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On Generalized (k, μ) Space Forms

C. R. Premalatha and H. G. Nagaraja

Department of Mathematics,
Bangalore University, Central College Campus,
Bengaluru – 560 001, India
e-mail: premalathacr@yahoo.co.in, hgnraj@yahoo.com
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Abstract

In this paper we study generalized (k, μ) space forms by considering flat, symmetry and semi-symmetry conditions. We find relations among associated functions to prove conformal flatness, projective flatness of generalized (k, μ) space forms. Further we prove that in a projectively flat generalized (k, μ) space form the associated functions f_2 and f_3 are linearly dependent.

Keywords and Phrases : Generalized (k, μ) space forms, symmetry, semi-symmetry, $\xi-$conformally flat, projective curvature, Einstein.
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1. Introduction

A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{1.1}$$

where f_1, f_2, f_3 are some differentiable functions on M^{2n+1} and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y$$
$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z$$
$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$

for any vector fields X, Y, Z on M^{2n+1}.

In [5], the authors defined a generalized (k, μ) space form as an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \quad (1.2)$$

where $f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6$ are differentiable functions on M^{2n+1} and $R_1,\ R_2,\ R_3$ are tensors defined as above and

$$R_4(X, Y)Z = g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y,$$

$$R_5(X, Y)Z = g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX,$$

$$R_6(X, Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi,$$

for any vector fields $X,\ Y,\ Z$, where $2h = L_\xi\phi$ and L is the usual Lie derivative. This manifold was denoted by $M^{2n+1}(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)$.

Natural examples of generalized (k, μ) space forms are (k, μ) space forms and generalized Sasakian space forms. The authors in [1] proved that contact metric generalized (k, μ) space forms are generalized (k, μ) spaces and if dimension is greater than or equal to 5, then they are (k, μ) spaces with constant $\phi-$sectional curvature $2f_6 - 1$. The authors gave a method of constructing examples of generalized (k, μ) space forms and proved that generalized (k, μ) space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [2], it is proved that under D_a-homothetic deformation generalized (k, μ) space form structure is preserved for dimension 3, but not in general. In this paper, we study generalized (k, μ) space forms under the flatness, symmetry and semi symmetry conditions. The paper is organised as follows. After preliminaries in section 2, we study conformal curvature tensor in a generalized (k, μ) space form in section 3. Projective semi symmetric and projective Ricci symmetric generalized (k, μ) space forms are studied in section 4. We derived conditions for projective semi symmetric generalized (k, μ) space forms to be projectively flat.

2. Preliminaries

In this section, some general definitions and basic formulas are presented which will be used later. A $(2n+1)-$dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold [3] if it admits a tensor field ϕ of type (1, 1), a vector field ξ and a $1-$form η satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(X, \phi X) = 0,\ g(X, \xi) = \eta(X). \quad (2.3)$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is the fundamental 2-form of M^{2n+1}.

It is well known that on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the symmetric tensor h satisfies the following relations:

$$h\xi = 0,\ \ h\phi = -\phi h,\ \ trh = 0,\ \ \eta \circ h = 0, \tag{2.4}$$

$$\nabla_X \xi = -\phi X - \phi hX,\ \ (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.5}$$

In a $(2n+1)-$dimensional $(k, \mu)-$contact metric manifold, we have [4]

$$h^2 = (k-1)\phi^2,\ \ k \leq 1, \tag{2.6}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.7}$$

$$\begin{aligned}(\nabla_X h)(Y) =& [(1-k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi hX) \\ &- \mu\eta(X)\phi hY.\end{aligned} \tag{2.8}$$

Definition 1. A contact metric manifold M^{2n+1} is said to be
(i) Einstein if $S(X, Y) = \lambda g(X, Y)$, where λ is a constant and S is the Ricci tensor,
(ii) η-Einstein if $S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions on M^{2n+1}.

In a $(2n+1)$-dimensional generalized (k, μ) space form, the following relations hold.

