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Linear Gauge and the Linearization Probleem for Webs

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Abstract

Regarding the theory of foliation, some aspects of the theory of riemannian foliations have been brought in completion by the Molino theory. Such a structure is defined by some finite dimensional Lie subalgebra of the Lie algebra of transverse vector fields. The problem I am interested in is more modeste. It is to get sufficient conditions for a smooth manifold admitting foliations with transverse (pseudo) riemannanian metrics. The investigation is inspired by both methods of information geometry and the Hessian geometry.

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1. Introduction

This aim of this paper is to discuss two questions in a smooth manifold M. The first question is related to foliation with transverse riemannian metrics. Here riemannian metric means non degenerate quadratic form. My aim is to provide some sufficient conditions for the existence of such foliations. The second question is to investigate sufficient conditions for some webs being linearizable. In both cases the sufficient conditions involve the geometry of dual pairs of linear gauges in riemannian manifolds. Beside the introduction the paper is divided into five sections.

Section.1 is devoted to recall some differential topology notions such as foliations and webs. We give main definitions that will be needed in the sequel.

Section.2 deals with the geometry of linear gauges which are nothing than Koszul connections in vector bundles. We discuss some notions such as the dualistic equivalence in riemannian vector bundles. We point out some perspectives in the transverse geometry of riemannian foliations. For instance we discuss the dualistic equivalence for Bott connection.

Section.3 is a brief introduction to some methods of the geometric science of information. We define a remarkable short exact sequence in statistic manifolds and we discuss its relationships with foliations admitting transverse riemannian metrics. Those discussions yield the statements of the main results to be proved.

The aim of Section 4. is to describe some materials in both information geometry and Hessian geometry, The theory of statistical models for measurable set is overviewed. The comparison of statistical models is goal in statistics. We point out two comparistion problems for fixed dimension statistical models for the same measurable set. We recall their characteristic invariant which is an application of (convex) set of linear gauge in the set of random Hessian metrics in the base manifold. Another comparison problem deals with the types of probability density in a fixed base manifold of statistical models. Local answers of this problem is related to a cohomology vanishing theorem In Section 5. we describe some cochain complex and we point out its heplness to discuss many relevant problems in both information geometry and differential topology.

2. Foliations and webs

From now on all ojects defined in a smooth manifold are smooth as well. We deals connected finite dimensional smooth manifolds.

2.1. Foliation

Let M be an $m-$dimensional smooth manifold and let q be a non negative integer with $q \neq m$.

Definition 2.1. A $q-$codimensional foliation in M is a smooth $q-$codimension partition $\mathcal{F}$ of M by $q-$codimensional connected submanifolds.

The partition $\mathcal{F}$ is a subset of the set $\mathcal{P}(M)$ of subsets of M and its elements are called leaves of the foliation. Every $x \in M$ belongs to only one leaf F. Let

$$\mathcal{D} = \mathcal{T}\mathcal{F}$$

be the union of tangent bundle of leaves of $\mathcal{F}$. Then $\mathcal{D}$ is an involutive differential system in M. Every section X of $\mathcal{D}$ is smooth vector field which is tangent to leaves of $\mathcal{F}$. The couple $(M, \mathcal{F})$ is called a foliated manifold.

Definition 2.2. A transverse riemannian (pseudo) metric in $(M, \mathcal{F})$ is a quadratic form g in M such that for every section X of $\mathcal{D}$ one has

$$i_X g = 0,$$

$$L_X g = 0.$$

Regarding the notion of holonomy group of a leaf of foliation we refer the readers to The notations i_X and L_X stand for the inner product by X and the Lie derivative in the direction of X respectivly. The question whether a given manifold does admit a non trivial foliation is a differential topology question which has no general answer. Suppose a smooth manifold admits foliations, there are no creteria to decide whether M does admit a foliation with transverse metrics. In this paper we give sufficient conditions to those existence problems.

Definition 2.3. A structure of riemannian foliation in M is a triplet $(M, g, \mathcal{F})$ formed by a foliated manifold $(M, \mathcal{F})$ and a riemannian structure (M, g) whose riemannian metric g is invariant under the holonomy group of every leaf of $\mathcal{F}$.

Some references for a extensive study of riemannian foliations is Molino's book [9].

2.2. Webs

Let k be a positive integer and le $\mathcal{D}_{1\leq j\leq k}$ be k regular differential systems. Thus every $\mathcal{D}_j$ is a vector subbundle of the tangent bundle TM.

