

J. T. S.

Vol. 6 No.1 (2012), pp.35-41

<https://doi.org/10.56424/jts.v6i01.10462>

Conformal Correspondence of Finsler Spaces with Special (α, β) –Metric

H. S. Shukla and A. P. Tiwari

Department of Mathematics & Statistics
DDU Gorakhpur University, Gorakhpur, India

(Received: 30 April, 2011)

(Dedicated to Prof. K. S. Amur on his 80th birth year)

Abstract

The purpose of the present paper is to discuss the conformal transformation of a Finsler space with a special (α, β) metric given by $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_1, c_2, c_3 are constants, α is Riemannian metric and β is one form. We have proved that for such a Finsler metric, the Berwald spaces, the locally Minkowski spaces and the projectively flat spaces are not invariant under non-homothetic conformal transformation.

Keywords and Phrases : Randers space, Kropina space, conformal change of metric, generalized Randers space, locally Minkowski space.

2000 AMS Subject Classification : 53B40, 53C60.

1. Introduction

Let M^n be an n -dimensional differentiable manifold and TM^n be its tangent bundle. The manifold M^n is covered by a coordinate neighbourhood $\{U\}$; in each U of which we have a local coordinate system (x^i) . A tangent vector at a point $x = (x^i)$ of U is written as $y^i(\frac{\partial}{\partial x^i})_x$ and we have a local co-ordinate system (x^i, y^i) of TM^n over the U .

The manifold M^n equipped with a fundamental function $L(x, y)$ is called a Finsler space $F^n = (M^n, L)$, if L is defined for any point of $TM^n - \{0\}$ and is positively homogeneous of degree one in y^i . If L is positively homogeneous of degree one in α and β , where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on M^n , then F^n is called a Finsler space with (α, β) –metric ([2], [3]).

Various special (α, β) -metrics have been considered in the literature. For instance if $L = \alpha + \beta$, then this metric is called Randers metric and corresponding Finsler space is called Randers space [4]. If $L = \frac{\alpha^2}{\beta}$, then this metric is called Kropina metric and corresponding Finsler space is called Kropina space [8].

A Finsler space is called C-reducible Finsler space if its Cartan's C-tensor $C_{ijk} = \frac{1}{4} \frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}$ is written in the form

$$C_{ijk} = h_{ij}A_k + h_{jk}A_i + h_{ki}A_j, \quad (1.1)$$

where h_{ij} is the angular metric tensor given by $h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j}$ and A_i is a vector field given by $A_i = \frac{1}{n+1} C_{ijk} g^{jk}$. In [5], M. Matsumoto and S. Hojo have obtained that a Finsler space of dimension $n \geq 3$, is C-reducible if and only if the F^n is either Randers space or Kropina space.

A Finsler space is called generalized C-reducible [7] if there exists a covariant tensor field K_{ij} of order 2 and a covariant vector field B_i such that

$$C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j. \quad (1.2)$$

A Finsler space with (α, β) -metric is generalized C-reducible if and only if $\frac{1}{2} \frac{\partial^3 L^2(\alpha, \beta)}{\partial \beta^3} = 0$. Owing to this fact Park and Choi [6] introduced an (α, β) metric L given by

$$L^2 = c_1 \alpha^2 + 2c_2 \alpha \beta + c_3 \beta^2, \quad (1.3)$$

where c_1, c_2, c_3 are constants. In case of $c_1 = c_2 = c_3 = 1$, the metric $L(\alpha, \beta)$ given by (1.3) is a Randers metric, so that the metric $L(\alpha, \beta)$ given by (1.3) may be considered as a generalization of Randers metric. If $c_2 = 0$, the metric $L(\alpha, \beta)$ given by (1.3) is a Riemannian metric and if $c_1 c_3 - c_2^2 = 0$, then the metric is Randers metric. Therefore we shall assume that $c_2 \neq 0$ and $c_1 c_3 - c_2^2 \neq 0$. A Finsler space with the metric (1.3) is said to be a generalized Randers space.

A Finsler space F^n is called a Berwald space if Berwald's connection coefficient of F^n is linear and F^n is called locally Minkowski if there exist a coordinate system in which L is independent of coordinates (x^i) . The Finsler space F^n is called projectively flat if F^n is conformal to a locally Minkowski space.

