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On Generalized W_2-recurrent $(LCS)_n$-manifolds

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Abstract

The object of the present paper is to study generalized recurrent and generalized W_2-recurrent $(LCS)_n$-manifolds.

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1. Introduction

In 2003 A. A. Shaikh [8] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example. An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp. non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp. $\leq 0, = 0, > 0$) [1, 4].

Recurrent spaces have been of great interest and were studied by a large number of authors such as Ruse [7], Patterson [5], U. C. De and N. Guha [2], Y. B. Maralabhavi and M. Rathnamma [3] etc. In this paper, I have studied a special type of Lorentzian manifolds called $(LCS)_n$-manifolds with generalized recurrent and generalized W_2-recurrent $(LCS)_n$-manifolds. The paper is organized as follows: Section 2 is concerened about basic identities of $(LCS)_n$-manifolds. In section 3, we study generalized recurrent $(LCS)_n$-manifolds. Here it is proved that such a manifold is Einstein if and only if $\beta = 2\alpha\rho$. The last section deals with generalized W_2-recurrent $(LCS)_n$-manifold and proved that if such a manifold is Einstein with $r = n(n-1)(\alpha^2 - \rho)$, then it reduces to a W_2-recurrent manifold. Finally, sufficient condition for a generalized W_2-recurrent manifold to be a generalized recurrent manifold is given.

2. $(LCS)_n$-manifolds

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X) \tag{2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{3}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{4}$$

where ρ being a certain scalar function. By virtue of (2), (3) and (4), it follws that

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{5}$$

where $\beta = -(\xi\rho)$ is a scalar function. Next if we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{6}$$

then from (3) and (6) we have

$$\phi X = X + \eta(X)\xi, \tag{7}$$

from which it follws that ϕ is symmetric $(1,1)$ tensor and is called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefy $(LCS)_n$-manifold) [8, 9]. In a $(LCS)_n$-manifold, the following relations hold

$$(a) \qquad \eta(\xi) = -1, \qquad (b)\phi\xi = 0, \qquad (c)\eta(\phi X) = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{9}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi], \tag{10}$$

$$\eta(R(X, Y)Z) = (\rho - \alpha^2)[g(Y, Z)X - g(X, Z)Y], \tag{11}$$

$$S(X, \xi) = (n - 1)(\rho - \alpha^2)\eta(X), \tag{12}$$

$$R(X, Y)\xi = (\rho - \alpha^2)[\eta(Y)X - \eta(X)Y], \tag{13}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)(\rho - \alpha^2)\eta(X)\eta(Y), \tag{14}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

3. Generalized recurrent $(LCS)_n$-manifolds

Definition 3.1 : A $(LCS)_n$-manifold M^n is called generalized recurrrent if its curvature tensor R satisfies the condition([2])

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U + B(X)[g(Z, U)Y - g(Y, U)Z] \tag{15}$$

where, A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, \rho_1), \;\; B(X) = g(X, \rho_2), \tag{16}$$

ρ_1 and ρ_2 are vector fields associated with 1-froms A and B, respectively. If the 1-form B vanishes, then the manifold reduces to recurrent manifold.

This section deals with generalized recurrent $(LCS)_n$-manifolds.

Theorem 3.1 : A generalized recurrent $(LCS)_n$-manifold is Einstein if and only if $\beta = 2\alpha\rho$.

Let us consider a generalized recurrent $(LCS)_n$-manifold. From (15) it follows that

$$\begin{aligned} g((\nabla_X R)(Y, Z)U, V) &= A(X)g(R(Y, Z)U, V) \\ &+ B(X)[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)]. \end{aligned} \tag{17}$$

Let $\{e_i\}$, $i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y = V = e_i$, $1 \leq i \leq n$, we get

$$(\nabla_X S)(Z, U) = A(X)S(Z, U) + (n - 1)B(X)g(Z, U). \tag{18}$$

Replacing U by ξ in (18) and using (12) we have

$$(\nabla_X S)(Z, \xi) = (n - 1)[(\rho - \alpha^2)A(X) + B(X)]\eta(Z). \tag{19}$$

Now we have

$$(\nabla_X S)(Z,\xi) = \nabla_X S(Z,\xi) - S(\nabla_X Z,\xi) - S(Z,\nabla_X \xi). \tag{20}$$

which yields by virtue of (3), (4) and (12) that

$$(\nabla_X S)(Z,\xi) = (n-1)[(2\alpha\rho-\beta)\eta(X)\eta(Z)+\alpha(\alpha^2-\rho)g(X,Z)]-\alpha S(X,Z). \tag{21}$$