Definition 2.4. The $D_j'^s$ are in general position if for every fixed $i \leq k$ and at every point $x \in M$ one has

$$dim\,\big(\sum_j^i \mathcal{D}_j(x)\big) = min\,(dim\,(M),\, \sum_j^i dim\,(\mathcal{D}_j(x)).$$

Definition 2.5. Given k foliations $(\mathcal{F}_j, 1 \leq j \leq k)$ are in general position if they are defined by a family $\mathcal{D}_j$ of subbundles of the tangent bundle TM which are in general position.

Definition 2.6. A $k-$web in a manifold M is family $\mathcal{F}_j$, $1 \leq j \leq k$ of foliations which are in general position.

For example (i) a bilagrangian structure $(\mathcal{L}_1, \mathcal{L}_2)$ in a symplectique manifold (M, ω) is a 2-web; (ii) A parallelism $(X_1, \ldots, X_m)$ in a $m-$dimensional compact manifold M defines a $m-$web. It must be noticed that different foliations in a web may have different codimensions.

2.3. The linearization problem for webs

In the real affine space $\mathbb{R}^m$ a k web $(\mathcal{F}_j)_{1\leq j,k}$ is called a linear web if the leaves of every foliation $\mathcal{F}_j$ are affine subspaces.

Definition 2.7. A $k-$web in a $m-$dimensional manifold M is called an linearizable web if it is locally isomorphic to a linear $k-$web in $\mathbb{R}^m$.

Nowadays the problem of linearization of webs in $\mathbb{R}^2$ is discussed.

In [7] I have widely discussed the problem of symplectic linearization of lagrangian webs. This question is related to the geometry of locally flat manifolds, [2].

As it has been mentioned in the introduction I plan to discuss both existence problems for foliations admitting transverse metrics and for linearizable webs.

3. The gauge geometry

Le G be a Lie group whose Lie algebra is denoted by $\mathcal{G}$. Let $\mathcal{P}$ be a principal $G-$bundle over a manifold M.

In mathematical physics a gauge field in $\mathcal{P}$ is a principal connection 1-form ω in $\mathcal{P}$.

The $1-$form ω is a $\mathcal{G}-$valued differential of type $\mathrm{ad}(G)$.

Actually I intend to deal with the vector bundle versus of gauge fields. Thus linear counterparts of principal connection forms are Koszul connections in vector bundles. So given a vector bundle $\mathcal{V}$ over a manifold M a Koszul connection in $\mathcal{V}$ will be denoted by ∇.

The curvature of ∇ the $End(\mathcal{V})-$valued differential 2-form R_∇ defined in the base manifold M as it follows

$$R_\nabla(X,X').s = \nabla_X.(\nabla_{X'}.s) - \nabla_{X'}.(\nabla_X.s) - \nabla_{[X,X']}.s.$$

In the formula above X, X' are smooth vector fields in M and s is a smooth section of $\mathcal{V}$.

Let $(\mathcal{V},g)$ be a riemannian vector bundle. At every point $x \in Mg_x$ is an inner product in the fiber $\mathcal{V}_x$. Let us recall some method of information geometry.

3.1. The dualistic relation

The dualistic realtion in $(\mathcal{V},g)$ deals with pairs of Koszul connections in $\mathcal{V}$.

Definition 3.1. A pair (∇, ∇^*) of Koszul connections in $(\mathcal{V}, g)$ is called a dual pair if for all vector field X in M and for all sections s, s' of $\mathcal{V}$ the following equality holds

$$X.g(s, s') = g(\nabla_X.s, s') + g(s, \nabla^*_X.s').$$

The curvature forms R_∇, R_{∇^*} are in dualistic relation as well. This means that the following identity holds

$$g(R_\nabla(X, X').s, s') + g(s, R_{\nabla^*}(X, X').s') = 0.$$

Thereby $(\mathcal{V}, \nabla)$ is a flat vector bundle iff $(\mathcal{V}, \nabla^*)$ is a flat vector bundle.

A pair (∇, ∇^*) is called dually flat pair if R_∇ vanishes identically.

3.1.1. $\alpha-$connections

A given dual pair (g, ∇, ∇^*) generates a one (real) parameter family $(g, \nabla^\alpha, \nabla^{-\alpha})$ of dual pairs in $(\mathcal{V}, g)$. For $\alpha \in \mathbb{R}$ the connection ∇^α is defined by setting

$$\nabla^\alpha = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*.$$

Thus to every pair $(\alpha, -\alpha) \in \mathbb{R}^2$ is assigned the dual pair $(g, \nabla^\alpha, \nabla^{-\alpha})$.