Let $\nabla_k b_i$ denote covariant derivative of b_i with respect to associated Riemannian connection and R_{hjk}^i be the curvature tensor of Riemannian space (M^n, α) , then we have the following theorems [6]:

Theorem 1.1. If $c_1 \neq 0, c_3 \neq 0$, then F^n is a Berwald space if and only if $\nabla_k b_i = 0$.

Theorem 1.2. If $c_1 = c_3 = 0$, then F^n is a Berwald space if and only if there exists a covariant vector field (v_k) such that

$$\nabla_k b_i = \frac{1}{2}(v^r b_r a_{ki} - v_i b_k - 2v_k b_i),$$

where $v^r = a^{ri} v_i$.

Theorem 1.3. The Finsler space with an (α, β) -metric given by (1.3), where $c_1 \neq 0$, $c_3 \neq 0$, is locally Minkowski space if and only if $R_{hjk}^i = 0$ and $\nabla_k b_i = 0$ are satisfied.

Theorem 1.4. A Finsler space with an (α, β) -metric given by (1.3), is projectively flat if and only if associated Riemannian space (M^n, α) is projectively flat and $\nabla_k b_i = 0$.

2. Conformal Change of Finsler Space

Let two distinct Finsler metric functions $L(x, y)$ and $L^*(x, y)$ be defined over an n -dimensional differentiable manifold M^n , then the two Finsler spaces (M^n, L) and (M^n, L^*) are said to be in conformal correspondence if there exist a function $\sigma(x)$ such that

$$L^* = e^\sigma L. \quad (2.1)$$

If we denote the quantities corresponding to (M^n, L^*) by putting ** on the superscript position of that quantity, then

$$\begin{aligned} l^{*i} &= \frac{y^i}{L^*} = e^{-\sigma} l^i, & l_i^* &= \frac{\partial L^*}{\partial y^i} = e^\sigma l_i, & g^{*ij} &= \frac{1}{2} \frac{\partial^2 L^{*2}}{\partial y^i \partial y^j} = e^{2\sigma} g_{ij}, \\ g^{*ij} &= e^{-2\sigma} g^{ij}, & C_{ijk}^* &= e^{2\sigma} C_{ijk}, & C_{jk}^{*i} &= g^{*ir} C_{rjk}^* = C_{jk}^i. \end{aligned} \quad (2.2)$$

If σ is constant, the conformal transformation is called homothetic. For homothetic transformation $\sigma_i = \frac{\partial \sigma}{\partial x^i} = 0$.

3. Conformal Change of (α, β) -metric

A conformal change $L \rightarrow L^* = e^\sigma L(\alpha, \beta)$ of (α, β) -metric L is expressed as $(\alpha, \beta) \rightarrow (e^\sigma \alpha, e^\sigma \beta)$. Thus a conformal change of (α, β) -metric is expressed as $(\alpha, \beta) \rightarrow (\alpha^*, \beta^*)$, where

$$\alpha^* = e^\sigma \alpha, \quad \text{and} \quad \beta^* = e^\sigma \beta. \quad (3.1)$$

Therefore, we have

$$a_{ij}^* = e^{2\sigma} a_{ij}, \quad b_i^* = e^\sigma b_i, \quad (3.2)$$

where $\alpha^* = \sqrt{a_{ij}^*(x) y^i y^j}$ and $\beta^* = b_i^*(x) y^i$.

Also we have $b^2 = a^{ij}b_i b_j = a^{*ij}b_i^* b_j^* = b^{*2}$.

Lemma 3.1. In a Finsler space with (α, β) -metric, the length b of b_i with respect to Riemannian α , is invariant under any conformal change of metric.

As to the metric (1.3) of generalized Randers space we have

$$L^{*2} = e^{2\sigma} L^2 = e^{2\sigma} (c_1 \alpha^2 + 2c_2 \alpha \beta + c_3 \beta^2) = c_1 \alpha^{*2} + 2c_2 \alpha^* \beta^* + c_3 \beta^{*2}.$$

Therefore we have the following:

Theorem 3.1. The conformal transformation of a generalized Randers space is a generalized Randers space with same constant coefficients.

If γ_{jk}^i denotes the Christoffel symbol of Riemannian space (M^n, α) , then the corresponding quantity of (M^n, α^*) is obtained from (3.2) and is given by

$$\gamma_{jk}^{*i} = \gamma_{jk}^i + (\sigma_k \delta_j^i + \sigma_j \delta_k^i - \sigma^i a_{jk}), \quad (3.3)$$

where $\sigma^i = a^{ij} \sigma_j$, $\sigma_j = \frac{\partial \sigma}{\partial x^j}$.

