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Geometry of Schwarzschild Soliton

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Abstract

It is shown that the gravitational field of Schwarzschild soliton differ by its original Schwarzschild metric. By using the technique of eigen value of characteristic equation of $\lambda-$tensor, the geometry of Schwarzschild soliton has been studied.

Keywords and Phrases : Schwarzschild metric, Ricci soliton, curvature.
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1. Introduction

Mathematical models are involved in theories of gravitational Physics. Sometimes these models are defined under ideal conditions by a set of differential equations and governed by some rules for translating the mathematical results into physical world with meaningful statements. In general relativity our main motive is to solve the Einstein's field equations. There are so many exact and non-exact solutions for these equations in the literature (c.f., [6]). In Einstein's theory of general relativity, the Schwarzschild solution discovered by Karl Schwarzschild in 1916, describes the gravitational field outside a spherically symmetric, uncharged, non-rotating gravitational object such as a (non-rotating) star, planet, or black hole. The cosmological constant is assumed to equal zero. If we suppose the gravitational mass as sun, then the field outside the sun is called the Schwarzschild solution, given by the metric

$$ds^2 = \left(\frac{r^2}{r^2 - 2mr}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - \left(\frac{r^2}{r^2 - 2mr}\right)^{-1} dt^2. \tag{1}$$

The corresponding solution for a charged, spherical, non-rotating body, the ReissnerNordström metric is

$$ds^2 = \left(\frac{r^2}{r^2 + e^2 - 2mr}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)^{-1} dt^2. \tag{2}$$

In 1982, Hamilton [5] introduced the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{3}$$

to study compact three-manifolds with positive Ricci curvature and he call equation (3) as evolution equation. Hamilton proved many important and remarkable theorems for the Ricci flow, and laid the foundation for the program to approach the Poincare's conjecture and Thurstons geometrization conjecture via the Ricci flow. Further the idea was extended to Ricci soliton by pulling back the solutions of Ricci flow along a λ-dependent diffeomorphism. The Ricci soliton is a manifold $(M,\ g_{ij})$ whose metric tensor for a vector field ξ on it satisfy the equation

$$R_{ij} - \frac{1}{2}\pounds_\xi g_{ij} = k g_{ij}. \tag{4}$$

Here k is a constant and R_{ij} is the Ricci tensor for metric g_{ij}. The soliton is gradient if $\xi = \nabla\phi$, for some function ϕ and steady if $k = 0$. If $k < 0$ the soliton is called an expander; if $k > 0$ it is a shrinker.

For four dimensional case Akbar and Woolger [3] have given a local $k = 0$ soliton, named as Schwarzschild soliton. Further the Ricci soliton for Lorentzian signature has been studied by Ali and Ahsan [2] and they have explored the case of Riesner-Nordström metric as a soliton. The metric of the Schwarzschild soliton is obtained by deforming the original Schwarzschild metric for a proper substitution of functions and vector fields, for which the new metric tensor satisfy the equation (4). The Schwarzschild soliton is given by the following equation (c.f., [2])

$$ds^2 = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$

Motivated by the all important role of Ricci soliton in differential geometry and relativity, we have studied this concept for the spacetime of general relativity. We have chosen the Schwarzschild metric and studied its soliton in detail. By using the 6-dimensional formalism, the characteristic values of λ-tensor (i.e. $R_{AB} - \lambda g_{AB}$) has been given in this paper and an example of canonical form of the system is shown. Further the cases of 2 and 3-dimension for Schwarzschild soliton are discussed, in which Gaussian curvature is calculated and shown its dependence on characteristic value of λ-tensor. Finally the discussion on geometry of Schwarzschild metric and Schwarzschild soliton is made.

2. Schwarzschild soliton

Equation (5) for signature (1, 1, 1, −1) can also be written in the following form

$$ds^2 = dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta d\phi^2) - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2. \tag{6}$$

The components of the potential for the gravitation or the metric tensor for Schwarzschild soliton (6) in spherical coordinates $x^\alpha \equiv (r,\ \theta,\ \phi,\ t)$ are given by

$$g_{ij}(x^\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 - 2mr & 0 & 0 \\ 0 & 0 & (r^2 - 2mr)\sin^2\theta & 0 \\ 0 & 0 & 0 & -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \end{pmatrix} \tag{7}$$

or

$$g_{11} = 1,\ g_{22} = r^2 - 2mr,\ g_{33} = (r^2 - 2mr)\sin^2\theta,\ g_{44} = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}. \tag{8}$$

The Christoffel symbols, can be calculated from the formula [1]

$$\begin{aligned} \Gamma^i_{jk} &= g^{il}\Gamma_{ljk} \\ &= \tfrac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j}\right). \end{aligned} \tag{9}$$

Thus the non-zero components of the Christoffel symbols for metric (6), by using equation (8) are

$$\begin{aligned} &\Gamma^1_{22} = (m - r),\ \Gamma^1_{33} = (m - r)\sin^2\theta \\ &\Gamma^1_{44} = \frac{\sqrt{2}m}{r^2 - 2mr}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}},\ \Gamma^2_{12} = \Gamma^2_{21} = \frac{r - m}{r^2 - 2mr} \\ &\Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{13} = \Gamma^3_{31} = \frac{r - m}{r^2 - 2mr} \\ &\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta,\ \Gamma^4_{14} = \Gamma^4_{41} = \frac{\sqrt{2}m}{r^2 - 2mr}. \end{aligned} \tag{10}$$