3.2. Some short exact sequences

Let (∇, ∇^*) be a dual pair in $(\mathcal{V}, g)$. The vector bundle $End(\mathcal{V})$ of the vector bundle endomorphisms of $\mathcal{V}$ contains the following remarkable vector subbundle.

The first vector subbundle is denoted by $J(\nabla, \nabla^*)$ which stands for the subbundle of gauge homomorphism of $(\mathcal{V}, \nabla)$ in $(\mathcal{V}, \nabla^*)$. Thus a section ϕ of $J(\nabla, \nabla^*)$ is a vector bundle homomorphism subject to the following requirement

$$\nabla^*_X.\phi(s) = \phi(\nabla_X.s).$$

Those $\phi \in J(\nabla, \nabla^*)$ are nothing than infinitesimal gauge homomorphisms.

The second vector subbundle is denoted by $J(\nabla, \nabla^*, g)$. Its sections are those sections *psi* of $J(\nabla, \nabla^*)$ subject to the following requirement

$$g(\psi(s), s') + g(s, \psi(s')) = 0.$$

Let $S_2(\mathcal{V}^*)$ be the vector bundle of symmetric bi-linear functions in $\mathcal{V}$. Its sections are inner products in the vector bundle $\mathcal{V}$. Now let us consider the map q from $J(\nabla, \nabla^*)$ to $S_2(\mathcal{V}^*)$ which is defined by setting

$$q(\phi).(s, s') = \frac{1}{2}\left(g(\phi(s), s') + g(s, \phi(s'))\right).$$

3.2.1. Parallel symmetric forms

Definition 3.2. A section Q of $S^2(V^*)$ is $\nabla-$parallel if the following identity holds

$$X.Q(s,s') = Q(\nabla_X.s, s') + Q(s, \nabla_X.s')$$

for all vector field X and for all sections s, s' of $\mathcal{V}$.

It is easy to prove the following claim.

Lemma 3.1. The image of the map

$$q : J(\nabla, \nabla^*) \to S_2(\mathcal{V}^*)$$

is the subbunde $S_2^{\nabla}(\mathcal{V}^*)$ of parallel forms.

3.2.2. Parallel skew symmetric forms

Let $\wedge_2(\mathcal{V}^*)$ be the bundle of skew symmetric bilinear forms in $\mathcal{V}$. We define the map ω of $J(\nabla, \nabla^*)$ in $\wedge_2(\mathcal{V}^*)$ by putting

$$\omega(\phi).(s,s') = \frac{1}{2}(g(\phi(s), s') - g(s, \phi(s')).$$

Lemma 3.2. The image of ω is the subbundle $\wedge_2^{\nabla}(\mathcal{V}^*)$ of parallel skew symmetric bilinear forms in the vector bundle $\mathcal{V}$.

Proposition 3.3. The following short sequence of vector bundles is exact

$$0 \to J(\nabla, \nabla^*, g) \to J(\nabla, \nabla^*) \to S_2^{\nabla}(\mathcal{V}^*) \to 0.$$

Actually it is a easy to see the induced map

$$\omega : J(\nabla, \nabla^*) \to \wedge_2^{\nabla}(\mathcal{V}^*)$$

is an isomorphism. Therefore one can regard $\wedge_2^{\nabla}(\mathcal{V}^*)$ as a subbundle of $J(\nabla, \nabla^*)$. The viewpoint yields the following short exact sequence

$$0 \to \wedge_2^{\nabla}(\mathcal{V}^*) \to J(\nabla, \nabla^*) \to S_2^{\nabla}(\mathcal{V}^*) \to 0.$$

Let us fix a couple $(\mathcal{V}, \nabla)$. Then up to isomorphism the vector bundle $J(\nabla, \nabla^*)$ does not depend on the riemannian vector bundle structure $(\mathcal{V}, g)$ of course, given two dual pairs $(\mathcal{V}, g, \nabla, \nabla^*)$ and $(\mathcal{V}, g', \nabla, \nabla'^*)$ is easy to see that the couples $(\mathcal{V}, \nabla^*)$ and $(\mathcal{V}, \nabla'^*)$ are canonically isomorphic. In other words there is an automorphism ϕ of $\mathcal{V}$ such that

$$\phi(\nabla'^*_X.s) = \nabla_X.\phi(s)$$

for every vector field X in the base manifold of $\mathcal{V}$ and for every section s of $\mathcal{V}$.