Thus $\nabla_k^* b_i^* = \frac{\partial b_i^*}{\partial x^k} - b_h^* \gamma_{ik}^{*h}$ is transformed to

$$\nabla_k^* b_i^* = e^\sigma (\nabla_k b_i + b_r \sigma^r a_{ik} - \sigma_i b_k). \quad (3.4)$$

In view of theorem (1.1) and equations (1.3), (3.4) it follows that a generalized Randers-Berwald space is conformally transformed to a generalized Randers-Berwald space if and only if

$$b_r \sigma^r a_{ij} = \sigma_i b_j. \quad (3.5)$$

Contracting (3.5) with a^{ij} , we get $(n-1) b_r \sigma^r = 0$ which implies that $b_r \sigma^r = 0$ for $n > 1$. Hence from (3.5) we get $\sigma_i b_j = 0$. Since $b_i \neq 0$ we have $\sigma_i = 0$, i.e. the transformation is homothetic. Thus

Theorem 3.2. For $c_1 \neq 0$, $c_3 \neq 0$, a generalized Randers-Berwald space is conformally transformed to a generalized Randers-Berwald space if and only if the transformation is homothetic.

Let F^n with metric (1.3), where $c_1 = c_3 = 0$, be a Berwald space. Then there exists a covariant vector field $v_i(x)$ such that [6]

$$(a) \quad a_{ij|k} = v_k a_{ij}, \quad (b) \quad b_{i|k} = -\frac{1}{2} v_k b_i, \quad (3.6)$$

where $|_k$ denote the covariant derivative with respect to Berwald connection $(G_{jk}^i, G_j^i, 0)$. From (3.6)(a), we get

$$G_{jk}^i = \gamma_{jk}^i - \frac{1}{2}(v_k \delta_j^i + v_j \delta_k^i - v^i a_{jk}), \quad (3.7)$$

where $v^i = a^{ij} v_j$. From (3.6)(b) and (3.7), we obtain

$$\nabla_k b_i = \frac{1}{2}(v^r b_r a_{ki} - v_i b_k - 2v_k b_i). \quad (3.8)$$

Now let us suppose that the conformally transformed generalized Randers space F^{*n} is also a Berwald space. Then there exist covariant vector field v_i^* such that

$$\nabla_k^* b_i^* = \frac{1}{2}(v^{*r} b_r^* a_{ki}^* - v_i^* b_k^* - 2v_k^* b_i^*), \quad (3.9)$$

where $v^{*i} = a^{*ij} v_j^*$. From (3.2) we have

$$v^{*r} b_r^* a_{ki}^* = e^\sigma v_r^* b^r a_{ki}. \quad (3.10)$$

From (3.4), (3.8) and (3.10), we get

$$v_r b^r a_{ki} - v_i b_k - 2v_k b_i = v_r^* b^r a_{ki} - v_i^* b_k - 2v_k^* b_i. \quad (3.11)$$

Contracting (3.11) with a^{ki} we get $v_r b^r = v_r^* b^r$ and putting this value in (3.11), we have $(v_i^* - v_i) b_k + 2(v_k^* - v_k) b_i = 0$.

Contracting this equation by b^k , and using the fact that $v_r b^r = v_r^* b^r$, we get

$$v_i^* = v_i. \quad (3.12)$$

Proposition 3.1. If the generalized Randers- Berwald space for $c_1 = c_3 = 0$, is conformally transformed to the generalized Randers-Berwald space, then the vector field v_i occurring in theorem (1.2) is conformally invariant.

Now we consider the generalized Randers space F^n for $c_1 = c_3 = 0$ and discuss the case that the Berwald space F^n is conformally transformed to a Berwald space or not. In this case from theorem (1.2), F^n is Berwald space if and only if there exists a vector field v_i such that (3.8) holds.

Now from (3.2), (3.8) and (3.12) we get

$$\nabla_k^* b_i^* - \frac{1}{2}(v^{*r} b_r^* a_{ki}^* - v_i^* b_k^* - 2v_k^* b_i^*) = e^\sigma (b_r \sigma^r a_{ki} - \sigma_i b_k).$$

Therefore by theorems (1.2) and (3.1), F^{*n} is also a Berwald space if and only if $b_r \sigma^r a_{ki} - \sigma_i b_k = 0$, which gives $\sigma_i = 0$, i.e. the conformal transformation is homothetic.

