J. T. S.
Vol. 6 No.1 (2012), pp.27-33
https://doi.org/10.56424/jts.v6i01.10463

The Study of Locally Dually Flat Finsler Space with Special $(\alpha, \beta)-$metric

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(Received: 25 April, 2011)

(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In this paper, we study the locally dually flatness of the special (α, β)-metric L satisfying $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\,\alpha\beta + c_3\beta^2$, where c_i are constants, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a differential 1-form.

Key Words : Finsler space, $(\alpha, \beta)-$metric, Special $(\alpha, \beta)-$metric, Locally dually flatness.
2010 AMS Subject Classification : 53B40, 53C60.

1. Introduction

The concept of the (α, β)-metric was introduced in 1972 by M. Matsumoto and has been studied by M. Hashiguchi, Y. Ichijyo, S. Kikuchi, C. Shibata and others ([2]-[6] and [8, 9]). The well known examples of the (α, β)-metric are the Randers metric, Kropina metric and the Matsumoto metric.

Locally dually flat Finsler metrics are studied in information geometry and the notion of dually flat Finsler metrics was first introduced by S. I. Amari and H. Nagaoka [1] when they studied the information geometry on Riemannian spaces. Z. Shen [7] extended the notion of dually flatness to Finsler metrics. The authors Xinyue Cheng, Zhongmin Shen and Yusheng Zhou [11] studied on locally dually flat Randers metrics and the same team of authors [12] studied on a class of locally dually flat Finsler metrics. Xinyue Cheng and Yanfang Tian [10] studied locally dually flat Finsler metrics with special curvature properties.

A Finsler metric F is said to be locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j}, \tag{1.1}$$

where $H = H(x, y)$ is a local scalar function on the tangent bundle TM of M. Such a coordinate system is called an adapted coordinate system.

2. Preliminaries

Definition 2.1. A Finsler metric on a manifold M is a C^∞ function $L : TM\backslash\{0\} \to [0, \infty)$ satisfying the following conditions:

(1) Regularity: L is C^∞ on $TM\backslash\{0\}$.
(2) Positive homogeneity: $L(x, \lambda y) = \lambda L(x, y), \ \ \lambda > 0$.
(3) Strong convexity: The fundamental tensor $g_{ij}(x, y)$ is positive definite for all $(x, y) \in TM\backslash\{0\}$, where $g_{ij} = \frac{1}{2}[L^2]_{y^i y^j}(x, y)$.

Definition 2.2. The fundamental function L of a Finsler space $F^n = (M, D, L)$ is called an (α, β)-metric, if L is a positively homogeneous function of degree one in two arguments $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$, where α is a Riemannian metric and β is a differential 1-form. The space $R^n = (M, \alpha)$ is called the associated Riemannian space with F^n.

Lemma 2.1. ([7]) A Finsler metric $L = L(x, y)$ on an open subset $\mho \subset R^n$ is dually flat if and only if it satisfies the following conditions:

$$[L^2]_{x^k y^l} y^k - 2[L^2]_{x^l} = 0. \tag{2.1}$$

In this case, $H = H(x, y)$ in (1.1) is given by $H = -\frac{1}{6}[L^2]_{x^m} y^m$.

3. Locally Dually Flat Special $(\alpha, \beta)-$Metric

In this section, we study the locally dually flat special (α, β)-metric L satisfying $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constants, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a differential 1-form. We prove the following theorem.

Theorem 3.1. Let L be a special (α, β)-metric on a manifold M satisfying $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constants, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a differential 1-form. The special (α, β)-metric L is locally dually flat if and only if in an adapted coordinate system, the following conditions are satisfied by α and β.

$$r_{00} = \frac{2}{3}\theta\beta - \frac{5}{3}\tau c_2 c_3 \beta^2 + \left[\tau c_1 c_2 + \frac{2}{3}\left(\tau c_2 c_3 b^2 - b_m \theta^m\right)\right]\alpha^2, \tag{3.1}$$

$$s_{k0} = \frac{\beta\theta_k - \theta b_k}{3}, \tag{3.2}$$

$$G^m_\alpha = \frac{1}{3}\left(2\theta + \tau c_2 c_3 \beta\right) y^m - \frac{1}{3}\left(\tau c_2 c_3 b^m - \theta^m\right)\alpha^2, \tag{3.3}$$

where $\tau = \tau(x)$ is a scalar function and $\theta = \theta_k y^k$ is a 1-form on M and $\theta^m = a^{im}\theta_i$.

