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On the C-Bochner Curvature Tensor of Generalized Sasakian-Space-Forms

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Abstract

The object of the present paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on C-Bochner curvature tensor.

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1. Introduction

Studying almost Hermitian manifold, Alfred Gray, a well Known geometrician, formulated a principle according to which the so called curvature identities for the Riemann-Christoffel tensor are the keys to understanding differential-geometric properties of such manifolds [20]. Many papers are devoted to the study of geometric consequences of these identities and to their analogs for almost contact metric structures. As a continuation of this line of research, we consider in the present paper some curvature properties of generalized Sasakian space forms regarding C-Bochner curvature tensor.

A generalized Sasakian-space-form was defined by P. Alegre, D. E. Blair and A. Carriazo [1] as that almost contact metric manifold (M^{2n+1}, ϕ, ξ, η, g) whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{1.1}$$

where f_1, f_2, f_3 are some differentiable functions on M^{2n+1} and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$
$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

for any vector fields X, Y, Z on M^{2n+1}. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well known Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. It is known that any three dimensional $(\alpha, \beta)-$trans-Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-form [2]. P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon give results in [3] about B. Y. Chen's inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid analyse the CR-submanifolds of generalized Sasakian-space-forms ([4], [5]). In [15], U. K. Kim studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. U. C. De and A. Sarkar [10] have studied generalized Sasakian-space-forms regarding projective curvature tensor. In [11], U. C. De, R. N. Singh and S. Pandey studied conharmonic curvature tensor of generalized Sasakian-space-forms. R. N. Singh and S. K. Pandey [21] have studied generalized Sasakian-space-forms regarding W_2-curvature tensor.

On the other hand, S. Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which now well known as the Bochner curvature tensor [8]. A geometric meaning of the Bochner curvature tensor is given by D. E. Blair [7]. By using the Boothby-Wang's fibration [9], M. Matsumoto and G.Chūman constructed C-Bochner curvature tensor [17] from the Bochner curvature tensor and studied in the context of Sasakian geometry. In a Sasakian manifold, the Ricci operator Q commutes with the structure tensor ϕ, but in general $Q\phi \neq \phi Q$. Thus the definition of C-Bochner curvature tensor seems not to include all the non-Sasakian cases [12]. Keeping this view in mind, Endo ([13],[14]) defined a curvature tensor B^{es} on a contact metric manifold, which coincides with the C-Bochner curvature tensor when the manifold is Sasakian. C-Bochner curvature tensor have studied by Kim, Tripathi and Choi [16], Pathak, De and Kim [19] and many others.

Motivated by the above studies, in the present paper, we study flatness and symmetry property of generalized Sasakian-space-forms regarding C-Bochner curvature tensor. The present paper is organized as follows:

In this paper, we study the C-Bochner curvature tensor of generalized Sasakian-space-forms. In section 2, some preliminary results are recalled. In section 3, we study C-Bochner- semisymmetric generalized Sasakian-space-forms. Section 4 deals with generalized Sasakian-space-forms with vanishing C-Bochner curvature tensor. In section 5, generalized Sasakian-space-forms satisfying $B.S = 0$ are studied. Section 6 is devoted to study generalized Sasakian-space-forms satisfying $B.R = 0$. The last section contains the generalized Sasakian-space-forms satisfying $B.B = 0$, where R, S and B are Riemannian curvature tensor, Ricci tensor and C-Bochner curvature tensor respectively of the space form.

2. Preliminaries

If on an odd dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \phi(\xi) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, g(X, \xi) = \eta(X), \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifold [6] and the structure (ϕ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of equations (2.1), (2.2) and (2.3), we have

$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \tag{2.5}$$

Again we know [1] that in a $(2n+1)$-dimensional generalized Sasakian-space-form

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ & - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.6}$$

for all vector fields X, Y, Z on M^{2n+1}, where R denotes the curvature tensor of M^{2n+1}.

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.8}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3). \tag{2.9}$$

We also have for a generalized Sasakian-space-forms

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)], \tag{2.12}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.13}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.14}$$

where Q is the Ricci operator, i.e. $g(QX,Y) = S(X,Y)$. A generalized Sasakian space-form is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.15}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on M^{2n+1}.