3.3. The linear gauges

In this subsection I shall be dealing with Koszul connections in the tangent bundle of smooth manifolds. So it will not be necessary to explicit the vector bundles.

Let (g, ∇, ∇^*) be a dual pair in a riemannian manifold (M, g). The mean

$$\tilde{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$$

of the couple (∇, ∇^*) is the Levi-Civita connection of (M, g). This implies that their torsion tensors are related as it follows

$$T_{\nabla} + T_{\nabla^*} = 0.$$

Now $S_2(M)$ and $\wedge_2(M)$ stand for $S_2(T^*M)$ and for $\wedge_2(T^*M)$ respectively.

I am interested in the following short exact sequence

$$0 \to \wedge_2^{\nabla}(M) \to J(\nabla, \nabla^*) \to S_2^{\nabla}(M) \to 0.$$

Every section ω of $\wedge_2^{\nabla}(M)$ yields the decomposition of the tangent bundle as it follows

$$TM = ker(\omega) \oplus ker(\omega)^+$$

where $ker(\omega)^+$ stands for the $g-$orthogonal to $ker(\omega)$

Mutatis muntandis every section Q of $S_2^{\nabla}(M)$ yields the orthogonal decomposition of the tangent bundle TM, namely

$$TM = ker(Q) \oplus ker(Q)^+.$$

The orthogonal decompositions above are of importance for discussing the transverse geometry of foliations.

4. Some Techniques of information geometry

I am interested in pairs (g, ∇, ∇^*) where both ∇ and ∇^* are symmetric. This means that both ∇ and ∇^* are torsion free. To (∇, ∇^*) one assigns the following short exact sequences

$$0 \to \wedge_2^{\nabla}(M) \to J(\nabla, \nabla^*) \to S_2^{\nabla}(M) \to 0,$$
$$0 \to \wedge_{*2}^{\nabla}(M) \to J(\nabla^*, \nabla) \to S_{*2}^{\nabla}(M) \to 0.$$

The context above yields the following statements.

Theorem 4.1. (i) For every section Q of $S_2^{\nabla}(M)$ the subbundles $\mathcal{K}er(Q)$ and $\mathcal{K}er(Q)^+$ are completely integrable. (ii) For every section ω of $\wedge_2^{\nabla}(M)$

the subbundles $\mathcal{K}er(\omega)$ and $\mathcal{K}er(\omega)^+$ are completely integrable. (iii) The foliation $\mathcal{K}er(Q)$ is transversaly (pseudo) riemannian. (iv) The foliation $\mathcal{K}er(\omega)$ is transversaly symplectic.

The statement above shows how the information geometry techniques help to construct foliations admitting prescribed transverse geometry. Moreover the couple $(\mathcal{K}er(Q), \mathcal{K}er(Q)^+)$ and the couple $(\mathcal{K}er(Q), \mathcal{K}er(Q)^+)$ are 2-webs.

4.0.1. The case of dually flat pairs

I intend to use information geometry methods to build examples of linearizable webs. From now on I shall be dealing with dually flat pairs.

Let (g, ∇, ∇^*) a dually flat pair. This means the torsion tensors and the curvature tensors of both ∇ and ∇^* vanish identically.

Theorem 4.2. (i) For every section Q of $S_2^{\nabla}(M)$ the 2-web $(\mathcal{K}er(Q), \mathcal{K}er(Q)^+)$ is linearizable

Taking into accoumpt the orthogonal decomposition

$$TM = ker(Q) \oplus ker(Q)^+.$$

Every vector field in M is presented as it follows

$$X = (X_1, X_2)$$

with $(X_1, X_2) \in ker(Q) \times ker(Q)^+$. Let D stand for the symmetric Koszul connection defined by the formula

$$D_{(X_1,X_2)}.(Y_1, Y_2) = (\nabla_{X_1}.Y_1 + [X_2, Y_1]_1, \nabla^*_{X_2}.Y_2 + [X_1, Y_2]_2).$$

Lemma 4.3. (i) The Koszul connection D is locaaly flat. (ii) The 2-web $(\mathcal{K}er(Q), \mathcal{K}er(Q)^+)$ is totaly geodesic w.r.t. the connection D.

A straightforward consequence of the last lemma is

Theorem 4.4. The 2-web $(\mathcal{K}er(Q), \mathcal{K}er(Q)^+)$ is linearizable.

