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On Contravariant Almost Analytic and Strictly Almost Analytic Vector Field

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In this paper we have studied and obtained necessary and sufficient conditions for a contravariant almost analytic vector field to be strictly almost analytic vector field in Kenmotsu manifold as well as in trans-Sasakian manifold.

Keywords : Contravariant almost analytic vector field; Contravariant strictly almost analytic vector field.
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1. Introduction

In 1972 Kenmotsu introduced a new type of almost contact manifold in [4] which was named after him. Many authors have studied various properties on this type of manifold and obtained interesting results in [2], [8]. In 1985 J.Oubina introduced another new class of almost contact manifold, called trans-Sasakian manifold in [6] which is the generalization of α-Sasakian and β-Kenmotsu manifold. J.C.Marrero in [5] studied local structures on this manifold and Tripathi, De in [3] gave some relations on curvature identities. A.Bhattacharyya, M.Tarafdar studied a special type of trans-Sasakian manifolds in [1].

This paper deals with contravariant almost analytic vector field and strictly almost analytic vector field in Kenmotsu manifold and in trans-Sasakian manifold. Here we have obtained necessary and sufficient conditions for an almost analytic vector field to be a strictly almost analytic vector field in both type of manifolds.

2. Preliminaries

Let $M^n(n = 2m + 1, m \geq 1)$ be an odd dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is (1,1) tensor field, ξ is a contravariant vector field, η is a 1-form and g is metric tensor satisfying the relations

$$\phi^2 X = -X + \eta(X)\xi,\ \phi \circ \xi = 0,\ \eta(\xi) = 1. \tag{2.1}$$

$$g(X, \xi) = \eta(X),\ \eta \circ \phi = 0. \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

$$g(X, \phi Y) = -g(\phi X, Y). \tag{2.4}$$

M^n is called Kenmotsu manifold [4] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X. \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.6}$$

In Kenmotsu manifold we have the following relations [2]

$$S(X, \xi) = -2m\eta(X). \tag{2.7}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.8}$$

Again M^n is called a trans-Sasakian manifold of type (α, β) [6] provided,

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}. \tag{2.9}$$

From (2.9) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.10}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.11}$$

In [3] De and Tripathi obtained some results which are useful for next sections. These are

$$S(X, \xi) = \{2n(\alpha^2 - \beta^2) - \xi\beta\}\eta(X) - (2n - 1)X\beta - (\phi X)\alpha \tag{2.12}$$

and

$$S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X) \tag{2.13}$$

provided

$$(2n - 1)grad\ \beta = \phi(grad\ \alpha). \tag{2.14}$$

On the other hand, Lie derivative along a vector field V is a type-preserving mapping such that [7]

$$L_V f = V f, \tag{2.15}$$

where f is a scalar function on M^n.

$$L_V X = [V, X] \ \forall X \tag{2.16}$$

and

$$(L_V B)(X) = V(B(X)) - B([V, X]) \ \forall \ B \tag{2.17}$$

where B is a 1-form.

V is said to be contravariant almost analytic if

$$(L_V \phi)(X) = 0. \tag{2.18}$$

V is said to be strictly almost analytic if together with (2.18)

$$(L_{\phi V} \phi)(X) = 0. \tag{2.19}$$

A vector valued trilinear function defined by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.20}$$

is called curvature tensor.

In M^n, we have

$$(\nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{[X,Y]} \phi) Z = {}'R(X, Y, \phi Z) - \phi R(X, Y, Z) \tag{2.21}$$

where

$$'R(X, Y, Z, U) = g(R(X, Y, Z), U) \tag{2.22}$$

also

$$\phi[X, Y] = \phi \nabla_X Y - \phi \nabla_Y X = \nabla_X \phi Y - \nabla_Y \phi X - (\nabla_X \phi) Y + (\nabla_Y \phi) X. \tag{2.23}$$

The torsion tensor T is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{2.24}$$

All these results will be useful in next sections.

3. Almost analytic vector field in Kenmotsu manifold

In this section we consider a vector field V in a Kenmotsu manifold is almost analytic.

