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# On the Hypersurface of a Finsler Space with the Special Metric $\alpha + \frac{\beta^{n+1}}{(\alpha-\beta)^n}$

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**Abstract**

In the present paper, we consider a $n-$dimentional Finsler space $F^n = (M^n, L)$ with $(\alpha, \beta)-$metric L$(\alpha, \beta) = \alpha + \frac{\beta^{n+1}}{(\alpha-\beta)^n}$ which is a generalization of the metric $\alpha + \frac{\beta^2}{(\alpha-\beta)}$ considered in [9] and the hypersurface of $F^n$ with $b_i(x) = \partial_i b$ being the gradient of a scalar function $b(x)$. We find the conditions for this hypersurface to be a hyperplane of 1st kind, 2nd kind and we also show that this hypersurface is a hyperplane of 3rd kind if and only if it is a hyperplane of first kind.

**Keywords and Phrases :** Hypersurface, Hyperplane, $(\alpha, \beta)$-metric, Normal curvature vector, Second fundamental tensor.
**2010 AMS Subject Classification :** 53B40, 53C60.

## 1. Introduction

We consider a $n$-dimensional Finsler space i.e., a pair consisting of a $n-$dimensional differential manifold $M^n$ equipped with a fundamental function $L(x, y)$. The concept of the $(\alpha, \beta)-$metric $L(\alpha, \beta)$ was introduced by M. Matsumoto ([5]) and has been studied by many authors ([1], [2], [7]). A Finsler metric L$(x, y)$ is called an $(\alpha, \beta)-$metric L$(\alpha, \beta)$ if L is a positively homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a_{ij}(x)\ y^i\ y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M^n$.

A hypersurface $M^{n-1}$ of the $M^n$ may be represented parametrically by the equation $x^i = x^i(u^\alpha)$, $\alpha = 1,...,n-1$, where $u^\alpha$ are Gaussian coordinates on $M^{n-1}$. The following notations are also employed [3]: $B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$, $B_{0\beta} =$

$v^\alpha B^i_{\alpha\beta}, B^{ij...}_{\alpha\beta...} = B^i_\alpha B^j_\beta....$ If the supporting element $y^i$ at a point $(u^\alpha)$ of $M^{n-1}$ is assumed to be tangential to $M^{n-1}$, we may then write $y^i = B^i_\alpha(u)v^\alpha$, so that $v^\alpha$ is thought of as the supporting element of $M^{n-1}$ at the point $(u^\alpha)$. Since the function $\underline{L}(u,v) = L(x(u), y(u,v))$ gives rise to a Finsler metric of $M^{n-1}$, we get a $(n-1)-$dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ .

## 2. Preliminaries

Let $F^n = (M^n, L)$ be a special Finsler space with the metric

$$\alpha + \frac{\beta^{n+1}}{(\alpha-\beta)^n}. \tag{2.1}$$

The derivatives of the (2.1) with respect to $\alpha$ and $\beta$ are given by

$$L_\alpha = \frac{(\alpha-\beta)^{n+1} - n\beta^{n+1}}{(\alpha-\beta)^{n+1}}, L_\beta = \frac{((n+1)\alpha-\beta)\beta^n}{(\alpha-\beta)^{n+1}}, L_{\alpha\alpha} = \frac{n(n+1)\beta^{n+1}}{(\alpha-\beta)^{n+2}},$$

$$L_{\beta\beta} = \frac{n(n+1)\alpha^2\beta^{n-1}}{(\alpha-\beta)^{n+2}}, L_{\alpha\beta} = \frac{-n(n+1)\alpha\beta^n}{(\alpha-\beta)^{n+2}},$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In the special Finsler space $F^n = (M^n, L)$ the normalized element of support $l_i = \dot{\partial}_i L$ and the angular metric tensor $h_{ij}$ are given by [7]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i$$

and

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i Y_j + b_2 Y_i Y_j) + Q_2 Y_i Y_j,$$