Mutatis mutandis one gets skew symmetric versus of the statements above by considering a section ω of $\wedge_2^{\nabla}(M)$.

Let $Q_1, \ldots, Q_k$ be k sections of $S_2^{\nabla}(M)$. One says that the sections Q_j are in general position if their kernels $ker(Q_j)$ are in general position. Thus one gets the $k-$web $(\mathcal{K}er(Q_j)$.

Theorem 4.5. Every finite family $(Q_j)_{1\leq j\leq k}$ of sections of $S_2^{\nabla})(M)$ which are in general position defines a linearizable $k-$web.

The proof of the theorem above is based on the flatness of the Koszul connection ∇.

5. Statistical manifolds

In this section we apply the materials of the previous sections to the information geometry.

Definition 5.1. A statistical manifold is a dual pair $(M, g; \nabla, \nabla^*)$ where the Koszul connections ∇, ∇^* are torsion free.

According to the sections above a statistical manifold structure yields the following remarkable short exact sequences

$$0 \to \wedge_2^{\nabla}(M) \to J(\nabla, \nabla^*) \to S_2^{\nabla}(M) \to 0,$$
$$0 \to \wedge_2^{\nabla^*}(M) \to J(\nabla^*, \nabla) \to S_2^{\nabla^*}(M) \to 0.$$

5.1. Gauge morphims and transverse geometry of foliations in statistical manifolds

Let us assign to a gauge morphism $\phi \in J(\nabla, \nabla^*)$ the pair $(Q, \omega) \in (S_2^{\nabla}(M), \wedge_2^{\nabla}(M))$ which is defined as it follows

$$Q(X, X') = \frac{1}{2}(g(\phi(X), X') + g(X, \phi(X'))),$$
$$\omega(X, X') = \frac{1}{2}(g(\phi(X), X') - g(X, \phi(X'))).$$

The differential $2-$form ω is de Rham closed. Since g is non degenerate there exists a pair (Phi, Psi) of elements of $J(\nabla, \nabla^*)$ suject to the following requirements

$$Q(X, X') = g(\Phi(X), X'),$$
$$\omega(X, X') = g(\Psi(X), X').$$

Thereby one has

$$\mathcal{K}er(Q) = \mathcal{K}er(\Phi),$$
$$\mathcal{K}er(\omega) = \mathcal{K}er(\Psi),$$

Those requirements yield

$$\mathcal{K}er(Q)^+ = im(\Phi),$$
$$\mathcal{K}er(\omega) = im(\Psi).$$

Since ∇ and ∇^* are torsion free the distributions $im(\Phi)$ and $im(\Psi)$ are completely integrable. Of course $\mathcal{K}er(Q)$ admits the transverse riemannian metric Q and $\mathcal{K}er(\omega)$ admits the transverse symplectic form ω.

Proposition 5.1. Suppose the statistical manifold (M, g, ∇, ∇^*) to be flat. Then the $2-$webs ($\mathcal{K}er(\Phi)$, $im(\Phi)$ and $\mathcal{K}er(\Psi)$, $im(\Psi)$) are linearizable.

Many objects we have discussed through here have their homological versus. In the next section I shall introduce the cochain complex which is related to those previous discussions.

6. A simplicial KV modules attached to a flat statistical manifold structure

Let (M, g, ∇, ∇^*) be a flat statistical manifold. Let $\mathcal{F}(M)$ be the associative commutative algebra of real valued smooth functions in M.

I intend to deal with the real algebra $\mathcal{A}$ whose underling vector space is the vector space of derivations of the algebra $\mathcal{F}(M)$. Those derivations are nothing but the vector space of smooth vector fields in M. The multiplication

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

is defined by putting

$$X.X' = \nabla_X.X'.$$

We consider the $\mathbb{Z}-$graded vector C^* whose homogeneous vector subspace C^q is defined as it follows, (i) if q is a negative integer then

$$C^q = 0.$$

(ii) for $q = 0$

$$C^0 = J(\mathcal{F}(M))$$

with $f \in \mathcal{F}(M)$ iff $\nabla^2(f) = \nabla(df) = 0$.
(iii) if q is a positive integer then

$$C^q = Hom_{\mathbb{R}}(A^{\otimes q}, \mathcal{F}(M)).$$

Let $q \in \mathbb{Z}$ with $3 \leq q$ and to every couple $(i, j) \in \mathbb{Z}^2$ with $i < j \leq q+1$ one assigns the linear map