While Riemann tensor for the Schwarzschild soliton (4) can be calculated from the formula [1]

$$R_{ijkl} = \tfrac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) + g_{mn}(\Gamma^m_{jk}\Gamma^n_{il} - \Gamma^m_{jl}\Gamma^n_{ik}) \tag{11}$$

and the non-zero components of Riemann tensor, by using equation (8) are

$$\begin{aligned}
R_{1212} &= \frac{m^2}{r^2 - 2mr} \\
R_{1414} &= \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}[m + \sqrt{2}(m - r)] \\
R_{2323} &= -m^2 \sin^2\theta \\
R_{2424} &= \frac{-\sqrt{2}m(m - r)}{(r^2 - 2mr)}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \\
R_{3131} &= \frac{m^2 \sin^2\theta}{r^2 - 2mr} \\
R_{3434} &= \frac{-\sqrt{2}m(m - rs)\sin^2\theta}{(r^2 - 2mr)}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}.
\end{aligned} \tag{12}$$

We now use the 6-dimensional formalism in the pseudo-Euclidean space $\mathbb{R}^6$ by making the identification [4]

$$\begin{array}{rcccccc} ij: & 23 & 31 & 12 & 14 & 24 & 34 \\ A: & 1 & 2 & 3 & 4 & 5 & 6. \end{array} \tag{13}$$

We also make use of the identification as

$$g_{ik}g_{jl} - g_{il}g_{jk} = g_{ijkl} \to g_{AB}, \tag{14}$$

where A, B= 1, 2, 3, 4, 5, 6 and g_{ij} are the components of the metric tensor at an arbitrary point (x^α) of the Schwarzschild soliton, whose metric is given by equation (6). The new metric tensor g_{AB} (A, B=1, 2, 3, 4, 5, 6) is symmetric and non-singular.

The non-zero components of the metric tensor g_{AB} for equation (6) in 6-dimensional formalism, by using formulation (14) are as

$$\begin{aligned}
&g_{11}(x^\alpha) = (r^2-2mr)^2\sin^2\theta,\ g_{22}(x^\alpha) = (r^2-2mr)\sin^2\theta \\
&g_{33}(x^\alpha) = (r^2-2mr),\ g_{44}(x^\alpha) = -\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \\
&g_{55}(x^\alpha) = -(r^2-2mr)\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \\
&g_{66}(x^\alpha) = -(r^2-2mr)\sin^2\theta\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}.
\end{aligned} \tag{15}$$

Similarly, we can transform the components of the Riemann tensor as $R_{ijkl} \to R_{AB}$. Thus, for example R_{1212} can be written as R_{33} [using identification (13)]. The non-zero components of the tensor R_{AB} under the identification (13) are

$$\begin{aligned}
&R_{11}(x^\alpha) = -m^2\sin^2\theta \\
&R_{22}(x^\alpha) = \frac{m^2\sin^2\theta}{r^2-2mr},\ R_{33}(x^\alpha) = \frac{m^2}{r^2-2mr} \\
&R_{44}(x^\alpha) = \frac{2m}{(r^2-2mr)^2}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}[m+\sqrt{2}(m-r)] \\
&R_{55}(x^\alpha) = \frac{-\sqrt{2}m}{(r^2-2mr)}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \\
&R_{66}(x^\alpha) = \frac{-\sqrt{2}m(m-r)\sin^2\theta}{(r^2-2mr)}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}
\end{aligned} \tag{16}$$

Further we use all these values to find a canonical form of the λ-tensor $R_{AB} - \lambda g_{AB}$. Next, we will be interested in eigen values for the Schwarzschild soliton (4), That is the solutions of the characteristic equation $\mid R_{AB} - \lambda g_{AB} \mid = 0$. By using equations (15) and (14) easily, we calculate these eigen values and those are given by

$$\begin{aligned}
\lambda_1(r) &= \frac{m^2}{(r^2-2mr)^2} \\
\lambda_2(r) &= \frac{m^2}{(r^2-2mr)^2} = \lambda_3(r) \\
\lambda_4(r) &= \frac{-2m}{(r^2-2mr)^2}[m+\sqrt{2}(m-r)] \\
\lambda_5(r) &= \frac{-\sqrt{2}m(m-r)}{(r^2-2mr)^2} = \lambda_6(r).
\end{aligned} \tag{17}$$

$\lambda_i,\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$, are the solution of the character equation $\mid R_{AB} - \lambda g_{AB} \mid = 0$ which depend on m and r. In other words we can say that for λ_i [equation (17)], the determinant of λ-tensor $R_{AB} - \lambda g_{AB}$ is zero. Thus we can transform the system in canonical form for values of λ_i as

$$g_{A'B'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and (18)

$$R_{A'B'} = \begin{pmatrix} \lambda_1(r) & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2(r) & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4(r) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_5(r) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_6(r) \end{pmatrix}.$$

Thus in our case (for Schwarzschild soliton) the gravitational field determined by λ- tensor is of the type $G_1[(1)(1)(11)(11)]$ in Segre symbols. From equation (18), we note that even if mass $m = 0$, the Schwarzschild soliton is flat.

