J. T. S.
Vol. 7 (2013), pp.9-16
https://doi.org/10.56424/jts.v7i01.10471

Lorentzian $\alpha-$Sasakian Manifolds Satisfying Certain Condition on the Concircular Curvature Tensor

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(Received: November 27, 2012)

Abstract

In this paper we have shown that Lorentzian $\alpha-$Sasakian manifold are Einstein manifold if they satisfy the condition $R(X,Y).S = 0$, $C(\xi, X).S = 0$, $C(\xi, X).C = 0$, $C(\xi, X).R = 0$, $R.C = R.R$ and $\phi^2((D_X Q)(Y)) = 0$.

Keywords and Phrases : Lorentzian $\alpha-$Sasakian manifold, Concircular curvature tensor, Einstien manifold.
2000 AMS Subject Classification : 53C25.

1. Introduction

The product of an almost contact manifold M and the real R carries a natural almost complex structure. However, if one takes M to be an almost contract metric manifold and suppose that the product metric G on $M \times R$ is Käehlerian, the structure on M is cosymplectic [3] and not Sasakian. Tanno S. [7] classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ, say 'c'. They showed that they can be divided into three classes:

(i) homogeneous normal contact Riemannian manifold with $c > 0$.
(ii) global Riemannian product of a line or a circle with a Käehler manifold of constant holomorphic sectional curvature if $c = 0$.
(iii) a warped product space if $c < 0$.

It is known that the manifold of class (i) are characterized by admitting a Sasakian structure.

In the Gray-Hervella classification of almost Hermitian manifolds [3], there appears a class, W_4 of Hermitian manifolds which are closely related to local conformal Käehler manifolds. An almost contact metric structure on a manifold M is called trans-Sasakian structure [5]. If the product manifold $M \times R$ belongs to the W_4. The class $C_6 \otimes C_5$ coincide with the class of the trans-Sasakian structure of type (α, β).

We note that trans-Sasakian structure of the type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic, $\beta-$Kenmotsu and $\alpha-$Sasakian repectivily. Yildiz and Murathan [10, 11] introduced Lorentzian $\alpha-$Sasakian manifolds. Many other author De and Tripathi, De and Sarakar, De and Shaikh, Prakasha, Bagewadi and Basavarajappa [1, 6, 7, 10, 11] studied and obtain interesting results.

A $(1,3)-$type of tensor $C(X,Y)\,Z$ which remains invariant under concircular transformation for $n-$dimensional Riemanniam manifold is given by Yano and Kon

$$C(X,Y)\,Z = R(X,Y)\,Z - \frac{r}{n(n-1)}[g(Y,Z)\,X - g(X,Z)\,Y],$$

where R is the Riemannian curvature tensor, 'r' is the scalar curvature tensor.

This paper is organized as follows. After introduction, we give a brief account of Lorentzian $\alpha-$Sasakian manifolds. In section 3, we study Lorentzian $\alpha-$Sasakian manifolds satisfying the condition $(X,Y).S = 0$, $C(\xi,X).S = 0$, $C(\xi,X).C = 0$, $C(\xi,X).R = 0$, $R.C = R.R$ and $\phi^2(D_X Q)(Y)) = 0$ are Einstein manifold.

2. Preliminaries

A differentiable manifold M of dimension $(2n+1)$ is called a Lorentzian $\alpha-$Sasakian manifold if it admits a tensor field ϕ of type $(1,1)$, a contravarint vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy

$$\phi^2 = I + \eta \otimes \xi \tag{2.1}$$

$$\eta(\xi) = -1 \tag{2.2}$$

$$g(\phi X,\, \phi Y) = g(X,Y) + \eta(X)\,\eta(Y) \tag{2.3}$$

$$g(X,\xi) = \eta(X) \tag{2.4}$$

$$\phi\xi = 0, \qquad \eta(\phi X) = 0 \tag{2.5}$$

$$(D_X\phi)Y = \alpha g(X,Y)\,\xi + \alpha\eta(Y)\,X, \tag{2.6}$$

for all $X, Y \in Tm$.

Also a Lorentzian $\alpha-$Sasakian manifold M satisfies

$$(D_X \xi) Y = \alpha \phi X \tag{2.7}$$

$$(D_X \eta) Y = \alpha g(X, \phi Y), \tag{2.8}$$

where D denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is constant.

Also on a Lorentzian $\alpha-$Sasakian manifold the following relation hold [2, 3]

$$R(X,Y)\,\xi = \alpha^2 \left(\eta(Y)X - \eta(X)Y\right) \tag{2.9}$$

$$R(\xi,X)\,Y = \alpha^2 (g(X,Y)\xi - \eta(Y)X) \tag{2.10}$$

$$R(\xi,X)\,\xi = \alpha^2 (\eta(X)\xi + X) \tag{2.11}$$

$$S(X,\xi) = 2n\alpha^2 \eta(X) \tag{2.12}$$

$$\phi\xi = 2n\alpha^2 \xi \tag{2.13}$$

$$S(\xi,\xi) = -2n\alpha^2. \tag{2.14}$$

For any vector field X, Y, Z where S is the Ricci curvature and Q is the φ Ricci operator given by

$$S(X,Y) = g(\varphi X, Y).$$

Definition 2.1. The concircular curvature Tensor C on Lorentzian $\alpha-$Sasakian manifold M of dimensional $(2n+1)$ is given by $C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]$ for any vector fields X, Y, Z where R is the curvature tensor and r is the scalar curvature.

