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Hypersurface of a Special Finsler Space with metric $\alpha+\beta+\frac{\alpha^2}{(\alpha-\beta)}$

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Abstract

In the present paper our study confines to the hypersurface of a Finsler space with (α, β) metric $\alpha+\beta+\frac{\alpha^2}{\alpha-\beta}$. We have examined the hypersurfaces as a hyperplane of first, second and third kinds.

1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$ i.e.,a pair consisting of an n-dimensional differentiable manifoldM^n equipped with a Fundamental function L. The concept of (α, β), metric $L(\alpha, \beta)$ was introduced first of all by M. Matsumoto [5] and has been studied by many authors [1, 2, 3, 5, 8, 7, 9]. As well known examples are Rander's metric $(\alpha+\beta)$, Kropina metric $\frac{\alpha^2}{\beta}$ and generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}(m \neq 0, -1)$ whose studies have greatly contributed a lot to the growth of Finsler geometry. A Finsler metric $L(x, y)$ is called an (α, β) metric if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a $1-$ form on M^n.

2. Preliminaries

We devote to a special Finsler Space $F^n = \{M^n, L(\alpha, \beta)\}$ with the metric

$$L(\alpha, \beta) = \alpha+\beta+\frac{\alpha^2}{(\alpha-\beta)} \tag{2.1}$$

Partial derivative of (2.1) w.r.t α and β are given by

$$L_\alpha = \frac{2\alpha^2+\beta^2-4\alpha\beta}{(\alpha-\beta)^2}, \quad L_\beta = \frac{2\alpha^2+\beta^2-2\alpha\beta}{(\alpha-\beta)^2}$$

$$L_{\alpha\alpha} = \frac{2\beta^2}{(\alpha-\beta)^3} \qquad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}, \qquad L_{\alpha\beta} = \frac{-2\alpha\beta}{(\alpha-\beta)^3}$$

where $L_\alpha = \dfrac{\partial L}{\partial \alpha}$, $L_\beta = \dfrac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \dfrac{\partial L\alpha}{\partial \alpha}$, $L_{\beta\beta} = \dfrac{\partial L\beta}{\partial \beta}$, $L_{\alpha\beta} = \dfrac{\partial L_\alpha}{\partial \beta}$.

In the Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor h_{ij} are given by [5]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j$$

where $Y_i = a_{ij} y^j$. For the fundamental function (2.1) above constants are

$$p = L L_\alpha \alpha^{-1} = \frac{4\alpha^4 - \beta^4 - 8\alpha^3\beta + 4\alpha\beta^3}{\alpha(\alpha-\beta)^3} \tag{2.2}$$

$$q_0 = L L_{\beta\beta} = \frac{4\alpha^4 - 2\alpha^2\beta^2}{(\alpha-\beta)^4}, \quad q_{-1} = L L_{\alpha\beta}\alpha^{-1} = \frac{2\beta^3 - 4\alpha^2\beta}{(\alpha-\beta)^4}$$

$$q_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{-4\alpha^5 - 2\alpha^2\beta^3 + 8\alpha^4\beta + \alpha\beta^4 - \beta^5}{\alpha^3(\alpha-\beta)^4}$$

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha,\beta)$ are given by [4, 5]

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \tag{2.3}$$

where

$$p_0 = q_0 + L_\beta^2 = \frac{8\alpha^4 + \beta^4 + 6\alpha^2\beta^2 - 8\alpha^3\beta - 4\alpha\beta^3}{(\alpha-\beta)^4}, \tag{2.4}$$

$$p_{-1} = q_{-1} + L^{-1} p L_\beta = \frac{2\alpha\beta^3 - 4\alpha^3\beta + (2\alpha^2 + \beta^2 - 2\alpha\beta)^2}{\alpha(\alpha-\beta)^4}$$

$$p_{-2} = q_{-2} + p^2 L^{-2} = \frac{2\beta^4 + 8\alpha^2\beta^2 - 6\alpha\beta^3 + \frac{\beta^5}{\alpha}}{\alpha^2(\alpha-\beta)^4}$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \tag{2.5}$$