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \tag{2.9}$$

$$\begin{aligned}QX =& [2nf_1 + 3f_2 - f_3]X + [(2n-1)f_4 - f_6]hX \\ &- [3f_2 + (2n-1)f_3]\eta(X)\xi,\end{aligned} \tag{2.10}$$

$$\begin{aligned}S(X, Y) =& [2nf_1 + 3f_2 - f_3]g(X, Y) + [(2n-1)f_4 - f_6]g(hX, Y) \\ &- [3f_2 + (2n-1)f_3]\eta(X)\eta(Y),\end{aligned} \tag{2.11}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.12}$$

$$r = 2n[(2n+1)f_1 + 3f_2 - 2f_3], \tag{2.13}$$

where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M^{2n+1}(f_1, ..., f_6)$.

The relation between the associated functions $f_i, i = 1, ..., 6$ of $M^{2n+1}(f_1, ..., f_6)$ was recently discussed by Carriazo et al. [5].

Remark 2.1. (Carriazo et al. [5]) A contact metric generalized (k,μ) space form $M^{2n+1}(f_1,...,f_6)$ is η-Einstein if and only if $(2n-1)f_4 - f_6 = 0$. In particular $M^3(f_1,...,f_6)$ is η-Einstein if and only if $f_4 - f_6 = 0$.

Proposition 2.1. (Carriazo et al. [5]) Let $M^3(f_1,...,f_6)$ be a contact metric generalized (k,μ) space form. If $f_1 - f_3 \neq 1$ and $f_4 - f_6 = 0$, then $f_1 + f_3$ and $f_1 + 3f_2$ are constants and $2f_1 + 3f_2 - f_3 = 0$ holds.

3. Conformal Curvature Tensor in Generalized (k,μ) Space Forms

The conformal curvature tensor C is defined by [6]

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & - g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

Definition 1.1. A generalized (k,μ) space form $M^{2n+1}(f_1,...,f_6)$ is

(1) ξ-conformally flat if $C(X,Y)\xi = 0$.
(2) conformally Ricci symmetric if $C.S = 0$.

Theorem 3.1. A $(2n+1)-$dimensional generalized (k,μ) space form is $\xi-$conformally flat if and only if

(i) $2nf_1 + 3f_2 - f_3 = 0$, when $k = 1$.
(ii) $2nf_1 + 3f_2 - f_3 = 0$ and $f_4 - f_6 = 0$ when $k \neq 1$.

Proof. Let $M^{2n+1}(f_1,...,f_6)$ be ξ-conformally flat. Then from (3.1), we obtain

$$\begin{aligned} R(X,Y)\xi =& \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + \eta(Y)QX - \eta(X)QY] \\ & - \frac{r}{2n(2n-1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.2}$$

We prove this theorem by considering two cases.

Case (i). If $k = 1$, then $h = 0$.

Using (2.1), (2.9), (2.10) and (2.12) in (3.2), we obtain

$$[2nf_1 + 3f_2 - f_3]\frac{(2n-2)}{2n-1}[\eta(Y)X - \eta(X)Y] = 0. \tag{3.3}$$

which implies that

$$2nf_1 + 3f_2 - f_3 = 0. \tag{3.4}$$

Case (ii). If $k \neq 1$, then $h \neq 0$.

Using (2.1), (2.9), (2.10) and (2.12) in (3.2), we obtain

$$\begin{aligned}&[2nf_1+3f_2-f_3]\frac{(2n-2)}{2n-1}[\eta(Y)X-\eta(X)Y]\\&+(f_4-f_6)[\eta(Y)hX-\eta(X)hY]=0.\end{aligned} \tag{3.5}$$

If X is orthogonal to ξ, then (3.5) reduces to

$$[2nf_1+3f_2-f_3]\frac{(2n-2)}{2n-1}\eta(Y)X+(f_4-f_6)\eta(Y)hX=0. \tag{3.6}$$

$$[2nf_1+3f_2-f_3]\frac{(2n-2)}{2n-1}X+(f_4-f_6)hX=0. \tag{3.7}$$

Contracting the above with W, we get

$$[2nf_1+3f_2-f_3]\frac{(2n-2)}{2n-1}g(X,W)+(f_4-f_6)g(hX,W)=0. \tag{3.8}$$

Taking $X=W=e_i$, where $\{e_i,\, i=1,...,2n+1\}$ is an orthonormal basis of T_pM and taking summation over $i=1,...,2n+1$, we obtain

$$2nf_1+3f_2-f_3=0. \tag{3.9}$$

Using (3.9) in (3.8), we obtain $f_4-f_6=0$.