Definition 2.2. An $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold is said to be Ricci semi-symmetric if $R(X,Y).S = 0$ where R is the curvature tensor and S is the Ricci tensor.

3. Main Results

In this section we prove the following theorem:

Theorem 3.1. Let M be an $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold then M is Ricci semi-symmetric if and only if it is an Einstein manifold.

Proof. It is well known that every Einstein manifold is Ricci semi-symmetric but converse is not true in general. Here we prove that in $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold $R(X,Y).S = 0$ implies that manifold is an Einstein manifold.

Now from definition (2.2), it follows that

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{3.1}$$

Putting $X=\xi$ in above, we get

$$S(R(\xi,Y)U,V)+S(U,R(\xi,Y)V)=0. \tag{3.2}$$

Using (2.10) and (2.13) in (3.2), we get

$$2n\alpha^4 g(Y,U)\eta(V)-\alpha^2\eta(U)S(Y,V)+2n\alpha^4 g(Y,V)\eta(U)-\alpha^2\eta(V)S(U,Y)=0. \tag{3.3}$$

Putting $U=\xi$ in (3.3) and using (2.4), (2.12), we get

$$2n\alpha^4\eta(Y)\eta(V)+\alpha^2 S(Y,V)-2n\alpha^4 g(Y,V)-2n\alpha^4\eta(Y)\eta(V)=0$$

which implies

$$S(Y,V)=2n\alpha^2 g(Y,V) \tag{3.4}$$

Therefore, M is Einstein manifold. This completes the proof of the theorem.

Theorem 3.2. Let M be an $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold then M satisfies the condition $C(\xi,X).S=0$ if and only if either M is Einstein manifold or M has scalar curvature $r=\alpha^2 2n\,(2n+1)$.

Proof. Since $C(\xi,X).S=0$, then we have $C(\xi,X).S(Y,\xi)=0$ which implies

$$S(C(\xi,X)Y,\xi)+S(Y,C(\xi,X)\xi)=0 \tag{3.5}$$

Using (2.12) and definition (2.2), in (3.5), we have

$$S\left(\left(\alpha^2-\frac{r}{2n(2n+1}\right)[g(Y,X)\xi-\eta(Y)X],\,\xi\right)$$
$$+S\left(Y,\left(\alpha^2-\frac{r}{2n(2n+1)}[\eta(X)\xi+X]\right)\right)=0,$$

which implies

$$\left(\alpha^2-\frac{r}{2n(2n+1)}\right)[g(X,Y)S(\xi,\xi)-\eta(Y)S(X,\xi)+\eta(X)S(Y,\xi)+S(Y,X)]=0. \tag{3.6}$$

Using (2.12) and (2.14) in (3.5), we have

$$\left(\alpha^2-\frac{r}{2n(2n+1)}\right)[-2n\alpha^2 g(X,Y)+S(X,Y)]=0. \tag{3.7}$$

This implies $S(X,Y)=2n\alpha^2 g(X,Y)$. Therefore M is an Einstein manifold with scalar curvature $r=\alpha^2 2n(2n+1)$. Converse is trivial. Therefore proof of the theorem is complete.

Theorem 3.3. An $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold M satisfies

$$C(\xi, X).C = 0.$$

If and only if either the scalar curvature r of M is $r = \alpha^2 2n(2n+1)$ or M is locally isometric to the Hyperbolic sphere $H^{2n+1}\alpha^2$.

Proof. In an $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold M, we have

$$C(\xi, X)Y = \left(\alpha^2 - \frac{r}{2n(2n+1)}\right)\{g(X,Y)\xi - \eta(X)Y\}, \tag{3.8}$$

$$C(X,Y)\xi = \left(\alpha^2 - \frac{r}{2n(2n+1)}\right)\{\eta(Y)X - \eta(X)Y\}. \tag{3.9}$$

The condition $C(\xi, X).C = 0$ implies that

$$C(\xi, U)C(X,Y)\,\xi - C(C(\xi,U)X, Y)\,\xi - C(X, C(\xi,U)Y)\,\xi = 0.$$

Then in view of (3.9), we get

$$\left(\alpha^2 - \frac{r}{2n(2n+1)}\right) \times [g(U, C(X,Y)\xi)\xi - C(X,Y)\,\xi\eta(U) - g(U,X)C(\xi,Y)\xi$$
$$+\eta(X)C(U,Y)\xi - g(U,Y)C(X,\xi)\xi + \eta(Y)C(X,U)\xi - C(X,Y)U] = 0.$$

Using (3.8) in above, we get

$$\left(\alpha^2 - \frac{r}{2n(2n+1)}\right) \times \left[C(X,Y)U - \left(\alpha^2 - \frac{r}{2n(2n+1)}\right)\{g(U,Y)X - g(U,X)Y\}\right],$$

which implies the scalar curvature $r = \alpha^2 2n(2n+1)$ or

$$\left[C(X,Y)U - \left(\alpha^2 - \frac{r}{2n(2n+1)}\right)\{g(U,Y)X - g(U,X)Y\}\right] = 0.$$

Then in view of definition (2.1), we have

$$R(X,Y)U = \alpha^2[g(Y,Z)X - g(X,Z)Y].$$

The above expression implies that M is of constant curvature α^2. Consequently, it is locally isometric to the hyperbolic space $H^{2n+1}\alpha^2$.