$$\sigma^1_{i.j} : C^q \to C^{q+1}$$

as it follows. Let $f \in C^q$ and $\xi = X_1 \otimes X_2 \otimes \ldots \otimes X_{q+1} \in A^{\otimes q+1}$. (a) If $j < q+1$ then

$$\begin{aligned}\sigma^1_{ij}(f).\xi =&(-1)^i\,[X_i.f(X_1 \otimes \ldots \otimes \hat{X}_i \otimes \ldots \otimes X_j \otimes \ldots \otimes X_{q+1})\\ &-\ qf(X_1 \otimes \ldots \otimes \hat{X}_i \otimes \ldots \otimes X_i.X_j \otimes \ldots \otimes X_{q+1})]\\ &(-1)^j[Xj.f(X_1 \otimes \ldots \otimes X_i \otimes \ldots \otimes \hat{X}_j \otimes \ldots \otimes X_{q+1})\\ &-\ qf(X_1 \otimes \ldots \otimes X_j.X_i \otimes \ldots \otimes \hat{X}_j \otimes \ldots \otimes X_{q+1})].\end{aligned}$$

(b) If $j = q+1$, then

$$\begin{aligned}\sigma^1_{i,q+1}(f).\xi =&(-1)^i\,[X_i.f(X_1 \otimes \ldots \otimes \hat{X}_i \otimes \ldots \otimes \ldots \otimes X_{q+1})\\ &-\ qf(X_1 \otimes \ldots \otimes \hat{X}_i \otimes \ldots \otimes X_i.X_{q+1})].\end{aligned}$$

Using the family $\sigma^1_{i,j}$ one defines the linear map

$$d : C^q \to C^{q+1}$$

by putting

$$df = \sum_{i<j} \sigma^1_{i,j}(f).$$

I am going to express the square $d^2 = d \circ d$, namely $d^2(f)$ as function depending linearly on anomaly function $KV(.,.,.)$ of the couple $\mathcal{A}, \mathcal{F}(M))$. Given $X, X', X'' \in \mathcal{A}$ and $h \in \mathcal{F}(M)$ let one set

$$KV(X, X', h) = (X.X').h - X.(X'.h) - (X'.X).h + X'.(X.h),$$

$$KV(X, X', X'') = (X.X').X'' - X.(X'.X'') - (X'.X).X'' + X'.(X.X'').$$

The smooth function $X.h \in \mathcal{F}(M)$ is defined by

$$X.h(x) = dh(X)$$

where the differential 1-form dh is the differential of the smooth function h.

Given $f \in C^q$ and $\xi = X_1 \otimes X_2 \otimes \ldots \otimes X_{q+2} \in A^{\otimes q+2} d^2(f).\xi$ is related to the anomaly functions as it follows

$$\begin{aligned}d^2(f).\xi = \sum_{i<j<k} &[(-1)^{i+j}[KX(X_i, X_j, f(X_1 \otimes \ldots \hat{X}_i, \ldots \hat{X}_j \ldots \otimes X_k \ldots \otimes X_{q+2})\\ &+ f(X_1 \otimes \ldots \hat{X}_i, \ldots \hat{X}_j \ldots \otimes KV(X_j, X_i, X_k) \otimes \ldots \otimes X_{q+2})]\\ &+ (-1)^{1+k}[KV(X_i, X_k, f(X_1 \otimes \ldots \hat{X}_i \ldots \otimes X_j \otimes \hat{X}_k \ldots \otimes X_{q+2}))\end{aligned}$$

$$+ f(X_1 \otimes \ldots \hat{X}_i \ldots \otimes KV(X_k, X_i, X_j) \otimes \ldots \hat{X}_k \otimes \ldots \otimes X_{q+2})]$$
$$+ (-1)^{j+k}[KV(X_j, X_k, f(X_1 \otimes \ldots \otimes Xi \otimes \ldots \hat{X}_j \otimes \ldots \hat{X}_k \otimes \ldots \otimes X_{q+2}))$$
$$+ f(X_1 \otimes \ldots \otimes KV(X_k, X_j, X_i) \otimes \ldots \hat{X}_j \otimes \ldots \hat{X}_k \otimes \ldots \otimes Xq+2)]].$$

Without any assumption neither on the multiplication

$$(X, X') \to X.X'$$

of the algebra $\mathcal{A}$ nor on the action

$$(X, h) \to X.h$$

one sees that $d^2(f)$ depends lineraly on the anomaly functions of the couple $(\mathcal{A}, \mathcal{F}(M))$. When (M, ∇) is a locally flat manifold the anomaly function vanishes identically. Thereby the couple (C^*, d) is a cochain complex whose cohomology space is denoted by $H^*(\nabla, \mathbb{R})$.