Proof. First, we prove the necessary condition. Assume that the special (α, β)-metric L satisfying $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constants, is locally dually flat on an open subset $\mho \subset R^n$. We have the following identities:

$$\alpha_{x^k} = \frac{y_m}{\alpha}\frac{\partial G^m_\alpha}{\partial y^k}, \quad \beta_{x^k} = b_{m|k} y^m + b_m \frac{\partial G^m_\alpha}{\partial y^k}, \quad s_{y^k} = \frac{\alpha b_k - s y_k}{\alpha^2}, \tag{3.4}$$

where $s = \beta/\alpha$ and $y_k = a_{jk}y^j$. By direct computation, we obtain

$$\begin{aligned} [L^2]_{x^k} &= \left(2c_1 y_m + 2c_2\alpha b_m + 2c_2 s y_m + 2c_3\beta b_m\right)\frac{\partial G^m_\alpha}{\partial y^k} \\ &+ \left(2c_2\alpha + 2c_3\beta\right) b_{m|k} y^m, \end{aligned} \tag{3.5}$$

$$\begin{aligned} [L^2]_{x^l y^k} y^l &= 2\left[2c_1 a_{mk} + \frac{2c_2 b_m y_k}{\alpha} + \frac{2c_2\beta a_{mk}}{\alpha} + 2c_2 y_m\left(\frac{\alpha b_k - s y_k}{\alpha^2}\right)\right. \\ &+ \left. 2c_3 b_m b_k\right] G^m_\alpha + \left(2c_1 y_m + 2c_2\alpha b_m + \frac{2c_2\beta}{\alpha} y_m + 2c_3\beta b_m\right)\frac{\partial G^m_\alpha}{\partial y^k} \\ &+ \left(2c_2\alpha + 2c_3\beta\right) b_{k|0} + \left(\frac{2c_2 y_k}{\alpha} + 2c_3 b_k\right) r_{00}. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (2.1), we obtain

$$\frac{2}{\alpha^3} A_1 G^m_\alpha - B_1 \frac{\partial G^m_\alpha}{\partial y^k} + C_1 r_{00} + D_1 (b_{k|0} - 2b_{0|k}) = 0, \tag{3.7}$$

where,

$A_1 = 2c_1 a_{mk}\alpha^3 + 2c_2 b_m y_k \alpha^2 + 2c_2 a_{mk}\alpha^2\beta + 2c_2 y_m(\alpha^2 b_k - \beta y_k) + 2c_3 b_m b_k \alpha^3,$

$B_1 = 2c_1 y_m + 2c_2\alpha b_m + 2c_2\dfrac{\beta}{\alpha} y_m + 2c_3\beta b_m,$

$C_1 = \dfrac{2c_2 y_k}{\alpha} + 2c_3 b_k,$

$D_1 = 2c_2\alpha + 2c_3\beta.$

Multiplying (3.7) by α^3 yields

$$\begin{aligned}&\left[2c_1a_{mk}\alpha^3+2c_2b_my_k\alpha^2+2c_2a_{mk}\alpha^2\beta+2c_2y_mb_k\alpha^2-2c_2y_my_k\beta\right.\\&\left.+2c_3b_mb_k\alpha^3\right]G^m_\alpha-\left[c_1y_m\alpha^3+c_2b_m\alpha^4+c_2y_m\alpha^2\beta+c_3b_m\alpha^3\beta\right]\frac{\partial G^m_\alpha}{\partial y^k}\\&+\left[c_2y_k\alpha^2+c_3b_k\alpha^3\right]r_{00}+\left(c_2\alpha^4+c_3\alpha^3\beta\right)(3s_{k0}-r_{k0})=0.\end{aligned}\tag{3.8}$$

Rewriting (3.8) as a polynomial in α, we have

$$A_2\alpha^4+B_2\alpha^3+C_2\alpha^2-D_2=0,\tag{3.9}$$

where,

$$\begin{aligned}A_2&=c_2\left(3s_{k0}-r_{k0}\right)-c_2b_m\frac{\partial G^m_\alpha}{\partial y^k},\\B_2&=\left(2c_1a_{mk}+2c_3b_mb_k\right)G^m_\alpha-\left(c_1y_m+c_3b_m\beta\right)\frac{\partial G^m_\alpha}{\partial y^k}+c_3b_kr_{00}\\&\quad+c_3\beta\left(3s_{k0}-r_{k0}\right),\\C_2&=\left(2c_2a_{mk}\beta+2c_2b_my_k+2c_2y_mb_k\right)G^m_\alpha-\left(c_2y_m\beta\right)\frac{\partial G^m_\alpha}{\partial y^k}+c_2y_kr_{00},\\D_2&=2c_2y_my_k\beta G^m_\alpha.\end{aligned}$$