where

$$Y_i = a_{ij} y^j,$$

$$p = LL_\alpha \alpha^{-1} = \frac{\{\alpha(\alpha-\beta)^n + \beta^{n+1}\}\{(\alpha-\beta)^{n+1} - n\beta^{n+1}\}}{\alpha(\alpha-\beta)^{(2n+1)}},$$

$$q_0 = LL_{\beta\beta} = \frac{n(n+1)\{\alpha(\alpha-\beta)^n + \beta^{n+1}\}\alpha^2\beta^{n-1}}{(\alpha-\beta)^{2n+2}},$$

$$q_1 = LL_{\alpha\beta}\alpha^{-1} = \frac{-n(n+1)\{\alpha(\alpha-\beta)^n + \beta^{n+1}\}\beta^n}{(\alpha-\beta)^{2(n+1)}},$$

$$\begin{aligned} q_2 &= L\alpha^{-2}(L_{\alpha\alpha}) \\ &= \frac{\{\alpha(\alpha-\beta)^n + \beta^{n+1}\}[n\beta^{n+1}\{(n+2)\alpha-\beta\} - (\alpha-\beta)^{n+2}]}{\alpha^3(\alpha-\beta)^{2(n+1)}}. \end{aligned} \tag{2.2}$$

The fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor $g^{ij}$ are given, respectively by [7]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,$$

where

$$\begin{aligned} p_0 &= q_0 + L_\beta^2 \\ &= \frac{(n+1)[\{n\alpha^3(\alpha-\beta)^n\beta^{n-1} + (n+1)\alpha^2\beta^{2n} + n\alpha^2\beta^{2n} - 2\alpha\beta^{2n+1}\} + \beta^{2n+2}]}{(\alpha-\beta)^{2n+2}}, \\ p_1 &= q_1 + L^{-1}pL_\beta \\ &= \frac{\{(1-n)\alpha-\beta\}(n+1)\alpha\beta^n(\alpha-\beta)^n - 2n(n+1)\alpha\beta^{2n+1} + n\beta^{2(n+1)} - \beta^{n+1}(\alpha-\beta)^{n+1}}{\alpha(\alpha-\beta)^{2n+2}}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} p_2 &= q_2 + p^2L^{-2} \\ &\frac{\{\alpha(\alpha-\beta)^n + \beta^{n+1}\}[n\beta^{n+1}\{(n+2)\alpha-\beta\} - (\alpha-\beta)^{n+2}] + \{(\alpha-\beta)^{2n+2} + n^2\beta^{2n+2} - 2n(\alpha-\beta)^{n+1}\beta^{n+1}\}}{\alpha^3(\alpha-\beta)^{2(n+1)}}, \end{aligned} \tag{2.4}$$

and

$$g^{ij} = p^{-1}a^{ij} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j \tag{2.5}$$

where

$$b^i = a^{ij}b_j,\ S_0 = \frac{[pp_0 + (p_0p_2 - p_1^2)\alpha^2]}{\zeta},\ S_1 = \frac{[pp_1 + (p_0p_2 - p_1^2)\beta]}{\zeta p},\ S_2 = \frac{[pp_2 + (p_0p_2 - p_1^2)b^2]}{\zeta p},$$

$$b^2 = a_{ij}b^ib^j, \qquad \zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2) \tag{2.6}$$

The hv-torsion tensor $c_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [7]

$$2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{2.7}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i.$$

Here $m_i$ is a non-vanishing covariant vector orthogonal to the element of support $y^i$. Let $\left\{ {i \atop jk} \right\}$ be the components of Christoffel symbols of the associated Riemannian space $R^n$ and $\nabla_k$ be covariant differentiation with respect to $x^k$ relative to this Christoffel symbols. We put