For a $(2n+1)$-dimensional $(n > 1)$ almost contact metric manifold the C-Bochner curvature tensor is given by [17]

$$\begin{aligned}B(X,Y)Z &= R(X,Y)Z - \frac{m-4}{(2n+4)}R_0(Y,X)Z + \frac{1}{(2n+4)}\{R_0(QY,X)Z \\ &- R_0(QX,Y)Z + R_0(Q\phi Y,\phi X)Z - R_0(Q\phi X,\phi Y)Z + 2g(Q\phi X,Y)\phi Z \\ &+ 2g(\phi X,Y)Q\phi Z + \eta(Y)R_0(QX,\xi)Z + \eta(X)R_0(\xi,QY)Z\} \\ &- \frac{m+2n}{(2n+4)}\{R_0(\phi Y,\phi X)Z + 2g(\phi X,Y)\phi Z\} + \frac{m}{(2n+4)}\{\eta(Y)R_0(\xi,X)Z \\ &+\eta(X)R_0(Y,\xi)Z\},\end{aligned} \tag{2.16}$$

where Q is the Ricci-operator, r is the scalar curvature,

$$R_0(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \ \forall X,Y,Z \in TM \tag{2.17}$$

and $m = \frac{2n+r}{2n+2}$.

By virtue of equations (2.6) and (2.17) equation (2.16) takes the form

$$\begin{aligned}B(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\} - \frac{m-4}{(2n+4)}\{g(X,Z)Y - g(Y,Z)X\} + \frac{1}{2n+4}[g(X,Z)QY \\ &- g(QY,Z)X - g(Y,Z)QX + g(QX,Z)Y + g(\phi X,Z)Q\phi Y - g(Q\phi Y,Z)\phi X \\ &- g(\phi Y,Z)Q(\phi X) + g(Q\phi X,Z)\phi Y + 2g(Q\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z \\ &+ \eta(Y)\{g(\xi,Z)QX - g(QX,Z)\xi\} + \eta(X)\{g(QY,Z)\xi - g(\xi,Z)QY\}] \\ &- \frac{m+2n}{(2n+4)}\{g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z\} + \frac{m}{(2n+4)}[\eta(Y) \\ &\times\{g(X,Z)\xi - g(\xi,Z)X\} + \eta(X)\{g(\xi,Z)Y - g(Y,Z)\xi\}].\end{aligned} \tag{2.18}$$

Putting $Z = \xi$ in above equation, we get

$$B(X,Y)\xi = \frac{4}{(2n+4)}(f_1 - f_3 - 1)R_0(X,Y)\xi, \tag{2.19}$$

which gives

$$\eta(B(X,Y)\xi) = 0. \tag{2.20}$$

$$B(\xi,Y)Z = -B(Y,\xi)Z = \frac{4}{(2n+4)}(f_1 - f_3 - 1)R_0(\xi,Y)Z, \tag{2.21}$$

$$\eta(B(\xi,Y)Z) = -\eta(B(Y,\xi)Z) = \frac{4}{(2n+4)}(f_1 - f_3 - 1)\eta(R_0(\xi,Y)Z) \tag{2.22}$$

and

$$\eta(B(X,Y)Z) = \frac{4}{(2n+4)}(f_1 - f_3 - 1)\eta(R_0(X,Y)Z). \tag{2.23}$$

3. C-Bochner- semisymmetric generalized Sasakian-space-forms

Definition 3.1. A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form is said to be C-Bochner- semisymmetric [10] if it satisfies $R.B = 0$, where R is the Riemannian curvature tensor and B is the C-Bochner curvature of the space-form.

Theorem 3.1. If a $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form satisfying $R.B = 0$, then either $f_1 = f_3$ or M^{2n+1} is an $\eta-$Einstein manifold.