Hence the proof.

Theorem 3.2. A $(2n+1)$-dimensional generalized (k,μ) space form is conformally Ricci symmetric if and only if it is an Einstein manifold.

Proof. Suppose $C.S=0$. i.e.

$$S(C(W,X)Y,Z)+S(Y,C(W,X)Z)=0. \tag{3.10}$$

Then taking $W=Z=\xi$ in (3.10) and using (2.3) and (2.9)– (2.12), we obtain

$$\begin{aligned}&\frac{2n(f_1-f_3)}{(2n-1)}S(X,Y)\\&=\left[2n(f_1-f_3)\left(\frac{a}{2n-1}\right)-b(k-1)\left(\frac{b}{2n-1}-(f_4-f_6)\right)\right]g(X,Y)\\&+\left[2n(f_1-f_3)(f_4-f_6)+a\left(\frac{b}{2n-1}-(f_4-f_6)\right)\right]g(X,hY)\\&+\left[2n(f_1-f_3)^2(2n(f_1-f_3)-1)+b(k-1)\left(\frac{b}{2n-1}-(f_4-f_6)\right)\right.\\&\left.+c\frac{2n(f_1-f_3)}{2n-1}\right]\eta(X)\eta(Y),\end{aligned} \tag{3.11}$$

where

$$a = 2nf_1 + 3f_2 - f_3,\ b = (2n-1)f_4 - f_6,\ \ c = -[3f_2 + (2n-1)f_3]. \tag{3.12}$$

If Y is orthogonal to ξ, then (3.11) reduces to

$$\frac{2n(f_1 - f_3)}{(2n-1)} S(X,Y) = pg(X,Y) + qg(X,hY). \tag{3.13}$$

where

$$p = 2n(f_1 - f_3)\left(\frac{a}{2n-1}\right) - b(k-1)\left(\frac{b}{2n-1} - (f_4 - f_6)\right),$$
$$q = 2n(f_1 - f_3)(f_4 - f_6) + a\left(\frac{b}{2n-1} - (f_4 - f_6)\right).$$

Replacing Y by hY in (3.11) and using (2.6) and (2.11), we obtain

$$g(X,hY) = \frac{2n(f_1 - f_3)b - q(2n-1)}{2n(f_1 - f_3)a - p(2n-1)}(k-1)g(X,Y). \tag{3.14}$$

Now substituting for $g(X,hY)$ in (3.11), we obtain

$$S(X,Y) = \alpha g(X,Y), \tag{3.15}$$

where

$$\alpha = \frac{(2n-1)(p+qs)}{2n(f_1 - f_3)},$$
$$s = \frac{2n(f_1 - f_3)b - q(2n-1)}{2n(f_1 - f_3)a - p(2n-1)}(k-1).$$

Thus $M^{2n+1}(f_1, ..., f_6)$ is an Einstein manifold.

Converse is obvious.

Hence the theorem is proved.

4. Projective Curvature Tensor in Generalized (k, μ) Space Forms

Let $M^{2n+1}(f_1, ..., f_6)$ be a generalized (k, μ) space. The projective curvature tensor P is given by [6]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\left(S(Y,Z)X - S(X,Z)Y\right). \tag{4.1}$$

Suppose $R.P = 0$. Then we have

$$\begin{aligned} R(\xi,X)P(Y,Z)W - P(R(\xi,X)Y,Z)W - P(Y,R(\xi,X)Z)W \\ - P(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{4.2}$$

for all vector fields X, Y, Z and W on M^{2n+1}.