From (2.17) and (2.18) we have

$$[V,\phi X]=\phi[V,X]. \tag{3.1}$$

From (3.1), (2.24) and (2.16), we get

$$\begin{aligned}\nabla_V\phi^2X-(\nabla_V\phi)\phi X-\nabla_{\phi X}\phi V+(\nabla_{\phi X}\phi)V-\phi T(V,\phi X)\\=\phi\nabla_VX-\phi\nabla_XV-\phi T(V,X).\end{aligned} \tag{3.2}$$

Using (2.1) and (2.23) in (3.2), we obtain

$$\begin{aligned}&(\nabla_V\eta)(X)\xi+\eta(X)\nabla_V\xi-\nabla_VX-(\nabla_V\phi)\phi X-\nabla_{\phi X}\phi V+(\nabla_{\phi X}\phi)V\\&-\phi T(V,\phi X)=\nabla_V\phi X-\nabla_X\phi V-(\nabla_V\phi)X+(\nabla_X\phi)V-\phi T(V,X).\end{aligned} \tag{3.3}$$

Using (2.5) in (3.3), we have

$$\begin{aligned}\{(\nabla_V\eta)(X)&-\eta(X)\eta(V)-g(V,\phi X)+g(X,\phi V)+g(V,\phi^2X)-g(\phi X,\phi V)\}\xi\\&+V\eta(X)-\eta(X)\phi V-\nabla_VX+\eta(\phi X)\phi V-\nabla_{\phi X}\phi V-\eta(V)\phi^2X\\&-\nabla_V\phi X+\nabla_X\phi V+\eta(V)\phi X=\phi T(V,\phi X)-\phi T(V,X).\end{aligned} \tag{3.4}$$

Using (2.1), (2.2), (2.3) and (2.8) in (3.4) we get

$$\begin{aligned}T(V,X)=\eta(T(V,X))\xi&-\phi T(V,\phi X)+V\eta(X)+X\eta(V)-\eta(X)\phi V-\nabla_VX\\&-\nabla_{\phi X}\phi V-\nabla_V\phi X+\nabla_X\phi V+\eta(V)\phi X+\{g(X,\phi V)\\&-g(V,\phi X)-g(X,V)-\eta(X)\eta(V)\}\xi.\end{aligned} \tag{3.5}$$

Now taking $X=\xi$ in (3.5) and using (2.1),(2.2) and (2.6) we get

$$T(V,\xi)=\eta(T(V,\xi))\xi+\nabla_\xi\phi V-\phi V. \tag{3.6}$$

Thus we can state the following theorem-

Theorem 3.1 The necessary and sufficient condition that a vector field V is contravariant almost analytic in Kenmotsu manifold is

$$T(V,\xi)=\eta(T(V,\xi))\xi+\nabla_\xi\phi V-\phi V$$

where T is a torsion tensor.

4. Almost analytic vector field in trans-Sasakian manifold

In this section we consider a vector field V in trans-Sasakian manifold is almost analytic.

Using (2.1), (2.3) and (2.9) in (3.3) we get

$$\begin{aligned}&(\nabla_V\eta)(X)\xi+\eta(X)\nabla_V\xi-\nabla_VX-\beta g(V,X)\xi-\nabla_{\phi X}\phi V-\alpha\eta(V)\phi X\\&\quad-\beta g(X,V)\xi+\beta\eta(X)g(\xi,V)\xi+\beta\eta(V)X-\phi T(V,\phi X)\\&=\nabla_V\phi X-\nabla_X\phi V-\alpha\eta(X)V-\beta\eta(X)\phi V-\alpha\eta(V)X-\beta\eta(V)\phi X\\&\quad-\phi T(V,X).\end{aligned}\tag{4.1}$$

Using (2.11) in (4.1) we obtain

$$\begin{aligned}&-\alpha g(X,\phi V)\xi-\beta g(X,V)\xi-\alpha\eta(X)\phi V+\eta(X)\beta(V-\eta(V)\xi)-\nabla_VX\\&\quad-\nabla_{\phi X}\phi V-\alpha\eta(V)\phi X+\beta\eta(V)X-\phi T(V,\phi X)\\&=\nabla_V\phi X-\nabla_X\phi V-\alpha\eta(X)V-\beta\eta(X)\phi V-\alpha\eta(V)X-\beta\eta(V)\phi X\\&\quad-\phi T(V,X)\end{aligned}\tag{4.2}$$

i.e.

$$\begin{aligned}\phi T(V,X)-\phi T(V,\phi X)\ &=\nabla_V\phi X+\nabla_VX+\nabla_{\phi X}\phi V-\nabla_X\phi V+\alpha[g(X,\phi V)\xi\\&\quad+\eta(X)\phi V+\eta(V)\phi X-\eta(X)V-\eta(V)X]\\&\quad+\beta[g(X,V)\xi-\eta(V)X-\eta(X)\phi V-\eta(V)\phi X]\\&\quad-\eta(X)\beta(V-\eta(V)\xi).\end{aligned}\tag{4.3}$$

Now putting $X=\xi$ in (4.3) and using (2.1), (2.2) and (2.10) we get

$$\phi T(V,\xi)=-\nabla_\xi\phi V-\alpha[V+\eta(V)\xi]-\beta\phi V.\tag{4.4}$$

Thus we can state-

Theorem 4.1 The necessary and sufficient condition that a vector field V is contravariant almost analytic in trans-Sasakian manifold is
$\phi T(V,\xi)=-\nabla_\xi\phi V-\alpha[V+\eta(V)\xi]-\beta\phi V$.