Proof. Let

$$R(\xi,X).B(Y,Z)U = 0. \tag{3.1}$$

In this case we can write

$$\begin{aligned} R(\xi,X)B(Y,Z)U - B(R(\xi,X)Y,Z)U - B(Y,R(\xi,X)Z)U \\ - B(Y,Z)R(\xi,X)U = 0. \end{aligned} \tag{3.2}$$

In view of equation (2.11) the above equation reduces to

$$\begin{aligned} &(f_1 - f_3)[g(X,B(Y,Z)U)\xi - \eta(B(Y,Z)U)X - g(X,Y)B(\xi,Z)U \\ &+ \eta(Y)B(X,Z)U - g(X,Z)B(Y,\xi)U + \eta(Z)B(Y,X)U \\ &- g(X,U)B(Y,Z)\xi + \eta(U)B(Y,Z)X] = 0. \end{aligned} \tag{3.3}$$

Now, taking the inner product of above equation with ξ and using equation (2.2), we get

$$\begin{aligned} &(f_1 - f_3)[g(X,B(Y,Z)U) - \eta(B(Y,Z)U)\eta(X) - g(X,Y)\eta(B(\xi,Z)U) \\ &+ \eta(Y)\eta(B(X,Z)U) - g(X,Z)\eta(B(Y,\xi)U) + \eta(Z)\eta(B(Y,X)U) \\ &+ \eta(U)\eta(B(Y,Z)X)] = 0, \end{aligned} \tag{3.4}$$

which on using equations (2.18), (2.20) and (2.22) gives

$$
\begin{aligned}
&(f_1 - f_3)[{}'R(Y,Z,U,X) - \frac{m-4}{(2n+4)}\{g(Y,U)g(X,Z) - g(Z,U)g(X,Y)\} \\
&+ \frac{1}{(2n+4)}\{g(Y,U)S(Z,X) - g(X,Y)S(Z,U) + g(Z,X)S(Y,U) \\
&- g(Z,U)S(X,Y) + g(\phi Y,U)S(\phi Z,X) - S(\phi Z,U)g(\phi Y,X) \\
&- g(\phi Z,U)S(\phi Y,X) + S(\phi Y,U)g(\phi Z,X) + 2S(\phi Y,Z)g(\phi U,X) \\
&+ 2g(\phi Y,Z)S(\phi U,X) + \eta(Z)\eta(U)S(X,Y) - \eta(Z)\eta(X)S(Y,U) \\
&+ \eta(Y)\eta(X)S(Z,U) - \eta(Y)\eta(U)S(Z,X)\} - \frac{m+2n}{(2n+4)}\{g(\phi Y,U)g(\phi Z,X) \\
&- g(\phi Z,U)g(\phi Y,X) + 2g(\phi Y,Z)g(\phi Y,X)\} + \frac{m}{2n+4}\{\eta(Z)\eta(X)g(Y,U) \\
&- \eta(Z)\eta(U)g(X,Y) + \eta(Y)\eta(U)g(X,Z) - \eta(Y)\eta(X)g(Z,U)\} \\
&- \frac{4}{(2n+4)}(f_1 - f_3 - 1)\{g(Z,U)g(X,Y) - g(Y,U)g(X,Z)\}] = 0. \qquad (3.5)
\end{aligned}
$$

Putting $Z = U = e_i$ in above equation and summing over i, $1 \leq i \leq 2n+1$, we obtain

$$
\begin{aligned}
&(f_1 - f_3)[6S(X,Y) - \frac{6n(2n+3) - 3r}{(2n+2)}g(X,Y) - 8n(f_1 - f_3 - 1)g(X,Y) \\
&+ \frac{1}{(2n+2)}\{3r - 2n(2n+1) - 4n(2n+2)(f_1 - f_3)\}\eta(X)\eta(Y)] = 0, \qquad (3.6)
\end{aligned}
$$

which shows that either $f_1 = f_3$ or

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad (3.7)$$

where

$$a = \frac{6n(2n+3) - 3r - 8n(2n+2)(f_1 - f_3 - 1)}{6(2n+2)}$$

and

$$b = \frac{2n(2n+1) + 4n(2n+2)(f_1 - f_3) - 3r}{6(2n+2)}.$$

This shows that M^{2n+1} is an $\eta-$Einstein manifold. This completes the proof.

4. Generalized Sasakian-space-forms with vanishing C-Bochner curvature tensor

Theorem 4.2. A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form with vanishing C-Bochner curvature tensor is an $\eta-$Einstein manifold.