Taking $Y = W = \xi$ in (4.2) and using (2.9), (2.12) and (4.1) we get

$$\begin{aligned} &(f_1 - f_3)^2 g(X, Z)\xi + (f_4 - f_6)(f_1 - f_3) g(Z, hX)\xi \\ &- \frac{(f_1 - f_3)}{2n} S(Z, X)\xi - \frac{(f_4 - f_6)}{2n} S(Z, hX)\xi = 0. \end{aligned} \tag{4.3}$$

If X is orthogonal to ξ, then in view of (2.11), contraction of the above equation with ξ yields

$$\begin{aligned} S(Z, X) = &\frac{1}{(f_1 - f_3)} \left([2n(f_1 - f_3)^2 + (f_4 - f_6)(k - 1)((2n - 1)f_4 \right. \\ &\left. - f_6)] g(Z, X) - (f_4 - f_6)[3f_2 + (2n - 1)f_3] g(Z, hX) \right). \end{aligned} \tag{4.4}$$

Replacing X by hX in (4.4) and using (2.6) and (2.11), we obtain

$$g(Z, hX) = \frac{(k - 1)[(f_1 - f_3)b - c(f_4 - f_6)]}{a(f_1 - f_3) - v}, \tag{4.5}$$

where

$$v = 2n(f_1 - f_3)^2 + (f_4 - f_6)(k - 1)((2n - 1)f_4 - f_6). \tag{4.6}$$

Substituting for $g(Z, hX)$ in (4.4), we obtain

$$S(X, Z) = \gamma g(X, Z), \tag{4.7}$$

with

$$\gamma = \frac{1}{(f_1 - f_3)} \left[v + \left(\frac{b(f_1 - f_3) - c(f_4 - f_6)}{a(f_1 - f_3) - v} \right) (k - 1)c(f_4 - f_6) \right]. \tag{4.8}$$

From (4.7), we conclude that $M^{2n+1}(f_1, ..., f_6)$ is an Einstein manifold if $f_1 \neq f_3$.

Using (4.7) in (4.1), we obtain

$$'P(X, Y, Z, W) =' R(X, Y, Z, W) - \frac{\gamma}{2n}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \tag{4.9}$$

Hence we can state the following:

Theorem 4.1. A $(2n + 1)-$dimensional projectively semi symmetric contact metric generalized (k, μ) space form with $f_1 \neq f_3$ is projectively flat if it is of sectional curvature $\frac{\gamma}{2n}$.

If $P.S = 0$, then we have

$$S(P(W, X)Y, Z) + S(Y, P(W, X)Z) = 0. \tag{4.10}$$

Taking $W = Y = \xi$ in (4.10) and using (2.1), (2.6), (2.12) and (4.1), we get

$$\begin{aligned} S(Z,X) = &\frac{1}{(f_1-f_3)}\big([2n(f_1-f_3)^2-(f_4-f_6)(k-1)((2n-1)f_4 \\ &-f_6)]g(Z,X)+(f_4-f_6)[(1-2n)f_3-3f_2]g(Z,hX) \\ &-(f_4-f_6)(k-1)((2n-1)f_4-f_6)\eta(X)\eta(Z)\big)\,. \end{aligned} \tag{4.11}$$

If X is orthogonal to ξ, then (4.11) reduces to

$$\begin{aligned} S(Z,X) = &\frac{1}{(f_1-f_3)}\big([2n(f_1-f_3)^2-(f_4-f_6)(k-1)((2n-1)f_4 \\ &-f_6)]g(Z,X)-(f_4-f_6)[3f_2+(2n-1)f_3]g(Z,hX)\big)\,. \end{aligned} \tag{4.12}$$

Replacing X by hX in (4.11) and using (2.6), (2.11) we obtain

$$g(Z,hX) = \frac{(k-1)[b(f_1-f_3)-c(f_4-f_6)]g(Z,X)}{a(f_1-f_3)-e}, \tag{4.13}$$

where

$$e = 2n(f_1-f_3)^2-(f_4-f_6)(k-1)((2n-1)f_4-f_6).$$

Substituting for $g(Z,hX)$ in (4.11), we obtain

$$S(X,Z) = \frac{1}{(f_1-f_3)}\left(\rho g(X,Z)\right), \tag{4.14}$$

with

$$\rho = e + \left(\frac{c(f_4-f_6)(k-1)[b(f_1-f_3)-c(f_4-f_6)]}{a(f_1-f_3)-e}\right).$$

From (4.14), it follows that $M^{2n+1}(f_1,...,f_6)$ is an Einstein manifold if $f_1 \neq f_3$.