5. Strictly almost analytic vector field in Kenmotsu manifold

In this section we consider a vector field V in Kenmotsu manifold is strictly almost analytic.

We know that

$$(L_{\phi V}\phi)(X)=L_{\phi V}\phi X-\phi L_{\phi V}X.\tag{5.1}$$

$$\begin{aligned}(L_{\phi V}\phi)(X)&=-\nabla_{\phi X}\phi V-T(\phi V,\phi X)+(\nabla_{\phi V}\phi)X-\nabla_XV+(\nabla_X\eta)(V)\xi\\&\quad+\eta(\nabla_XV)\xi+\eta(V)\nabla_X\xi-(\nabla_X\phi)\phi V+\phi T(\phi V,X).\end{aligned}\tag{5.2}$$

$$\phi(L_V\phi)(X) = (\nabla_V\eta)(X)\xi + \eta(X)\nabla_V\xi - (\nabla_V\phi)\phi X - \nabla_{\phi X}\phi V$$
$$+(\nabla_{\phi X}\phi)V - \nabla_X V + \eta(\nabla_X V)\xi - \phi T(V,\phi X) + \phi^2 T(V,X). \tag{5.3}$$

Subtracting (5.3) from (5.2) and using (2.2),(2.3),(2.5) and (2.6) we obtain

$$(L_{\phi V}\phi)(X) - \phi(L_V\phi)X = \phi T(\phi V,X) + \phi T(V,\phi X) - T(\phi V,\phi X)$$
$$-\phi^2 T(V,X). \tag{5.4}$$

When V in a Kenmotsu manifold is strictly almost analytic then left hand side of (5.4) becomes zero. Therefore we have

$$T(\phi V,\phi X) + \phi^2 T(V,X) = \phi T(\phi V,X) + \phi T(V,\phi X). \tag{5.5}$$

Putting $X = \xi$ in (5.5) and using (2.1) we get

$$\phi^2 T(V,\xi) = \phi T(\phi V,\xi). \tag{5.6}$$

Hence we can state-

Theorem 5.1 An almost analytic vector field V in a Kenmotsu manifold is strictly almost analytic if and only if

$$\phi^2 T(V,\xi) = \phi T(\phi V,\xi).$$

6. Strictly almost analytic vector field in trans-Sasakian manifold

In this section we consider a vector field V in trans-Sasakian manifold is strictly almost analytic.

Subtracting equation (5.3) from (5.2) and using (2.2),(2.4),(2.9),(2.10) and (2.11), we get

$$(L_{\phi V}\phi X) - \phi(L_V\phi)(X) = -T(\phi V,\phi X) + \phi T(\phi V,X) + \phi T(V,\phi X)$$
$$-\phi^2 T(V,X) + 2\alpha g(X,\phi V)\xi + \beta[\{\eta(X)\}^2 - \{\eta(V)\}^2]. \tag{6.1}$$

When V in a trans-Sasakian manifold is strictly almost analytic then left hand side of (6.1) is zero. Therefore we obtain

$$T(\phi V,\phi X) + \phi^2 T(V,X) = \phi T(\phi V,X) + \phi T(V,\phi X) + 2\alpha g(X,\phi V)\xi$$
$$+\beta[\{\eta(X)\}^2 - \{\eta(V)\}^2]. \tag{6.2}$$

Putting $X = \xi$ in (6.2) and using (2.1),(2.2) we have

$$\phi^2 T(V,\xi) = \phi T(\phi V,\xi) + \beta[1 - \{\eta(V)\}^2]. \tag{6.3}$$

Thus we can state the following theorem-

Theorem 6.1 An almost analytic vector field V in a trans-Sasakian manifold is strictly almost analytic if and only if

$$\phi^2 T(V,\xi) = \phi T(\phi V,\xi) + \beta[1 - \{\eta(V)\}^2].$$

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