Proof. Suppose $B = 0$ on generalized Sasakian-space-form, then from equations (2.16) and (2.17), we have

$$\begin{aligned}
R(X,Y)Z = {} & \frac{m-4}{(2n+4)}\{g(X,Z)Y - g(Y,Z)X\} \\
& - \frac{1}{2n+4}[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX + S(X,Z)Y \\
& + g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q(\phi X) \\
& + S(\phi X,Z)\phi Y + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z \\
& + \eta(Y)\{g(\xi,Z)QX - S(X,Z)\xi\} + \eta(X)\{S(Y,Z)\xi - g(\xi,Z)QY\}] \\
& + \frac{m+2n}{(2n+4)}\{g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z\} \\
& - \frac{m}{(2n+4)}[\eta(Y)\{g(X,Z)\xi - \eta(Z)X\} + \eta(X)\{\eta(Z)Y - g(Y,Z)\xi\}].
\end{aligned} \tag{4.1}$$

By virtue of equations (2.7) and (2.8) above equation takes the form

$$\begin{aligned}
R(X,Y)Z = {} & \frac{m-4}{(2n+4)}\{g(X,Z)Y - g(Y,Z)X\} \\
& - \frac{1}{2n+4}[(2nf_1 + 3f_2 - f_3)\{2g(X,Z)Y - 2g(Y,Z)X \\
& + 2g(\phi X,Z)\phi Y - 2g(\phi Y,Z)\phi X + 4g(\phi X,Y)\phi Z \\
& + \eta(Y)\eta(Z)X - \eta(Y)g(X,Z)\xi + \eta(X)g(Y,Z)\xi - \eta(X)\eta(Z)Y\} \\
& - (3f_2 + (2n-1)f_3)\{\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi - \eta(Y)\eta(Z)X \\
& + \eta(X)\eta(Z)Y\}] + \frac{m+2n}{(2n+4)}\{g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\
& + 2g(\phi X,Y)\phi Z\} - \frac{m}{(2n+4)}[\eta(Y)\{g(X,Z)\xi - \eta(Z)X\} \\
& + \eta(X)\{\eta(Z)Y - g(Y,Z)\xi\}].
\end{aligned} \tag{4.2}$$

Now taking the inner product of above equation with U, we get

$$\begin{aligned}
R(X,Y)Z = {} & \frac{m-4}{(2n+4)}\{g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\} \\
& - \frac{1}{2n+4}[(2nf_1 + 3f_2 - f_3)\{2g(X,Z)g(Y,U) - 2g(Y,Z)g(X,U) \\
& + 2g(\phi X,Z)g(\phi Y,U) - 2g(\phi Y,Z)g(\phi X,U) + 4g(\phi X,Y)g(\phi Z,U) \\
& + \eta(Y)\eta(Z)g(X,U) - \eta(Y)\eta(U)g(X,Z) + \eta(X)\eta(U)g(Y,Z) \\
& - \eta(X)\eta(Z)g(Y,U)\} - (3f_2 + (2n-1)f_3)\{\eta(Y)\eta(U)g(X,Z) \\
& - \eta(X)\eta(U)g(Y,Z) - \eta(Y)\eta(Z)g(X,U) + \eta(X)\eta(Z)g(Y,U)\}]
\end{aligned} \tag{4.3}$$

$$+\frac{m+2n}{(2n+4)}\{g(\phi X,Z)g(\phi Y,U)-g(\phi Y,Z)g(\phi X,U)$$
$$+2g(\phi X,Y)g(\phi Z,U)\}-\frac{m}{(2n+4)}[\eta(Y)\{g(X,Z)\eta(U)-\eta(Z)g(X,U)\}$$
$$+\eta(X)\{\eta(Z)g(Y,U)-g(Y,Z)\eta(U)\}].$$

Putting $Y=Z=e_i$ in above equation and taking summation over i, $1\leq i\leq 2n+1$, we get

$$S(X,U)=ag(X,U)+b\eta(X)\eta(Y), \tag{4.4}$$

where

$$a=\frac{1}{(2n+4)}[-\frac{(2n-1)(2n+r)}{2n+2}+8n+2n(4n-1)f_1+6(2n-1)f_2-2(3n-1)f_3]$$

and

$$b=\frac{1}{(2n+4)}[\frac{(2n-1)(2n+r)}{2n+2}-2n(2n-1)f_1-6(2n-1)f_2+2(3n-2n^2-1)f_3],$$

which shows that M^{2n+1} is an $\eta-$Einstein manifold. This completes the proof.