Case I - $\theta = 0$ or $\theta = \pi$

When taking $\theta = 0$ or $\theta = \pi$ that is $d\theta = 0$, the Schwarzschild soliton, given by equation (6), reduces to the form

$$^{*}ds^2 = dr^2 - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2. \tag{19}$$

Now equation (19) is a 2-dimensional surface now. The metric tensor $^{*}g$ in coordinates $x^{\beta} \equiv (r,\ t)$ is given by

$$^{*}g_{ij}(x^{\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \end{bmatrix}, \tag{20}$$

here $i,\ j = 1,\ 4$. Thus the hypersurface for $\theta = 0$ or $\theta = \pi$ (i.e., $^{*}H_0$ or $^{*}H_{\pi}$) degenerates to two dimensional surface. The non-zero component of Riemann curvature tensor for equation (19) is unique and given by

$$^{*}R_{1414}(x^{\beta}) = \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}[m + \sqrt{2}(m - r)], \tag{21}$$

so the Gaussian curvature $^{*}K$ for surface $^{*}H_0$ or $^{*}H_{\pi}$ is

$$^{*}K(x^{\beta}) = \frac{2m}{r^2 - 2mr}[m + \sqrt{2}(m - r)]. \tag{22}$$

Equations (17) and (22) show that curvature of the 2-dimensional surface of the Schwarzschild soliton is related to the eigen value $\lambda_4(r)$.

Case II - $2m < r < \infty$, $0 < \theta < \pi$ and $\phi = 0$

For this case, equation (6) reduces to

$$ds^2 = dr^2 + (r^2 - 2mr)d\theta^2 - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2. \tag{23}$$

The metric tensor $^{**}g_{ij}$ for equation (23) in coordinate $x^{\gamma} \equiv (r,\ \theta,\ t)$ is given by

$$^{**}g_{ij}(x^{\gamma}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (r^2 - 2mr) & 0 \\ 0 & 0 & -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \end{bmatrix}. \tag{24}$$

The non-zero components of the Riemann curvature tensor for the metric (23) are as following

$$
\begin{aligned}
{}^{**}R_{1212}(x^\gamma) &= \frac{m^2}{r^2 - 2mr} \\
{}^{**}R_{1414}(x^\gamma) &= \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}[m + \sqrt{2}(m - r)] \qquad (25) \\
{}^{**}R_{2424}(x^\gamma) &= \frac{\sqrt{2}m}{(r^2 - 2mr)}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}.
\end{aligned}
$$

So for 3-dimensional space (23), the Gaussian curvature at each point $x^\gamma \equiv (r,\ \theta,\ t)$ is given by the following three physical quantities

$$
\begin{aligned}
{}^{**}K_1(x^\gamma) &= \frac{{}^{**}R_{2424}(x^\gamma)}{|{}^{**}g_{24}|} = \frac{-\sqrt{2}m}{(r^2 - 2mr)^2} \\
{}^{**}K_2(x^\gamma) &= \frac{{}^{**}R_{1414}(x^\gamma)}{|{}^{**}g_{14}|} = \frac{-2m}{(r^2 - 2mr)^2}[m + \sqrt{2}(m - r)] \qquad (26) \\
{}^{**}K_4(x^\gamma) &= \frac{{}^{**}R_{1212}(x^\gamma)}{|{}^{**}g_{12}|} = \frac{m^2}{(r^2 - 2mr)^2}.
\end{aligned}
$$

Here ${}^{**}g_{24}$ denotes the sub-matrix of ${}^{**}g_{ij}$ corresponding to $x^1 = r$. It is clear from equations (17) and (26) that the curvature of the 3-dimensional space of Schwarzschild soliton can be expressed in terms of a λ-tensor which happens to be the solutions (eigen-values) of the characteristic equation $|R_{AB} - \lambda g_{AB}| = 0$.

3. Discussion

In this paper we worked out on gravitational field of Schwarzschild soliton by using characteristic of λ-tensor $R_{AB} - \lambda g_{AB}$, we have also discussed 2 and 3-dimensional cases. It is seen that Schwarzschild soliton, given by Akbar and Woolger [3] has different geometry as that of Schwarzschild metric which is studied by Borgiel [4]. We see that the gravitational field for Schwarzschild soliton is of type $G_1[(1)(1)(11)(11)]$ [equation (18)] in Segre symbols while Borgiel has given type $G_1[(1111)(11)]$ for Schwarzschild metric. For Schwarzschild soliton, not only the Gaussian curvature differ with that of Schwarzschild metric but also the dependence of curvature on eigen values of $\lambda-$tensor $R_{AB} - \lambda g_{AB}$ is not similar. Thus the deformation in metric (along a $\lambda-$dependent diffeomorphism) of a spacetime is cause for change in geometry or gravitational field.

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