From (3.9), we know that the coefficients of α are zero. Hence the coefficients of α^3 must be zero too. That is

$$\begin{aligned}B_2&=\left(2c_1a_{mk}+2c_3b_mb_k\right)G^m_\alpha-\left(c_1y_m+c_3b_m\beta\right)\frac{\partial G^m_\alpha}{\partial y^k}\\&+c_3b_kr_{00}+c_3\beta\left(3s_{k0}-r_{k0}\right)=0.\end{aligned}\tag{3.10}$$

Thus, we have

$$A_2\alpha^4+C_2\alpha^2-D_2=0.\tag{3.11}$$

Note that

$$y_m\frac{\partial G^m_\alpha}{\partial y^k}=\frac{\partial\left(y_mG^m_\alpha\right)}{\partial y^k}-a_{mk}G^m_\alpha,\tag{3.12}$$

$$b_m\frac{\partial G^m_\alpha}{\partial y^k}=\frac{\partial\left(b_mG^m_\alpha\right)}{\partial y^k}.\tag{3.13}$$

Contracting (3.10) with b^k and by using (3.12) and (3.13), we get

$$\begin{aligned}&c_1\frac{\partial\left(y_mG^m_\alpha\right)}{\partial y^k}b^k+c_3\beta\frac{\partial\left(b_mG^m_\alpha\right)}{\partial y^k}b^k\\&=(3c_1+2c_3b^2)b_mG^m_\alpha+c_3b^2r_{00}+c_3\beta\left(3s_0-r_0\right).\end{aligned}\tag{3.14}$$

Again contracting (3.11) with b^k and by using (3.12) and (3.13), we obtain

$$c_2\beta\alpha^2\frac{\partial\left(y_mG^m_\alpha\right)}{\partial y^k}b^k+c_2\alpha^4\frac{\partial\left(b_mG^m_\alpha\right)}{\partial y^k}b^k=c_2\left(3s_0-r_0\right)\alpha^4$$
$$+\left(2c_2b^2y_mG^m_\alpha+5c_2\beta b_mG^m_\alpha+c_2\beta r_{00}\right)\alpha^2-2c_2\beta^2y_mG^m_\alpha. \quad (3.15)$$

Multiplying (3.15) by $c_3\beta$ and subtracting it from the product of (3.14) and $c_2\alpha^4$ yields

$$A_3c_2\alpha^2(c_1\alpha^2-c_3\beta^2)=c_2c_3B_3(b^2\alpha^2-\beta^2), \quad (3.16)$$

where,

$$\begin{aligned} A_3 &= \frac{\partial\left(y_mG^m_\alpha\right)}{\partial y^k}b^k-3b_mG^m_\alpha,\\ B_3 &= (2b_mG^m_\alpha\alpha^2+r_{00}\alpha^2-2\beta y_mG^m_\alpha). \end{aligned}$$

Since $(b^2\alpha^2-\beta^2)$, $(c_1\alpha^2-c_3\beta^2)$ and α^2 are all irreducible polynomials of (y^i) (provided $c_1\neq c_3$ where the case may be), we know that there is a function $\tau=\tau(x)$ on M such that

$$2b_mG^m_\alpha\alpha^2+r_{00}\alpha^2-2\beta y_mG^m_\alpha = \tau c_2\alpha^2\left(c_1\alpha^2-c_3\beta^2\right), \quad (3.17)$$
$$\frac{\partial\left(y_mG^m_\alpha\right)}{\partial y^k}b^k-3b_mG^m_\alpha = \tau c_2c_3\left(b^2\alpha^2-\beta^2\right). \quad (3.18)$$

Then equation (3.17) can be written as

$$2\beta y_mG^m_\alpha=\left(2b_mG^m_\alpha+r_{00}-\tau c_1c_2\alpha^2+\tau c_2c_3\beta^2\right)\alpha^2.$$

Since α^2 does not contain the factor β, then we have

$$y_mG^m_\alpha = \theta\alpha^2, \quad (3.19)$$
$$b_mG^m_\alpha = \beta\theta-\frac{1}{2}r_{00}+\frac{\tau c_1c_2\alpha^2}{2}-\frac{\tau c_2c_3\beta^2}{2}, \quad (3.20)$$

where $\theta=\theta_ky^k$ is a 1-form on M. Then we obtain the following

$$\frac{\partial\left(y_mG^m_\alpha\right)}{\partial y^k} = \theta_k\alpha^2+2\theta y_k, \quad (3.21)$$
$$\frac{\partial\left(b_mG^m_\alpha\right)}{\partial y^k} = \theta_k\beta+b_k\theta-r_{k0}+\tau c_1c_2y_k-\tau c_2c_3\beta b_k. \quad (3.22)$$