where $b^i = a^{ij} b_j$ and $b^2 = a_{ij} b^i b^j$

(2.6)
$$\begin{aligned} s_0 &= \frac{1}{\tau p}\{pp_0 + (p_0p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} &= \frac{1}{\tau p}\{pp_{-1} + (p_0p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} &= \frac{1}{\tau p}\{pp_{-2} + (p_0p_{-2} - p_{-1}^2)b^2\}, \\ \tau &= p(p + p_0b^2 + p_{-1}\beta) + (p_0p_{-2} - p_{-1}^2)(\alpha^2b^2 - \beta^2) \end{aligned}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [10]

(2.7)
$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k$$

where,

(2.8)
$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{{}^{\,i}_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space R^n and ∇_k be the covariant derivative with respect to x^k relative to this christoffel symbol. Now we define,

(2.9)
$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{{}^{\,i}_{jk}\}$ of the special Finsler space F^n is given by

(2.10)
$$\begin{aligned} D^i_{jk} = {} & B^iE_{jk} + F^i_kB_j + F^i_jB_k + B^i_jb_{0k} + B^i_kb_{0j} \\ & -b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_sg^{is} \\ & +\lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}) \end{aligned}$$

where

(2.11)
$$B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F_i^k = g^{kj} F_{ji}$$
$$B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B_i^k = g^{kj} B_{ji}$$
$$A_k^m = B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m$$
$$\lambda^m = B^m E_{00} + 2 B_0 F_0^m, \quad B_0 = B_i y^i$$

where '0' denote contraction with y^i except for the quantities p_0, q_0 and s_0.

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$ (where $\alpha = 1, 2, 3...(n-1)$). The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is [6],

(3.1)
$$y^i = B_\alpha^i(u) v^\alpha$$

the metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u,v), y(u,v)\} B_\alpha^i N^j = 0, \quad g_{ij}\{x(u,v), y(u,v)\} N^i N^j = 1$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface are given by

(3.2)
$$h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j, \quad h_{ij} B_\alpha^i N^j = 0, \quad h_{ij} N^i N^j = 1$$

(B_i^α, N_i) inverse of (B_α^i, N^i) is given by

$$B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_i^\alpha N^i = 0, \quad B_\alpha^i N_i = 0$$
$$N_i = g_{ij} N^j, \quad B_i^k = g^{kj} B_{ji}, \quad B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by [6].

$$\Gamma^{*\alpha}_{\beta\gamma} = B_i^\alpha (B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma$$
$$G^\alpha_\beta = B_i^\alpha (B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_\beta), \quad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^j_\beta B^k_\gamma$$

where

$$M_{\beta\gamma} = N_i C^i_{jk} B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma} M_{\beta\gamma}, \quad H_\beta = N_i (B^i_{0\beta} + \Gamma^{*i}_{oj} B^j_\beta)$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha$$

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M_\beta H_\gamma \tag{3.3}$$

where

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k \tag{3.4}$$

The relative h and v-covariant derivatives of projection factor B^i_α with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \qquad B^i_\alpha|_\beta = M_{\alpha\beta} N^i$$

It is obivious form the equation (3.3) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta \tag{3.5}$$

The above equation yield

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0 \tag{3.6}$$

We shall use following lemmas which are due to Matsumoto [6] in the coming section

Lemma 3.1. The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.

Lemma 3.2. A hypersurface F^{n-1} is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$.

Lemma 3.3. A hypersurface F^{n-1} is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma 3.4. A hypersurface F^{n-1} is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4. Hypersurface $F^{n-1}(c)$ of a special Finsler space

Let us consider a Finsler space with the metric $L = \alpha + \beta + \dfrac{\alpha^2}{\alpha - \beta}$, where, vector field $b_i(x) = \dfrac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{n-1}(c)$ given by equation $b(x) = c$, a constant [10].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0$$
$$\frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0$$
$$b_i B^i_\alpha = 0$$

Above shows that $b_i(x)$ are covarient component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad and \quad b_i y^i = 0 \quad i.e \quad \beta = 0 \tag{4.1}$$

and induced matric $L(u,v)$ of $F^{n-1}(c)$ is given by

$$L(u,v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \tag{4.2}$$

which is a Riemannian metric.

