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On Certain Subclasses of Meromorphic Functions with Positive Coefficients involving Generalized Liu-Srivastava Operator

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

In this paper, making use of the generalized Liu-Srivastava operator, we introduce and study a new subclass $\Sigma_P(\alpha, \beta, \gamma, \lambda, k)$ of meromorphically univalent functions. We obtain the coefficient estimates, distortion bounds, extreme points, radii of meromorphically starlikeness and convexity for functions in the class $\Sigma_P(\alpha, \beta, \gamma, \lambda, k)$. Furthermore we obtain the partial sums for the same.

Keywords and Phrases : Meromorphic functions, Hadamard product (or convolution), meromorphic starlike functions, meromorphic convex functions, Liu-Srivastava operator.
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1. Introduction

Let Σ denote the class of functions of the form

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$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $\Sigma_{\mathcal{S}}$, $\Sigma^*(\gamma)$ and $\Sigma_K(\gamma)$, $(0 \leq \gamma < 1)$ denote the subclasses of Σ that are meromorphically univalent, meromorphically starlike functions of order γ and meromophically convex functions of order γ respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if, f is of the form (1.1) and satisfies

$$-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

similarly, $f \in \Sigma_K(\gamma)$, if and only if, f is of the form (1.1) and satisfies

$$-\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas [2], Aouf [4], Mogra et al. [17], Uralegadi et al. [21, 22, 23] and others (see [3, 6, 18, 19]).

Let Σ_P be the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \;\; a_n \geq 0, \tag{1.2}$$

that are analytic and univalent in $\mathbb{U}^*$. For functions $f \in \Sigma$ given by (1.1) and $g \in \Sigma$ given

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \tag{1.3}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.4}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \;\; &:= \;\; \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\; l, m \in \mathbb{N}_0 \;\; &:= \;\; \mathbb{N} \cup \{0\}; z \in U), \end{aligned} \tag{1.5}$$

where N denotes the set of all positive integers and $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1, & n=0; \theta \in \mathbb{C} \setminus \{0\} \\ \theta(\theta+1)(\theta+2)\dots(\theta+n-1), & n \in \mathbb{N}; \theta \in C \ . \end{cases} \tag{1.6}$$

Corresponding to a function ${}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z)$ defined by

$$\mathcal{F}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z) := z^{-1}{}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z),$$

Liu and Srivastava [15] (see also [16]) considered a linear operator $\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) : \Sigma \to \Sigma$ defined by the following Hadamard product (or convolution):

$$\begin{aligned} \mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)f(z) &= \mathcal{F}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ &= z^{-1} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_{n+1} \dots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \dots (\beta_m)_{n+1}} \frac{a_n\ z^n}{(n+1)!}, \end{aligned} \tag{1.7}$$

where, $\alpha_i > 0, (i = 1, 2, ...l), \beta_j > 0, (j = 1, 2, ...m), l \leq m+1;\ l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For notational simplicity, we use a shorter notations $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)$, in the sequel. We note that the linear operator $\mathcal{H}_m^l[\alpha_1]$ was motivated essentially by Dziok and Srivastava [9].

Next, we define the linear operator $\mathcal{D}_{\lambda,k}^{l,m} : \Sigma \to \Sigma$ by

$$\mathcal{D}_{\lambda,0}^{l,m} f(z) = f(z),$$

$$\mathcal{D}_{\lambda,1}^{l,m} f(z) = (1-\lambda)\mathcal{H}_m^l[\alpha_1]f(z) + \frac{\lambda}{z}(z^2\mathcal{H}_m^l[\alpha_1]f(z))' = \mathcal{D}_{\lambda}^{l,m} f(z), (\lambda \geq 0).$$

and (in general),

$$\begin{aligned} \mathcal{D}_{\lambda,k}^{l,m} f(z) &= \mathcal{D}_{\lambda}^{l,m}(\mathcal{D}_{\lambda,k-1}^{l,m} f(z)) \\ \mathcal{D}_{\lambda,k}^{l,m} f(z) &:= \frac{1}{z} + \sum_{n=1}^{\infty} \Gamma_n(\alpha_1, k, \lambda)\ a_n\ z^n, \end{aligned} \tag{1.8}$$

where,

$$\Gamma_n(\alpha_1, k, \lambda) = \left(\frac{(\alpha_1)_{n+1} \dots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \dots (\beta_m)_{n+1}} \frac{[1+\lambda(n-1)]}{(n+1)!} \right)^k, \ (k \in \mathbb{N}_0,\ \lambda > 0). \tag{1.9}$$