6.1. The Hessian geometry in flat statistical manifolds

To a flat statistical manifold (M, g, ∇, ∇^*) one assigns the pair $(\mathcal{A}, \mathcal{A}^*)$ of algebras whose underlying vector space is the space of derivations of the associative commutative algebra $\mathcal{F}(M)$. Let us denote by $C^*(\nabla)$, $C^*(\nabla^*)$ their related cochain complex. Then the riemannian metric tensor g is a $2-$cocyle in both $C^*(\nabla)$ and $C^*(\nabla^*)$. Then this consideration yields a well known fact that (M, g, ∇) and (M, g, ∇^*) Hessian manifolds [8], [1]. Following [2] the cohomology class $[g] \in H^2(\nabla, \mathbb{R})$ an obstruction to (M, ∇) be a locally flat hyperbolic structure. Of course the same remark holds for (M, ∇^*).

There many other relationships between the Hessian geometry and the geometry of information, see [6], [4, 5], [9]. Regarding the relationships between the Hessian geometry and industry the reader is refered to [10]. To end the discussion about the information geometry I intend to define the notion of statistical model for measurable sets. I suppose the reader knows the notion of probabilized measurable sets.

6.2. Statistical models

From now on (Ξ, Ω) is a measurable set whose Boole algebra is $\Omega \subset \mathcal{P}(\Xi)$. At the moment Ξ is provided with the discrete topology. Let Γ be the group of measurable isomorphisms of (Ξ, Ω).

I am concerned with special locally trivial smooth $\Gamma-$bundles whose fiber type is Ξ and whose base manifolds are locally flat manifolds in the sense of [3], [7].

6.2.1. The dynamics of Γ

The starting data are the following:

(1) An $m-$dimensional locally flat structure (M, ∇) whose underlying manifod M is the base manifold of a $\Gamma-$bundle

$$\pi : \mathcal{E} \to M$$

which is locally modeled on Ξ.

(2) Both $\mathcal{E}$ and M admit $\Gamma-$actions

$$\Gamma\mathcal{E} \to \mathcal{E},$$

$$M\Gamma \to M$$

such that (3)

$$\pi(\gamma.e) = \pi(e).\gamma.$$

Of course the product $\mathcal{E} \times M$ admits a left $\Gamma-$action defined by

$$\gamma.(e, x) = (\gamma.e, x.\gamma^{-1}).$$

(4) Given an open subset $U \subset M$ and $\mathcal{E}_U = \pi^{-1}(U)$ a trivialization of π over U is a couple (Φ, ϕ) where ϕ is an affine diffeomorphism of (U, ∇) onto an open subset $\Theta_U \subset \mathbb{R}^m$ and Φ is a homeomorphism of $\mathcal{E}_U$ onto the product $\Theta_U \times \Xi$ such that

$$\phi(\pi(e)) = p_1(\Phi(e))$$

where $p_1(x, \xi) = x \, \forall (x, \xi) \in \Theta_U \times \Xi$.

The couple (Φ, ϕ) is called a fibered chart of π whose domain is $(\mathcal{E}_U, U)$.

Definition 6.1. Given a fibered chart

$$(\Phi \times \phi) : \mathcal{E}_U \times U) \to \Theta_U \times \Xi) \times U$$

a $(\Phi, \phi)-$supported local statistical model for Ξ is a real valued function

$$P : \Theta \times \Xi \to \mathbb{R}$$

subject the following requirements (i) If one fixes $\xi\Xi$ then the function

$$\theta \in \Theta_U \to P(\theta, \xi)$$

is smooth.

(ii) If one fixes $\theta In\Theta_U$ then the function

$$\xi \to P(\theta, \xi)$$

is a probability density, viz

$$\int_{\Xi} P(\theta, \xi)\, d\xi = 1.$$

(iii) the (so-caled horizontal) differentiation d_θ commutes to the integration w.r.t $\xi \in \Xi$.

To the local statistical model P is asigned the Fisher information which is the quadratic form g defined in Θ_U by

$$g(\theta).(X, X') = \int_{\Xi} [d_\theta \log (\mathrm{P}(\theta, \xi)]^2 \, (\mathrm{X}, \mathrm{X}')\, \mathrm{d}\xi$$

where $d\theta$ stands the differential w.r.t. θ. This quadratic form is positive semi definite.