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{2.8}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{ij}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the Cartan connection of $F^n$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \begin{Bmatrix} i \\ jk \end{Bmatrix}$ of the special Finsler space $F^n$ is given by [4]

$$\begin{aligned} D^i_{jk} = {} & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k \\ & - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (c^i_{jm} c^m_{sk} + c^i_{km} c^n_{sj} - c^m_{jk} c^i_{ms}), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} & B_k = p_0 b_k + p_1 Y_k, \qquad B^i = g^{ij} B_j, \qquad F^k_i = g^{kj} F_{ji}, \\ & B_{ij} = \frac{\{p_1(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \\ & B^k_i = g^{kj} B_{ji}, \\ & A^m_k = b^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ & \lambda^m = B^m E_{00} + 2B_0 F^m_0, B_0 = B_i y^i, \end{aligned} \tag{2.10}$$

where ‘0 ’denote contraction with $y^i$ except for the quantities $p_0$, $q_0$ and $S_0$.

## 3. Induced Cartan connection

Let $F^{n-1}$ be a hypersurface of $F^n$ given by the equations $x^i = x^i(u)$. The element of support $y^i$ of $F^n$ is to be taken tangential to $F^{n-1}$, that is

$$y^i = B^i_\alpha(u)\, v^\alpha. \tag{3.1}$$

The metric tensor $g$ and $v$-torsion tensor $C$ of $F^{n-1}$ are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma.$$

At each point $u$ of $F^{n-1}$, a unit normal vector $N^i(u,\, v)$ is defined by

$$g_{ij}(x(u,v), y(u,v)\, B^i_\alpha N^j = 0, \qquad g_{ij}(x(u,v), y(u,v))\, N^i\, N^j = 1,$$

and for the angular metric tensor $h_{ij}$, we have

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad h_{ij} B^i_\alpha N^j = 0, \qquad h_{ij} N^i N^j = 1. \tag{3.2}$$

If $(B^\alpha_i, N^i)$ denote the inverse of $(B^i_\alpha, N_i)$, then we have

$$\begin{aligned} & B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0, \quad N_i = g_{ij} N^j, \\ & B^k_i = g^{kj} B_{ji}, \qquad B^i_\alpha B^\alpha_j + N^i N^j = \delta^j_i. \end{aligned}$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of $F^{n-1}$ induced from the Cartan's connection $(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ is given by [6]

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,$$
$$G^\alpha_\beta = B^\alpha_i(B^i_{0\chi} + \Gamma^{*i}_{0j}B^i_\beta),$$
$$C^\alpha_{\alpha\beta} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma,$$

where

$$M_{\beta\gamma} = N_i C^i_{jk}B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta) \tag{3.3}$$

and $B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial U^r}$. The quantities $M_{\beta\gamma}$ and $H_\beta$ are called the second fundamental $v$-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]:

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{3.4}$$

where

$$M_\beta = N_i C^i_{jk}B^j_\beta N^k. \tag{3.5}$$

The relative h and v-covariant derivatives of projection factor $B^i_\alpha$ with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, B^i_\alpha|\beta = M_{\alpha\beta}N^i. \tag{3.6}$$

The equation (2.3) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{3.7}$$

The above equations yield

$$H_{0\gamma} = H_\gamma, H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{3.8}$$

We use following lemmas, which are due to Matsumoto [6]:

**Lemma 3.1.** The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector $H_\beta$ vanishes.

**Lemma 3.2.** A hypersurface $F^{n-1}$ is a hyperplane of the 1st kind if and only if $H_\alpha = 0$.

**Lemma 3.3.** A hypersurface $F^{n-1}$ is a hyperplane of the 2nd kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

**Lemma 3.4.** A hyperplane of the 3rd kind is characterized by $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.

## 4. Hypersurface $F^{n-1}(c)$ of the special Finsler space

Let us consider special Finsler metric $L = \alpha+\beta^{n+1}/(\alpha-\beta)^n$ with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ given by the equation $b(x) = c$(constant) [8]. From parametric equation $x^i = x^i(u)$ of $F^{n-1}(c)$, we get $\partial_\alpha B(x(u)) = 0 = b_i B^i_\alpha$ so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$b_i B^i_\alpha = 0, \qquad b_i\, y^i = 0. \tag{4.1}$$

The induced metric $L(u,\, v)$ of $F^{n-1}(c)$ is given by

$$L(u,v) = a_{\alpha\beta} v^\alpha v^\beta, \qquad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta. \tag{4.2}$$

which is the Riemannian metric.