We note that, for $k = 1$ and $\lambda = 0$ the operator $\mathcal{D}_{0,1}^{l,m} f(z) = H_m^l[\alpha_1]f(z)$ which was investigated by Liu and Srivastava [15], Aouf [5] (see also [7]), for $l = 2$, $m = 1$, $\alpha_2 = 1$, $\lambda = 0$ and $k = 1$ the operator $\mathcal{D}_{0,1}^{2,1}[\alpha_1, 1; \beta_1]f(z) = \mathcal{L}[\alpha_1; \beta_1]f(z)$ was introduced and studied by Liu and Srivastava [14] (see also [1],

[11] and [25]). Further,we remark in passing that this operator $\mathcal{L}[\alpha_1;\beta_1]$ is closely related to the Carlson-Shaffer operator $\mathcal{L}[\alpha_1;\beta_1]$ defined on the space of analytic and univalent functions in $\mathbb{U}$. For $l=2$, $m=1$, $\alpha_1=\delta+1$ $\beta_1=\alpha_2=1$, $\lambda=0$ and $k=1$, the operator $\mathcal{D}^{2,1}_{0,1}[\delta+1,1;1]f(z)=\mathcal{D}^{\delta}f(z)=\frac{1}{z(1-z)^{\delta+1}}*f(z)\,(\delta>-1)$, where $\mathcal{D}^{\delta}$ is the differential operator which was introduced by Ganigi and Uralegadi [10] (see also [7]) and then it was generalized by Yang [24].

Now by making use of the operator $\mathcal{D}^{l,m}_{\lambda,k}$, we define a new subclass of functions in Σ_P as follows.

Definition 1.1. For $0\le\alpha<1$ and $0<\beta\le1$, $\frac{1}{2}<\gamma\le1$ if $\alpha=0$, $\frac{1}{2}<\gamma\le\frac{1}{2\alpha}$ if $\alpha\neq0$, let $\Sigma(\alpha,\beta,\gamma,\lambda,k)$ denote a subclass of Σ consisting functions of the form (1.1) satisfying the condition that

$$\left|\frac{\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)}+1}{2\gamma\left[\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)}+\alpha\right]-\left[\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)}+1\right]}\right|<\beta,\quad z\in\mathbb{U}^*,\tag{1.10}$$

where $\mathcal{D}^{l,m}_{\lambda,k}f(z)$ is given by (1.8). Furthermore, we say that a function $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, whenever $f(z)$ is of the form (1.2).

We observe that by specializing the parameters involved in the operator and in the class, we obtain the classes studied by Aouf [3], Kulkarni and Joshi [12], Mogra et al., [17] and others. The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of functions belonging to the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Further we determine the radius of starlikeness and convexity for the functions in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. We also obtain the partial sums for the aforementioned class.

2. Coefficient Inequalities

In this section we assume that $0\le\alpha<1$ and $0<\beta\le1$, $\frac{1}{2}<\gamma\le1$ if $\alpha=0$, $\frac{1}{2}<\gamma\le\frac{1}{2\alpha}$ if $\alpha\neq0$, and $\Gamma_n(\alpha_1,k,\lambda)$ is given by (1.9). Our first result for the functions $f\in\Sigma$ is contained in the following theorem.

Theorem 2.1. Let $f(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}a_nz^n$ be analytic in U^*. If

$$\sum_{n=1}^{\infty}[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)\;|a_n|\le2\beta\gamma(1-\alpha),\tag{2.1}$$

then $f\in\Sigma(\alpha,\beta,\gamma,\lambda,k)$.