Converse is obvoius.

Hence we can state the following.

Theorem 4.2. A $(2n+1)$-dimensional generalized (k,μ) space form is projective Ricci symmetric if and only if it is an Einstein manifold for $f_1 \neq f_3$.

The following establishes relation between f_2 and f_3.

Theorem 4.3. In a projectively flat $M^{2n+1}(f_1,...,f_6)$, the associated functions f_2 and f_3 are linearly dependent.

Proof. If $M^{2n+1}(f_1,...,f_6)$ is projectively flat, then $P(X,Y)Z = 0$. From (4.1), we obtain

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{4.15}$$

Taking $X = \phi Y$ and using (1.2) and (2.11) in (4.15) , we obtain

$$f_1[g(Y,Z)\phi Y - g(\phi Y,Z)Y] + f_2[g(\phi Y,\phi Z)\phi Y - g(Y,\phi Z)\phi^2 Y + 2g(\phi Y,\phi Y)\phi Z]$$
$$+ f_3[g(\phi Y,Z)\eta(Y)\xi - \eta(Y)\eta(Z)\phi Y] + f_4[g(Y,Z)h\phi Y - g(\phi Y,Z)hY$$
$$+ g(hY,Z)\phi Y - g(h\phi Y,Z)Y] + f_5[g(hY,Z)h\phi Y - g(h\phi Y,Z)hY$$
$$+ g(\phi h\phi Y,Z)\phi hY - g(\phi hY,Z)\phi h\phi Y] + f_6[g(h\phi Y,Z)\eta(Y)\xi - \eta(Y)\eta(Z)h\phi Y]$$
$$= \frac{1}{2n}\left((2nf_1 + 3f_2 - f_3)[g(Y,Z)\phi Y - g(\phi Y,Z)Y]\right.$$
$$- (3f_2 + (2n-1)f_3)\eta(Z)\eta(Y)\phi Y$$
$$\left. + ((2n-1)f_4 - f_6)[g(hY,Z)\phi Y - g(h\phi Y,Z)Y]\right). \tag{4.16}$$

Taking $Y = Z = U$, where U is a unit vector orthogonal to ξ and using (2.1), (2.4) and (2.6) in (4.16), we obtain

$$f_1 g(U,U)\phi U + 3f_2 g(\phi U,\phi U)\phi U + f_4[g(U,U)h\phi U + g(hU,U)\phi U - g(\phi U,hU)U]$$
$$+ f_5[g(hU,U)h\phi U - g(\phi U,hU)hU + g(\phi h\phi U,U)\phi hU + g(hU,\phi U)\phi h\phi U]$$
$$= \frac{1}{2n}\big([2nf_1 + 3f_2 - f_3]g(U,U)\phi U + [(2n-1)f_4 - f_6]\times$$
$$[g(hU,U)\phi U - g(h\phi U,U)U]\big) \tag{4.17}$$

Let $\{e_i, i = 1,...,2n+1\}$ be an orthonormal basis of T_pM. Taking $U = e_i$ in (4.17) and summing over $i = 1,...,2n+1$, we obtain

$$[3(2n-2)f_2 + f_3]\phi e_i + 2nf_4 h\phi e_i = 0. \tag{4.18}$$

Contracting the above equation with respect to ϕe_i and using (2.4), we obtain

$$3(2n-2)]f_2 + f_3 = 0. \tag{4.19}$$

Hence the proof.

Combining theorem 4.1 and theorem 4.3, we have

Corollary 4.1. Let $M^{2n+1}(f_1,...,f_6)$ be a $(2n+1)$-dimensional projectively-semi-symmetric contact metric generalized (k,μ) space form with $f_1 \neq f_3$. If it is of sectional curvature $\frac{\gamma}{2n}$, then f_2 and f_3 are linearly dependent.

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