i - M^i, \text{ from which we get, } \sigma_i = \overline{M}_i - M_i.$$

Substituting this in (4.1) and putting $D^i_{hj} = \gamma^i_{hj} + \delta^i_h M_j - M^i a_{hj}$, we have

$$\overline{D}^i_{hj} = D^i_{hj} \tag{4.3}$$

D^i_{hj} is a symmetric conformally invariant linear connection on M. Thus we have the following proposition:

Proposition (4.1). In a Cartan space with generalized $(\alpha,\beta)-$ metric there exists a conformally invariant symmetric linear connection D^i_{hj}.

If we denote by D^i_{hjk}, the curvature tensor of D^i_{hj}, we have from (4.3)

$$\overline{D}^i_{hjk} = D^i_{hjk} \tag{4.4}$$

Since $b^{(r)i}_{:k} = 0$, we have $M^i = 0$. Hence, $D^i_{jk} = \gamma^i_{jk}$ and $D^i_{hjk} = R^i_{hjk}$. Thus, we have the next proposition:

Proposition (4.2). In a Cartan space with generalized $(\alpha,\beta)-$ metric admitting h-metrical d-connection $M^i = 0$ and there exists a conformally invariant symmetric linear connection D^i_{jk} such that $D^i_{jk} = \gamma^i_{jk}$ and $D^i_{hjk} = R^i_{hjk}$.

If the associated Riemannian space (M,α) is locally flat $\left(R^i_{hjk} = 0\right)$, then from (4.4) and proposition (4.2), we have $\overline{D}^i_{hjk} = 0$, i.e., the space C^n is conformally flat. Thus we conclude that

Theorem (4.1). A Cartan space C^n with generalized $(\alpha, \beta)-$ metric admitting h-metrical d-connection is conformally flat if and only if the associated Riemannian space (M, α) is locally flat $\left(R^i_{hjk} = 0\right)$.

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