Proof. Suppose (2.1) holds, for all admissible values of α, β and γ. Then we have

$$\left|z(\mathcal{D}^{l,m}_{\lambda,k}f(z))' + \mathcal{D}^{l,m}_{\lambda,k}f(z)\right| - \beta\left|(2\gamma-1)z(\mathcal{D}^{l,m}_{\lambda,k}f(z))' + (2\alpha\gamma-1)\mathcal{D}^{l,m}_{\lambda,k}f(z)\right|$$
$$= \left|\sum_{n=1}^{\infty}(n+1)\Gamma_n(\alpha_1,k,\lambda)a_nz^n\right| - \beta\left|2\gamma(\alpha-1)\frac{1}{z} + \sum_{n=1}^{\infty}(2\gamma-1)n\Gamma_n(\alpha_1,k,\lambda)a_nz^n\right.$$
$$\left. + \sum_{n=1}^{\infty}(2\alpha\gamma-1)\Gamma_n(\alpha_1,k,\lambda)a_nz^n\right|$$
$$\leq \sum_{n=1}^{\infty}[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)\ |a_n|z^{n+1} - 2\beta\gamma(1-\alpha).$$

Since the above inequality holds for all $r = |z|$, $0 < r < 1$, letting $r \to 1$, we have,

$$\sum_{n=1}^{\infty}[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)\ |a_n| \leq 2\beta\gamma(1-\alpha)$$

by (2.1). Hence it follows that $f \in \Sigma(\alpha,\beta,\gamma,\lambda,k)$.

In the next theorem, we obtain the necessary and sufficient condition for a function f to be in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Theorem 2.2. Let $f \in \Sigma_P$ be given by (1.2). Then $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ if and only if

$$\sum_{n=1}^{\infty}[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)\ a_n \leq 2\beta\gamma(1-\alpha), \qquad (2.2)$$

where $\Gamma_n(\alpha_1,k,\lambda)$ is given by (1.9).

Proof. By previous theorem, it is sufficient to show the only if part. Let us assume that the function $f(z)$ is of the form (1.2) is in $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Then

$$\left|\frac{\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)} + 1}{2\gamma\left[\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)} + \alpha\right] - \left[\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)} + 1\right]}\right|$$

$$= \left| \frac{\sum_{n=1}^{\infty}(n+1)\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1}}{2\gamma(1-\alpha) - \sum_{n=1}^{\infty}(1-2\gamma)n\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1} - \sum_{n=1}^{\infty}(1-2\alpha\gamma)\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1}} \right|$$

$$< \beta, \ z \in \mathbb{U}^*. \tag{2.3}$$

Using fact that $|\Re(z)| \leq |z|$ for any z, it follows that

$$\Re\left(\frac{\sum_{n=1}^{\infty}(n+1)\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1}}{2\gamma(1-\alpha) - \sum_{n=1}^{\infty}(1-2\gamma)n\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1} - \sum_{n=1}^{\infty}(1-2\alpha\gamma)\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1}} \right) < \beta, \ z \in \mathbb{U}^*. \tag{2.4}$$

Now choose the value of z on real axis so that $\frac{z(\mathcal{D}^{l,m}_{\lambda,k}f(z))'}{\mathcal{D}^{l,m}_{\lambda,k}f(z)}$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1$ through positive values, we obtain

$$\sum_{n=1}^{\infty}[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)\ a_n \leq 2\beta\gamma(1-\alpha).$$

Hence the result follows.

The coefficient estimate for functions in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ is stated in the following corollary.

Corollary 2.3. If $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ then

$$a_n \leq \frac{2\beta\gamma(1-\alpha)}{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}. \tag{2.5}$$

The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}, \ n \geq 1. \tag{2.6}$$

Next we obtain the distortion and growth results for the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ from the following theorem.