5. Generalized Sasakian-space-forms satisfying $B.S=0$.

Theorem 5.3. If a $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space-form satisfies $B.S=0$, then either $f_1-f_3=1$ or M^{2n+1} is an Einstein manifold.

Proof. Let us consider generalized Sasakian-space-form satisfying $B(\xi,X).S=0$. In this case we can write

$$S(B(\xi,X)Y,Z)+S(Y,B(\xi,X)Z)=0, \tag{5.1}$$

which on using equation (2.21) reduces to

$$\begin{aligned}&(f_1-f_3-1)[g(X,Y)S(\xi,Z)-\eta(Y)S(X,Z)\\&+g(X,Z)S(Y,\xi)-\eta(Z)S(X,Y)]=0.\end{aligned} \tag{5.2}$$

Now, putting $Z=\xi$ in above equation, we get

$$(f_1-f_3-1)\{2n(f_1-f_3)g(X,Y)-S(X,Y)\}=0, \tag{5.3}$$

which shows that either $f_1-f_3=1$ or

$$S(X,Y)=2n(f_1-f_3)g(X,Y).$$

Thus M^{2n+1} is an Einstein manifold. This completes the proof.

6. Generalized Sasakian-space-forms satisfying $B.R=0$

Theorem 6.4. If a $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space-form satisfies $B.R=0$, then either $f_1-f_3=1$ or it is an Einstein manifold.

Proof. Let $B(\xi, X).R(Y, Z)U = 0$, then we have

$$\begin{aligned} &B(\xi, X)R(Y, Z)U - R(B(\xi, X)Y, Z)U - R(Y, B(\xi, X)Z)U \\ &- R(X, Y)B(\xi, X)U = 0, \end{aligned} \tag{6.1}$$

which on using equation (2.21) takes the form

$$\begin{aligned} &(f_1 - f_3 - 1)[g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X - g(X, Y)R(\xi, Z)U \\ &+ \eta(Y)R(X, Z)U - g(X, Z)R(Y, \xi)U + \eta(Z)R(Y, X)U \\ &- g(X, U)R(Y, Z)\xi + \eta(U)R(Y, Z)X] = 0. \end{aligned} \tag{6.2}$$

Now taking the inner product of above equation with ξ, we get

$$\begin{aligned} &(f_1 - f_3 - 1)[{}'R(Y, Z, U, X) - \eta(R(Y, Z)U)\eta(X) - g(X, Y)\eta(R(\xi, Z)U) \\ &+ \eta(Y)\eta(R(X, Z)U) - g(X, Z)\eta(R(Y, \xi)U) + \eta(Z)\eta(R(Y, X)U) \\ &- g(X, U)\eta(R(Y, Z)\xi) + \eta(U)\eta(R(Y, Z)X)] = 0. \end{aligned} \tag{6.3}$$

Using equations (2.6), (2.11) and (2.12) in above equation, we get

$$\begin{aligned} (f_1 - f_3 - 1)[{}'R(Y, Z, U, X) - (f_1 - f_3)\{g(X, Y)g(Z, U) \\ - g(X, Z)g(Y, U)\}] = 0. \end{aligned} \tag{6.4}$$

Putting $X = Y = e_i$ in above equation and summing over i, $1 \leq i \leq 2n + 1$, we get either $f_1 - f_3 = 1$ or

$$S(Z, U) = 2n(f_1 - f_3)g(Z, U), \tag{6.5}$$

which shows that M^{2n+1} is an Einstein manifold. This completes the proof.

7. Generalized Sasakian-space-forms satisfying $B.B = 0$

Theorem 7.5. If a $(2n + 1)-$dimensional $(n > 1)$ generalized Sasakian-space-form satisfying $B.B = 0$, then either $f_1 - f_3 = 1$ or M^{2n+1} is an $\eta-$Einstein manifold.