6.2.2. Compatibility of local models

Let $((\Phi, \phi), \mathcal{E}_U, U; P)$ as in the last subsubsection and let $\gamma \in \Gamma$. To simplify let assume that $(\mathcal{E}_{U.\gamma}, U.\gamma)$ is the domain of a fibered trivialisation

$$\Psi \times \psi) : \mathcal{E}_{U.\gamma} \times U.\gamma \to \Theta_{U.\gamma} \times U.\gamma$$

supporting a local model

$$P^* : \Theta_{U.\gamma} \times U.\gamma \to \mathbb{R}.$$

It is obvious that for $(x, \xi) \in \Theta_U \times \Xi \gamma.(x, \xi) \in \Theta_{U.\gamma} \times \Xi$.

Definition 6.2. The local models P and P^* are compatible if for all $(x, \xi) \in \Theta_U \times \Xi$ one has

$$P^*(\gamma.(x, \xi)) = P(x, \xi).$$

6.2.3. Statistical atlas

Now we consider an open covering (U_i) of M with a family

$$\Phi_i \times \phi_i \mathcal{E}_i, \quad U_i \to (\Theta_i \times \Xi) \times U_i.$$

We assume that each (Φ, ϕ_i) supports a local model

$$P_i : \Theta_i \times \Xi \to \mathbb{R}.$$

For every every $e \in \mathcal{E}_{U_i}$ let us set

$$(x_i(e), \xi_i(e)) = \Phi_i(e).$$

Definition 6.3. The data $[\mathcal{E}_i, U_i, \Phi_i, \phi_i, P_i]$ define a structure of statistical model in the fibration π if the models P_i are pairwise compatible. Moreover if $U_i \cap U_j \neq \emptyset$ then there exists a mapping

$$\gamma_{ij} : U_i \cap U_j \to \Gamma$$

such that for $e \in \mathcal{E}_{U_i \cap U_j}$ one has

$$(x_j(e), \xi_j(e)) = \gamma_{ij}(x_i(e)).(x_i(e), \xi_i(e)).$$

The family $(U_i, \Phi_i, \phi_i, P_i)$ is called a statistical atlas.

Definition 6.4. The atlas $(U_i, \Phi_i, \phi_i, P_i)$ and the atlas $(U_j^*, \Phi_j^*, \phi_j^*, P_j^*)$ are compatible if their local charts are pairwise compatible.

A statistical model for (Ξ, Ω) is a class of compatible atlases.

Actually a statistical atlas $(U_i, \Phi_i, \phi_i, P_i)$ gives rise to the global function

$$p : \mathcal{E} \to \mathbb{R}$$

such that

$$P_i(\Phi_i(e)) = p(e)$$

for all $e \in \mathcal{E}_i$. If $(U_j^*, \Phi_j^*, \phi_j^*, P_j^*)$ and $(U_i, \Phi_i, \phi_i, P_i)$ are compatible then $p^* = p$. So the global function p depends only on the structure of (global) statistical model.

Since π is locally trivial the function p is ($\pi-$)horizontally smooth. So $d_\theta p$ stands for the horizontal differentiation in $\mathcal{E}$. Of course d_θ commutes to the integration along the fibers of π.

Let $\mathcal{L}(M)$ se the convex set of the Koszul connections in M. To every $D \in \mathcal{L}(M)$ is assigned the horizontal quadratic form $Q(D)$ which is defined as it follows. Given $\pi-$projetable vector fields X, X' let us set

$$Q(D).(X, X') = [D(d_H \log(p)).(X, X')].$$

Consider the category $E(M, \Xi)$ of structures of statistical models for (Ξ, Ω) with the same base manifold M.

Theorem 6.1. The map

$$D \in \mathcal{L}(M) \to Q(D)$$

is a characteristic invariant.

I am dealing with fiber morphism

$$\mathcal{R} \times r : \mathcal{E} \times M \to \mathcal{E}^* \times M.$$

In other words $\forall\, e \in \mathcal{E}$ one has

$$\pi^* \circ \mathcal{R} = r \circ \pi$$

where

$$\pi^* : \mathcal{E}^* \to M$$

is the fibration of the model $\mathcal{E}^*$. The theorem above means that a fiber isomorphism $\mathcal{R} \times r$ sending Q to Q^* is a isomorphism of statistical models, viz

$$(p^* \circ \mathcal{R}) = p.$$

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