Theorem 2.4. If $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, then

$$\frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} r \leq |f(z)| \leq \frac{1}{r} + \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} r \quad (|z|=r)$$

and

$$\frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} \quad (|z|=r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} z. \tag{2.7}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n. \tag{2.8}$$

Given that $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, from the equation (2.2), we have

$$\begin{aligned}[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)\sum_{n=1}^{\infty} a_n &\leq \sum_{n=1}^{\infty}[(n+1)+\beta((1-n)+2\gamma(n-\alpha))] \times \\ &\quad \Gamma_n(\alpha_1,k,\lambda)\ a_n \\ &\leq 2\beta\gamma(1-\alpha).\end{aligned}$$

That is,

$$\sum_{n=1}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)}.$$

Using the above equation in (2.8), we have

$$|f(z)| \leq \frac{1}{r} + \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} r$$

and

$$|f(z)| \geq \frac{1}{r} - \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)} z$. Similarly we have,

$$|f'(z)| \geq \frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)}$$

and

$$|f'(z)| \leq \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{[2+2\beta\gamma(1-\alpha)]\Gamma_1(\alpha_1,k,\lambda)}.$$

Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ be given by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \ n \in \mathbb{N}, n \geq 1. \tag{2.9}$$

We state the following closure theorem for the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ without proof.

Theorem 2.5. Let the function $f_j(z)$ defined by (2.9) be in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ for every $j = 1, 2, ..., m$. Then the function $f(z)$ defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

belongs to the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, where $a_n = \frac{1}{m}\sum_{j=1}^{m} a_{n,j}$, $(n = 1, 2, ..)$.

Theorem 2.6. (Extreme Points) Let

$$\begin{aligned} f_0(z) &= \frac{1}{z} \text{ and} \\ f_n(z) &= \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)} z^n, \ (n \geq 1). \end{aligned} \tag{2.10}$$

Then $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), \ \ (\mu_n \geq 0, \sum_{n=0}^{\infty} \mu_n = 1). \tag{2.11}$$

Proof. Suppose $f(z)$ can be expressed as in (2.11). Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \mu_n f_n(z) \\ &= \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \mu_n \frac{2\beta\gamma(1-\alpha)}{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)} z^n. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\mu_n\frac{2\beta\gamma(1-\alpha)}{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}\times$$
$$\frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}z^n$$
$$=\sum_{n=1}^{\infty}\mu_n-1=1-\mu_0\le 1.$$

So by Theorem 2.2, $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Conversely, suppose $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Since

$$a_n\le\frac{2\beta\gamma(1-\alpha)}{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)},\quad n\ge 1.$$

We set,

$$\mu_n=\frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}a_n,\quad n\ge 1$$

and $\mu_0=1-\sum\limits_{n=1}^{\infty}\mu_n$. Then we have, $f(z)=\sum\limits_{n=0}^{\infty}\mu_nf_n(z)=\mu_0f_0(z)+\sum\limits_{n=1}^{\infty}\mu_nf_n(z)$. Hence the results follows.

3. Radii of meromorphically starlikeness and meromorphically convexity

In this section, we obtain the radii of stralikeness and convexity of order δ for functions in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Theorem 3.1. Let $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Then f is meromorphically starlike of order δ $(0\le\delta<1)$ in the disc $|z|<r_1$, where

$$r_1=\inf_n\left[\left(\frac{1-\delta}{n+2-\delta}\right)\frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}\right]^{\frac{1}{n+1}},\quad (n\ge 1).$$

The result is sharp for the extremal function $f(z)$ given by (2.10).

Proof. The function $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ of the form (1.1) is meromorphically starlike of order δ in the disc $|z|<r_1$, if and only if it satisfies the condition

$$\left|\frac{zf'(z)}{f(z)}+1\right|<1-\delta.\tag{3.1}$$

Since

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq \left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}}{1+\sum_{n=1}^{\infty}a_n z^{n+1}}\right| \leq \frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{1-\sum_{n=1}^{\infty}|a_n||z|^{n+1}}.$$

The above expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n+2-\delta}{1-\delta}|a_n|\ |z|^{n-1} < 1.$$

Using the fact, that $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}a_n \ < 1.$$

We say (3.1) is true if

$$\frac{n+2-\delta}{1-\delta}|z|^{n+1} < \frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)}\frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}$$

which yields the starlikeness of the family.