At a point of $F^{n-1}(c)$, from (2.2), (2.3) and (2.5), we have

$$\begin{aligned} &p = 1, \quad q_0 = 0, \quad q_1 = 0, \quad q_2 = -\alpha^{-2}, \quad p_0 = 0, \\ &p_2 = 0, \quad \zeta = 1, \quad S_0 = 0, \quad S_1 = 0, \quad S_2 = 0. \end{aligned} \tag{4.3}$$

Therefore, from (2.4) we get

$$g^{ij} = a^{ij}. \tag{4.4}$$

Thus along $F^{n-1}(c)$, (4.3) and (4.1) lead to

$$g^{ij} b_i b_j = b^2 = b.b = bN^j b_j = \sqrt{b^2} g^{ij} g_{i\alpha} N^\alpha b_j = \sqrt{b^2} g^{ij} N_i b_j.$$

This implies

$$g^{ij} b_j (b_j - \sqrt{b^2} N_i) = 0.$$

Therefore, we get

$$b_i = \sqrt{b^2} N_i, b^2 = a^{ij} b_i b_j, \tag{4.5}$$

i.e., $b_i(x(u)) = bN_i$, where $b$ is the length of the vector $b^i$. Again from (4.4) and (4.5) we get

$$b^i = bN^i. \tag{4.6}$$

Thus we have

**Theorem 4.1.** In the special Finsler hypersurface $F^{n-1}(c)$, the Induced metric is a Riemannian metric given by (4.2) and the Scalar function $b(x)$ is given by (4.5) and (4.6).

The angular metric tensor and metric tensor of $F^n$ are given by

$$h_{ij} = a_{ij} - \frac{Y_i Y_j}{a^2}, g_{ij} = a_{ij}. \tag{4.7}$$

From (4.1), (4.7) and (3.2) it follows that if $h^a_{\alpha\beta}$ denotes the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}(c), h_{\alpha\beta} = h^a_{\alpha\beta}$. From (2.3), we get

$$\frac{\partial p_0}{\partial \beta} = \frac{[n(n+1)\alpha^3\{(n-1)\beta^n(\alpha-\beta)^{3n+2} + (n+2)\beta^{n-1}(\alpha-\beta)^{3n+1}\}+}{(\alpha-\beta)^{4n+4}}$$

$$\frac{\{n(n+1)\alpha^2 + (n+1)^2\alpha^2\}\left(\substack{2n(\alpha-\beta)^{2n+2}\beta^{2n-1}+2(n+1)(\alpha-\beta)^{2n+1}\beta^{2n} \\ -2\alpha(n+1)}\right)}{\left(\substack{(2n+1)\beta^{2n}(\alpha-\beta^{2n+2})+(2n+2)(\alpha-\beta)^{2n+1}\beta^{2n+1} \\ (\alpha-\beta)^{2n+2}(2n+2)\beta^{2n+1}+(2n+2)(\alpha-\beta)^{2n+1}\beta^{2n+2}}\right)}$$

Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = 0$ and therefore (1.6) gives $\gamma_1 = 0$, $m_i = b_i$. Therefore the hv-torsion tensor becomes

$$C_{ijk} = 0. \tag{4.8}$$

in a special Finsler hypersurface $F^{n-1}$ (c). Therefore, (3.3), (3.5) and (4.8) give

$$M_{\alpha\beta} = 0, \qquad M_\alpha = 0. \tag{4.9}$$

From (3.7) it follows that $H_{\alpha\beta} = 0$ is symmetric. Thus we have

**Theorem 4.2.** The second fundamental v-tensor of special Finsler hyper-surface $F^{n-1}(c)$ vanishes and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Next from (4.1), we get $b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$. Therefore, from (3.6) and Using, $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_j N^j H_\beta$, we get