Theorem 3.3. For $c_1 = c_3 = 0$, the generalized Randers-Berwald space is conformally transformed to the generalized Randers-Berwald space if and only if the transformation is homothetic.

Now let us suppose that the generalized Randers space F^n with $c_1 \neq 0$, $c_3 \neq 0$, is locally Minkowski space. Then in view of theorem (1.3) we have $R_{hjk}^i = 0$ and $\nabla_k b_i = 0$. For conformally transformed generalized Randers space F_n^* , from (3.4) we have

$$\nabla_k^* b_i^* = e^\sigma (b_r \sigma^r a_{ik} - \sigma_i b_k) \quad (3.13)$$

and [1]

$$R_{hjk}^{*i} = R_{hjk}^i + \delta_k^i \sigma_{hj} - \delta_j^i \sigma_{hk} + a^{ir} (a_{hj} \sigma_{rk} - a_{hk} \sigma_{rj}) + (\delta_k^i a_{hj} - \delta_j^i a_{hk}) \Delta_1 \sigma, \quad (3.14)$$

where

$$\sigma_{ij} = \nabla_i \sigma_j - \sigma_i \sigma_j, \quad \Delta_1 \sigma = a^{ij} \sigma_i \sigma_j. \quad (3.15)$$

From (3.13) and (3.14) it follows that F^{*n} is also locally Minkowski if and only if $\sigma_i = 0$ i.e. the conformal transformation is homothetic and in this case (3.14) gives $R_{hjk}^{*i} = 0$. Hence we have the following:

Theorem 3.4. If a generalized Randers space F^n with $c_1 \neq 0$, $c_3 \neq 0$, is locally Minkowski, its conformally transformed space F^{*n} is also locally Minkowski if and only if the transformation is homothetic.

If F^n is projectively flat then from theorem (1.4) we have $\nabla_k b_i = 0$ and associated Riemannian space is projectively flat.

The associated Riemannian space is projectively flat if and only if $\gamma_{00}^i = v_0 y^i$ [3], where '0' denote the contraction with y^i .

From (3.3) and (3.12) it follows that

$$\gamma_{00}^{*i} - v_0^* y^i = (\gamma_{00}^i - v_0 y^i) + 2\sigma_0 y^i - \alpha^2 \sigma^i. \quad (3.16)$$

Thus if F^n is projectively flat then F^{*n} is also projectively flat if and only if $\sigma_i = 0$ i.e. the conformal transformation is homothetic.

Theorem 3.5. If a generalized Randers space F^n is projectively flat then its conformally transformed space F^{*n} is also projectively flat if and only if the transformation is homothetic.

Conclusion. For a non-homothetic conformal transformation of generalized Randers space

- (i) a Berwald space is not conformal to a Berwald space,

- (ii) a locally Minkowski space is not conformal to a locally Minkowski space,
- (iii) a projectively flat space is not conformal to a projectively flat space.

REFERENCES

- [1] Eisenhart, L. P. : Riemannian Geometry, Princeton University Press, (1925).
- [2] Matsumoto, M. : On C-reducible Finsler spaces, Tensor, N. S. 24 (1972), 29-37.
- [3] Matsumoto, M. : Theory of Finsler spaces with (α, β) -metric, Reports on Mathematical Physics, 31 (1972), 43-83.
- [4] Matsumoto, M. : On Finsler spaces with Randers metric and special forms of important tensors, J. Math. Kyoto Univ., 14-3 (1974), 477-498.
- [5] Matsumoto, M. and Hojo, S. : A conclusive theorem on C-reducible Finsler space, Tensor, N. S., 32 (1978), 225-230.
- [6] Park, H. S. and Choi, E. S. : On a Finsler space with a special (α, β) -metric, Tensor, N. S., 56 (1995), 142-148.
- [7] Prasad, B. N. and Singh, J. N. : On C3-like Finsler spaces, Indian J. Pure Appl. Math., 19-5 (1988), 423-428.
- [8] Shibata, C. : On Finsler spaces with Kropina metric, Reports on Mathematical Physics, 13 (1978), 117-128.