Conversely, if it has the scalar curvature $r = \alpha^2 2n(2n+1)$, then from (3.9) it follows that $C(\xi, X) = 0$. Similarly in the second case, since constant $r = \alpha^2 2n(2n+1)$, therefore again we get $C(\xi, X) = 0$. This complete the proof of the theorem.

Using the fact $C(\xi, X).R = 0$, $C(\xi, X)$ is acting as a derivation, we state the following corollary.

Corollary 3.4. An $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold M satisfies

$$C(\xi, X).R = 0.$$

If and only if either the scalar curvature r of M is $r = \alpha^2 2n(2n+1)$ or M is locally isometric to the Hyperbolic sphere $H^{2n+1}\alpha^2$.

Theorem 3.5. Let M be an $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold then $R.C = R.R$.

Proof. We have

$$\begin{aligned}(R(X,Y).C)(U,V,W) = R(X,Y)C(U,V)W - C(R(X,Y)U,V)W \\ -C(U,R(X,Y)V)W - C(U,V)R(X,Y)W.\end{aligned}$$

In view of definition (2.1), from above we have

$$\begin{aligned}&(R(X,Y).C)(U,V,W)\\ &= R(X,Y)\left[R(U,V)W - \frac{r}{2n(2n+1)}(g(V,W)U - g(U,W)V)\right]\\ &-R(R(X,Y)U,V)W + \frac{r}{2n(2n+1)}[g(V,W)R(X,Y)U - g(R(X,Y)U,W)V]\\ &-R(U,R(X,Y)V)W + \frac{r}{2n(2n+1)}[g(R(X,Y)V,W)U - g(U,W)R(X,Y)V]\\ &-R(U,V)R(X,Y)W + \frac{r}{2n(2n+1)}[g(V,R(X,Y)W)U - g(V,R(X,Y)W)V].\end{aligned}$$

On simplification, we get

$$R(X,Y).C(U,V,W) = (R(X,Y).R)(U,V,W).$$

Therefore $R.C = R.R$.

This completes the proof of the theorem.

Definition 3.1. An $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold is said to be $\phi-$Ricci symmetric if the Ricci operator satisfies $\phi^2(D_XQ)(Y) = 0$ for all vector field X, Y on M and $S(X,Y) = g(QX,Y)$.

Theorem 3.6. An $(2n+1)-$dimensional Lorentzian $\alpha-$Sasakian manifold is $\phi-$Ricci symmetric if and only if manifold is Einstein manifold.

Proof. Let us suppose that manifold is $\phi-$Ricci symmetric then in view of definition (3.3), we have

$$\phi^2((D_XQ)(Y)) = 0.$$

Using (2.1), we have

$$((D_XQ)(Y) + \eta((D_XQ)(Y))\xi) = 0. \tag{3.10}$$

Taking inner product of (3.10) with Z, we get

$$g((((D_XQ)(Y), Z)) + \eta((D_XQ)(Y))\,\eta(Z)) = 0,$$

which implies

$$g(D_XQ(Y) - Q(D_XY).Z) + \eta(D_XQ)(Y)\eta(Z) = 0.$$

On simplification, we have

$$g(D_XQ(Y), Z) - S(D_XY, Z) + \eta(D_XQ)(Y)\eta(Z) = 0. \tag{3.11}$$

Putting $Y = \xi$, in (3.11) and using (2.7), (2.13), we get

$$2n\alpha^3 g(\phi X, Z) - \alpha S(\phi X, Z) + \eta((D_XQ)(\xi))\,\eta(Z) = 0.$$

Replacing Z by ϕZ, we get $S(\phi X, \phi Z) = 2n\alpha^2 g(\phi X, \phi Z)$

$$S(X, Z) + 2n\alpha^2\eta(X)\eta(Z) = 2n\alpha^2 g(X, Z) + 2n\alpha^2\eta(X)\,\eta(Z),$$

which implies

$$S(X, Z) = 2n\alpha^2 g(X, Z).$$

Therefore manifold is Einstein manifold.

Now let us suppose that manifold is Einstein manifold then in the view of definition (2.2), we have $S(X, Y) = \lambda g(X, Y)$ where $S(X, Y) = g(\phi X, Y)$ and λ is constant.

Hence $QX = \lambda X$. Therefore we obtain

$$\phi^2((D_XQ)(Y)) = 0.$$

This completes the proof.

Acknowledgement

The Author is grateful to Dr S. K. Srivastava for his valuable suggestation for the improve the paper.

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