From (19) and (21), it follows that

$$\begin{aligned}\alpha S(X,Z) &= (n-1)[(2\alpha\rho-\beta)\eta(X)\eta(Z)+\alpha(\alpha^2-\rho)g(X,Z)]\\ &\quad -(n-1)[(\alpha^2-\rho)A(X)+B(X)]\eta(Z).\end{aligned} \tag{22}$$

Hence setting $Z=\phi Z$ in (22) and then using (8c) we have

$$S(X,Z) = (n-1)(\alpha^2-\rho)g(X,Z). \tag{23}$$

If the manifold under consideration is Einstein, then (23) implies $(\alpha^2-\rho)=$ constant and hence $2\alpha\rho-\beta=0$. Conversely, if $2\alpha\rho-\beta=0$, then $\nabla_X(\alpha^2-\rho)=0$. Consequently $(\alpha^2-\rho)=$ constant. This result was proved by A.A. Shaikh [10] for generalized Ricci-recurrent $(LCS)_n$-manifolds.

Next, the nature of scalar curvature r in terms of contact forms $\eta(\rho_1)$ and $\eta(\rho_2)$ is discussed.

Theorem 3.2 : The scalar curvature r of a generalized recurrent $(LCS)_n$-manifold is related in terms of contact forms $\eta(\rho_1)$ and $\eta(\rho_2)$ as given by

$$r = [(n-1)/\eta(\rho_1)][2(\alpha^2-\rho)\eta(\rho_1)-(n-2)\eta(\rho_2)]. \tag{24}$$

Let us consider a generalized recurrent $(LCS)_n$-manifold. In (15) changing X, Y, Z; cyclically in and then adding the results, we obtain

$$\begin{aligned}&(\nabla_X R)(Y,Z)U+(\nabla_Y R)(Z,X)U+(\nabla_Z R)(X,Y)U\\ &\quad = A(X)R(Y,Z)U+B(X)[g(Z,U)Y-g(Y,U)Z]\\ &\quad +A(Y)R(Z,X)U+B(Y)[g(X,U)Z-g(Z,U)X]\\ &\quad +A(Z)R(X,Y)U+B(Z)[g(Y,U)X-g(X,U)Y].\end{aligned} \tag{25}$$

By virtue of second Bianchi identity, we have

$$\begin{aligned}&A(X)g(R(Y,Z)U,V)+B(X)[g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]\\ &+A(Y)g(R(Z,X)U,V)+B(Y)[g(X,U)g(Z,V)-g(Z,U)g(X,V)]\\ &+A(Z)g(R(X,Y)U,V)+B(Z)[g(Y,U)g(X,V)-g(X,U)g(Y,V)]=0.\end{aligned} \tag{26}$$

Contraction (26) with respect to Z and U, we get

$$\begin{aligned} &A(X)S(Y,V) + (n-1)B(X)g(Y,V) \\ &-A(Y)S(X,V) - (n-1)B(Y)g(X,V) \\ &-A(R(X,Y)V) + B(Y)g(X,V) - B(X)g(Y,V) = 0. \end{aligned} \tag{27}$$

Again, by contraction (26) with respect to Y and V, we get

$$A(X)r + (n-1)(n-2)B(X) - 2S(X,\rho_1) = 0. \tag{28}$$

Taking $X = \xi$ and then using (12) and (18), we have the required result.

4. Generalized W_2-recurrent $(LCS)_n$-manifolds

In 1970 G. P. Pokhariyal and R. S. Mishra [6] introduced the notion of a new curvature tensor, denoted by W_2 and studied its relativistic significance. The W_2-curvature tensor of type $(0,4)$ is defined by

$$W_2(Y,Z,U,V) = R(Y,Z,U,V) + \frac{1}{n-1}[g(Y,U)S(Z,V) - g(Z,U)S(Y,V)] \tag{29}$$

where S is the Ricci tensor of type $(0,2)$.

Definition 4.1 : A $(LCS)_n$-manifold M^n is called generalized W_2-recurrrent if its curvature tensor W_2 satisfies the condition

$$(\nabla_X W_2)(Y,Z)U = A(X)W_2(Y,Z)U + B(X)[g(Z,U)Y - g(Y,U)Z] \tag{30}$$

where A and B are as defined as in (16).

Theorem 4.1 : A generalized W_2-recurrent $(LCS)_n$-manifold is Einstein if and only if $\beta = 2\alpha\rho$.