Proof. Let

$$B(\xi, X).B(Y, Z)U = 0. \tag{7.1}$$

In this case we can write

$$\begin{aligned} B(\xi, X)B(Y, Z)U - B(B(\xi, X)Y, Z)U - B(Y, B(\xi, X)Z)U \\ - B(Y, Z)B(\xi, X)U = 0. \end{aligned} \tag{7.2}$$

In view of equation (2.21) the above equation reduces to

$$\begin{aligned}&(f_1 - f_3 - 1)[g(X, B(Y, Z)U)\xi - \eta(B(Y, Z)U)X - g(X, Y)B(\xi, Z)U\\&+ \eta(Y)B(X, Z)U - g(X, Z)B(Y, \xi)U + \eta(Z)B(Y, X)U\\&- g(X, U)B(Y, Z)\xi + \eta(U)B(Y, Z)X] = 0.\end{aligned} \tag{7.3}$$

Now, taking the inner product of above equation with ξ and using equation (2.2), we get

$$\begin{aligned}&(f_1 - f_3 - 1)[g(X, B(Y, Z)U) - \eta(B(Y, Z)U)\eta(X)\\&- g(X, Y)\eta(B(\xi, Z)U) + \eta(Y)\eta(B(X, Z)U) - g(X, Z)\eta(B(Y, \xi)U)\\&+ \eta(Z)\eta(B(Y, X)U) + \eta(U)\eta(B(Y, Z)X)] = 0,\end{aligned} \tag{7.4}$$

which on using equations (2.18), (2.20) and (2.22) gives

$$\begin{aligned}&(f_1 - f_3 - 1)[{}'R(Y, Z, U, X) - \frac{m-4}{(2n+4)}\{g(Y, U)g(X, Z) - g(Z, U)g(X, Y)\}\\&+ \frac{1}{(2n+4)}\{g(Y, U)S(Z, X) - g(X, Y)S(Z, U) + g(Z, X)S(Y, U)\\&- g(Z, U)S(X, Y) + g(\phi Y, U)S(\phi Z, X) - S(\phi Z, U)g(\phi Y, X)\\&- g(\phi Z, U)S(\phi Y, X) + S(\phi Y, U)g(\phi Z, X) + 2S(\phi Y, Z)g(\phi U, X)\\&+ 2g(\phi Y, Z)S(\phi U, X) + \eta(Z)\eta(U)S(X, Y) - \eta(Z)\eta(X)S(Y, U)\\&+ \eta(Y)\eta(X)S(Z, U) - \eta(Y)\eta(U)S(Z, X)\} - \frac{m+2n}{(2n+4)}\{g(\phi Y, U)g(\phi Z, X)\\&- g(\phi Z, U)g(\phi Y, X) + 2g(\phi Y, Z)g(\phi Y, X)\} + \frac{m}{2n+4}\{\eta(Z)\eta(X)g(Y, U)\\&- \eta(Z)\eta(U)g(X, Y) + \eta(Y)\eta(U)g(X, Z) - \eta(Y)\eta(X)g(Z, U)\}\\&- \frac{4}{(2n+4)}(f_1 - f_3 - 1)\{g(Z, U)g(X, Y) - g(Y, U)g(X, Z)\}] = 0.\end{aligned} \tag{7.5}$$

Putting $Z = U = e_i$ in above equation and summing over i, $1 \leq i \leq 2n + 1$, we obtain

$$\begin{aligned}&(f_1 - f_3 - 1)[6S(X, Y) - \frac{6n(2n+3) - 3r}{(2n+2)}g(X, Y)\\&- 8n(f_1 - f_3 - 1)g(X, Y) + \frac{1}{(2n+2)}\{3r - 2n(2n+1)\\&- 4n(2n+2)(f_1 - f_3)\}\eta(X)\eta(Y)] = 0,\end{aligned} \tag{7.6}$$

which shows that either $f_1 - f_3 = 1$ or

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{7.7}$$

where

$$a = \frac{6n(2n+3) - 3r - 8n(2n+2)(f_1 - f_3 - 1)}{6(2n+2)}$$

and

$$b = \frac{2n(2n+1) + 4n(2n+2)(f_1 - f_3) - 3r}{6(2n+2)}.$$

This shows that M^{2n+1} is an $\eta-$Einstein manifold. This completes the proof.

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