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^i H_\beta + b_i H_{\alpha\beta} N^i = 0. \tag{4.10}$$

Since $b_i|_j = -b_h C^h_{ij}$, we get

Thus (4.10) gives

$$bH_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \tag{4.11}$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (4.11) with $v^\beta$ and then with $v^\alpha$ and using (3.1), (3.8) and (4.9), we get

$$bH_\alpha + b_{i|j}B^i_\alpha y^j = 0. \tag{4.12}$$

$$bH_0 + b_{i|j}y^i y^j = 0. \tag{4.13}$$

In view of Lemmas (3.1) and (3.2), the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from (4.12) it follows that $F^{n-1}(c)$

is a hyperplane of the first kind if and only if $b_{i|j}y^i\,y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of $F^n$ depends on $y^i$. Since $b_i$ is a gradient vector, from (2.8) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ and $F^i_j = 0$. Thus (2.9) reduces to

$$D^i_{ij} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^n_k - C^i_{km} A^n_j + C_{jkm} A^n_s g^{is} + \lambda(C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}). \tag{4.14}$$

In view of (4.3) and (4.4), the relations in (2.10) become to

$$B_i = 0, \quad B^i = 0, \quad B_{ij} = 0, \quad B^i_j = 0, \quad A^m_k = 0, \quad \lambda^m = 0. \tag{4.15}$$

By virtue of (4.15) we have $B^i_0 = 0$, $B_{i0} = 0$ which leads $A^m_0 = 0$. Therefore we have

$$D^i_{j0} = 0, \qquad D^i_{00} = 0.$$

Thus from the relation (4.1), we get

$$b_i D^i_{j0} = 0. \tag{4.16}$$

$$b_i D^i_{00} = 0. \tag{4.17}$$

From (4.8) it follows that

$$b^m\, b_i\, C^i_{jm}\, B^j_\alpha = b^2\, M_\alpha = 0.$$

Therefore, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and equations (4.16), (4.17) give

$$b_{i|j} y^i y^j = b_{00}.$$

Consequently, (4.12) and (4.13) may be written as

$$bH_\alpha + b_{i|0} B^i_\alpha = 0, \qquad bH_0 + b_{00} = 0. \tag{4.18}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where $b_{ij}$ does not depend on $y^i$. Since $y^i$ is to satisfy (4.1), the condition is written as $b_{ij}y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{4.19}$$

From (4.1) and (4.19) it follows that $b_{00} = 0$, $b_{ij} B^i_\alpha B^j_\beta = 0, b_{ij} B^i_\alpha y^i = 0$. Hence (4.18) gives $H_\alpha = 0$. Again from (4.19) and (4.15) we get $b_{i0} b^i = \frac{b^2 c_0}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = 0$. Thus (3.4), (4.4), (4.5), (4.6), (4.9) and (4.14) give

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = 0. \tag{4.20}$$

Therefore (4.11) reduces to

$$H_{\alpha\beta} = 0. \tag{4.21}$$

Thus we have

**Theorem 4.3.** The special Finsler hypersurface $F^{n-1}(c)$ is hyperplane of 1st kind if and only if (4.19) holds.

From the Lemmas (3.1), (3.2) (3.3) and Theorem (4.3), we have the following:

**Theorem 4.4.** If the special Finsler hypersurface $F^{n-1}(c)$ is a hyperplane of the 1st kind then it becomes a hyperplane of the 2nd kind too.

Hence from (3.8), (4.21), Theorem (4.2), and Lemma (3.4) we have

**Theorem 4.5.** The special Finsler hypersurface $F^{n-1}(c)$ is hyperplane of the 3rd kind if and only if it is a hyperplane of 1st kind.

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