Let us consider a generalized W_2-recurrent $(LCS)_n$-manifolds. From (30) it follows that

$$\begin{aligned} g((\nabla_X W_2)(Y,Z)U,V) &= A(X)g(W_2(Y,Z)U,V) \\ &+B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)]. \end{aligned} \tag{31}$$

Let $\{e_i\}$, $i = 1,2,...,n$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y = V = e_i$, $1 \le i \le n$, we get

$$\begin{aligned} &(\nabla_X S)(Z,U) \\ &= A(X)S(Z,U) + \tfrac{1}{n-1}[S(Z,U) - rg(Z,U)]A(X) + (n-1)B(X)g(Z,U). \end{aligned} \tag{32}$$

Replacing U by ξ in (32) and using (12) we have

$$(\nabla_X S)(Z,\xi) = \left[\left\{n(\alpha^2-\rho) - \frac{r}{n-1}\right\} A(X) + (n-1)B(X)\right]\eta(Z). \tag{33}$$

From (33) and (21), it follows that

$$\alpha S(X,Z) = (n-1)[(\alpha^2-\rho)\alpha g(X,Z) + (2\alpha\rho-\beta)\eta(X)\eta(Z)]$$
$$+\left[\left\{\frac{r}{n-1} - n(\alpha^2-\rho)\right\} A(X) + (n-1)B(X)\right]\eta(Z). \tag{34}$$

Hence setting $Z=\phi Z$ in (22) and then using (8c) we have

$$S(X,\phi Z) = (n-1)(\alpha^2-\rho)g(X,\phi Z). \tag{35}$$

If the manifold under consideration is Einstein, then (35) implies $(\alpha^2-\rho)=$ constant and hence $2\alpha\rho-\beta=0$. Conversely, if $2\alpha\rho-\beta=0$, then $\nabla_X(\alpha^2-\rho)=0$. Consequently $(\alpha^2-\rho)=$ constant.

Theorem 4.2 : An Einstein generalized W_2-recurrent $(LCS)_n$-manifold with $r=n(n-1)(\alpha^2-\rho)$ is a W_2-recurrent $(LCS)_n-$ manifold.

If generalized W_2-recurrent $(LCS)_n$-manifold is Einstein, then $\alpha^2-\rho$ is constant and hence $2\alpha\rho-\beta=0$. Consequently, from (34) we have

$$\alpha S(X,Z) = (n-1)(\alpha^2-\rho)\alpha g(X,Z)$$
$$+\left[\left\{\frac{r}{n-1} - n(\alpha^2-\rho)\right\} A(X) + (n-1)B(X)\right]\eta(Z). \tag{36}$$

By putting $Z=\xi$ in (36), we obtain

$$B(X) = -\frac{1}{n-1}\left[\frac{r}{n-1} - n(\alpha^2-\rho)\right]A(X). \tag{37}$$

If $r=n(n-1)(\alpha^2-\rho)$, then from (37) we get $B(X)=0$. Hence, generalized W_2-recurrent $(LCS)_n-$manifold reduces to W_2-recurrent $(LCS)_n-$ manifold.

Sufficient condition for a generalized W_2-recurrent manifold to be a generalized recurrent manifold

Theorem 4.3 : An Einstein generalized W_2-recurrent manifold with vanishing scalar curvature is a generalized recurrent manifold.

If a generalized W_2-recurrent manifold is Einstein. So we have

$$S(X,Y) = \frac{r}{n}g(X,Y) \tag{38}$$

From which it follows that

$$dr(X) = 0 \quad \text{and} \quad (\nabla_Z S)(X, Y) = 0 \quad \text{for all} \quad \text{X, Y, Z}. \tag{39}$$

Using (38) and (39) in (29), we have

$$(\nabla_X W_2)(Y, Z, U, V) = (\nabla_X R)(Y, Z, U, V). \tag{40}$$

In view of (30), the relation (40) takes the form

$$\begin{aligned} &(\nabla_X R)(Y, Z, U, V) \\ &\quad = A(X) \left\{ R(Y, Z, U, V) + \tfrac{1}{n-1}[g(Y, U)S(Z, V) - g(Z, U)S(Y, V)] \right\} \\ &\quad +B(X)[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)]. \end{aligned} \tag{41}$$

Again, in an Einstein generalized W_2-recurrent $(LCS)_n-$manifold if $r = 0$, then we have $S(X, Y) = 0$ for all X, Y and hence (41) yields (15). This shows that the manifold if generalized recurrent.

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