By using (3.20)-(3.22), the equations (3.10) and (3.11) become

$$\begin{aligned} c_3\beta\,(3s_{k0} &+ \theta b_k - \beta\theta_k) + c_1\alpha^2\,(\tau c_2 c_3 b_k - \theta_k) \\ &+ 3c_1 a_{mk} G^m_\alpha - c_1 y_k\,(2\theta + \tau c_2 c_3 \beta) = 0, \end{aligned} \tag{3.23}$$

$$\begin{aligned} [(3c_2 s_{k0} &+ c_2\theta b_k - c_2\beta\theta_k) + (c_2 c_2 c_3 \tau b_k - c_2\theta_k)\beta]\,\alpha^2 \\ &- (2c_2\theta + c_2 c_2 c_3 \tau\beta)\beta y_k + 3c_2\beta a_{mk} G^m_\alpha = 0. \end{aligned} \tag{3.24}$$

Subtracting (3.24) from the product of (3.23) and β yields

$$3s_{k0} + \theta b_k - \beta\theta_k = 0. \tag{3.25}$$

This yields the required condition (3.1) of the theorem.

Contracting (3.23) with a^{lk} yields

$$c_3\beta\left(3s^l_0 + \theta b^l - \beta\theta^l\right) + c_1\left(\tau c_2 c_3 b^l - \theta^l\right)\alpha^2 + 3c_1 G^l_\alpha - c_1\,(2\theta + \tau c_2 c_3\beta)\,y^l = 0. \tag{3.26}$$

Contracting (3.25) with a^{lk} yields $3s^l_0 + \theta b^l - \beta\theta^l = 0$. Then, from (3.26), we obtain another necessary condition (3.3) of the theorem.

Substituting (3.26) into (3.20), we obtain final required condition (3.2) of the theorem.

By assuming that the three conditions (3.1), (3.2) and (3.3) hold true, we can prove the sufficient condition by direct computation. This completes the proof of the Theorem 3.1.

References

[1] Amari, S. I. and Nagaoka, H. : Methods of Information Geometry, AMS Translation of Math. Monographs, 191, Oxford University Press, (2000).

[2] Bacso, S., Cheng, X. and Shen, Z. : Curvature properties of (α,β)-metrics, (2006).

[3] Narasimhamurthy, S. K. and Latha Kumari, G. N. : On a hypersurface of a special Finsler space with a metric $L = \alpha + \beta + \frac{\beta^2}{\alpha}$, ADJM, 9 (1) (2010), 36–44.

[4] Narasimhamurthy, S. K. and Vasantha, D. M. : Projective change between two Finsler spaces with (α,β)-metric, Kyungpook Math. J., 52 (1) (2012), 81–89.

[5] Narasimhamurthy, S. K., Aveesh, S. T., Nagaraj, H. G. and Pradeep Kumar : On special hypersurface of a Finsler space with the metric $(\alpha + \frac{\beta^{n-1}}{\alpha^n})$, Acta Universitatis Apulensis, 17 (2009).

[6] Prasad, B. N., Gupta, B. N. and Singh, D. D. : Conformal transformation in Finsler spaces with (α,β)-metric, Indian J. Pure and Appl. Math., 18 (4) (1961), 290–301.

[7] Shen, Z. : Riemann-Finsler geometry with applications to information geometry, Chin. Ann. Math., 27B (1) (2006), 73–94.

[8] Shibata, C. : On Finsler spaces with an (α, β)-metric, J. Hokkaido Univ. of Education, IIA 35 (1984), 1–6.

[9] Shimada, H. and Sabau, S. V. : Introduction to Matsumoto metric, Nonlinear Analysis, 63 (2005), e165–e168.

[10] Xinyue Cheng and Yanfang Tian : Locally dually flat Finsler metrics with special curvature properties, Differential Geometry and its Applications, (2011).

[11] Xinyue Cheng, Zhongmin Shen and Yusheng Zhou : On locally dually flat Randers metrics, (2008).

[12] Xinyue Cheng, Zhongmin Shen and Yusheng Zhou : On a class of locally dually flat Finsler metrics, (2009).