Theorem 3.2. Let $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_2$, where

$$r_2 = \inf_n \left[\left(\frac{1-\delta}{n(n+2-\delta)}\right)\frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)}\right]^{\frac{1}{n+1}}, \quad (n \geq 1).$$

The result is sharp for the extremal function $f(z)$ given by (2.7).

Proof. The proof is analogous to that of Theorem 3.1, and we omit the details.

4. Partial Sums

Let $f \in \Sigma_P$ be a function of the form (1.1). Motivated by Silverman [20], Cho and Owa [8], Latha and Shivarudrappa [13], we define the partial sums $f_m(z)$ defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n \quad (m \in \mathbb{N}). \tag{4.1}$$

In this section, we consider partial sums of functions from the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 4.1. Let $f \in \Sigma_P(\alpha, \beta, \gamma, \lambda, k)$ be given by (1.2) and define the partial sums $f_1(z)$ and $f_m(z)$, by

$$f_1(z) = \frac{1}{z} \text{ and } f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n, \ (m \in \mathbb{N}/\{1\}). \tag{4.2}$$

Suppose also that

$$\sum_{n=1}^{\infty} d_n a_n \leq 1,$$

where

$$d_n \geq \begin{cases} 1 & \text{for } n = 1, 2, 3, \ldots, m \\ \frac{[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)} & \text{for } n = m+1, m+2, m+3 \cdots \end{cases} \tag{4.3}$$

Then $f \in \Sigma_P(\alpha, \beta, \gamma, \lambda, k)$. Furthermore,

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{4.4}$$

and

$$Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{4.5}$$

Proof. For the coefficients d_n given by (4.3) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{4.6}$$

Therefore we have

$$\sum_{n=1}^{m} a_n + d_{m+1} \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1 \tag{4.7}$$

by using the hypothesis (4.3). By setting

$$\begin{aligned} g_1(z) &= d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right) \\ &= 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=1}^{m} a_n z^{n-1}}, \end{aligned}$$

then it suffices to show that

$$\text{Re } (g_1(z)) \geq 0 \ \ (z \in \mathbb{U}^*)$$

or,

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1 \;\; (z \in \mathbb{U}^*)$$

and applying (4.7), we find that

$$\begin{aligned}\left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty} a_n}{2-2\sum\limits_{n=1}^{m} a_n - d_{m+1}\sum\limits_{n=m+1}^{\infty} a_n} \\ &\leq 1, \;\; z \in \mathbb{U}^*,\end{aligned}$$

which readily yields the assertion (4.4) of Theorem 4.1. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{4.8}$$

gives sharp result, we observe that for $z = re^{i\pi/m}$ that $\frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}}$ as $r \to 1^-$.

Similarly, if we take

$$g_2(z) \;=\; (1+d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right)$$

and making use of (4.7), we deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty} a_n}{2-2\sum\limits_{n=1}^{m} a_n - (1-d_{m+1})\sum\limits_{n=m+1}^{\infty} a_n}$$

which leads us immediately to the assertion (4.5) of Theorem 4.1. The bound in (4.5) is sharp for each $m \in \mathbb{N}$ with the extremal function $f(z)$ given by (4.8).

Theorem 4.2. If $f(z)$ of the form (1.2) satisfies the condition (2.2). Then

$$Re\left(\frac{f'(z)}{f_m'(z)}\right) \geq 1 - \frac{m+1}{d_{m+1}}$$

and

$$Re\left(\frac{f_m'(z)}{f'(z)}\right) \geq \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geq \begin{cases} n & \text{for } n = 2,3,\ldots,m \\ \frac{n[(n+1)+\beta((1-n)+2\gamma(n-\alpha))]\Gamma_n(\alpha_1,k,\lambda)}{2\beta\gamma(1-\alpha)} & \text{for } n = m+1, m+2, m+3\cdots \end{cases}.$$

The bounds are sharp, with the extremal function $f(z)$ of the form (2.7).

Proof. The proof is analogous to that of Theorem 4.1, and we omit the details.

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