Writing $\beta = 0$ in the equations (2.2), (2.3) and (2.5) we get

$$\begin{aligned} &p = 4, \quad q_0 = 4, \quad q_{-1} = 0 \quad q_{-2} = -4\alpha^{-2} \\ &p_0 = 8 \quad p_{-1} = 4\alpha^{-1} \quad p_{-2} = 0 \quad \tau = 16(1+b^2), \\ &s_0 = \frac{1}{4(1+b^2)} \quad s_{-1} = \frac{1}{4\alpha(1+b^2)} \quad s_{-2} = \frac{-b^2}{4\alpha^2(1+b^2)} \end{aligned} \tag{4.3}$$

from (2.4) we get,

$$g^{ij} = \frac{1}{4}a^{ij} - \frac{1}{4(1+b^2)}b^i b^j - \frac{1}{4\alpha(1+b^2)}(b^i y^j + b^j y^i) + \frac{b^2}{4\alpha^2(1+b^2)}y^i y^j \tag{4.4}$$

thus along $F^{n-1}(c)$, (4.4) and (4.1) leads to

$$g^{ij} b_i b_j = \frac{b^2}{4(1+b^2)}$$

So we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{4(1+b^2)}} N_i, \qquad b^2 = \alpha^{ij} b_i b_j \tag{4.5}$$

where b is the length of the vector b^i.

Again from (4.4) and (4.5), we get

$$b^i = a^{ij} b_j = \sqrt{\frac{4b^2(1+b^2)}{\{1+b^2(1-\alpha^2)\}^2}} N^i + \frac{\alpha b^2 y^i}{1+b^2(1-\alpha^2)} \tag{4.6}$$

thus we have,

Theorem 4.1. In a special Finsler hypersurface $F^{n-1}(c)$, the Induced Riemannian metric is given by (4.2) and the scalar function b(x)is given by (4.5) and (4.6).

Now the angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

$$(4.7) \quad h_{ij} = 4a_{ij} + 4b_ib_j - \frac{4}{\alpha^2}Y_iY_j \quad and \quad g_{ij} = 4a_{ij} + 8b_ib_j + \frac{4}{\alpha}(b_iY_j + b_jY_i$$

From equation (4.1), (4.7) and (3.2) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F^{n-1}_{(c)}$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

Thus along $F^{n-1}_{(c)}$, $\dfrac{\partial p_0}{\partial \beta} = \dfrac{24}{\alpha}$.

From equation (2.6) we get

$$Y_1 = \tfrac{48}{\alpha}, \qquad m_i = b_i$$

then hv-torsion tensor becomes

$$(4.8) \qquad C_{ijk} = \frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{6}{\alpha}b_ib_jb_k$$

in the special Finsler hypersurface $F^{n-1}_{(c)}$. Due to fact from (3.2), (3.3), (3.5), (4.1) and (4.8) we have

$$(4.9) \qquad M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{4(1+b^2)}}h_{\alpha\beta} \quad and \quad M_\alpha = 0$$

Therefore from equation (3.6) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem 4.2. The second fundamental v-tensor of the special Finsler hypersurface $F^{n-1}_{(c)}$ is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Now from (4.1) we have $b_iB^i_\alpha = 0$. Then we have

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0$$

Therefore, from (3.5) and using $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_jN^jH_\beta$, we have

$$(4.10) \qquad b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^jH_\beta + b_iH_{\alpha\beta}N^i = 0$$

since $b_i|_j = -b_hC^h_{ij}$, we get

$$b_{i|j}B^i_\alpha N^j = 0$$

Therefore from equation (4.10) we have,

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0 \tag{4.11}$$

because $b_{i|j}$ is symmetric. Now contracting (4.11) with v^β and using (3.1) we get

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0 \tag{4.12}$$

Again contracting by v^α equation (4.12) and using (3.1), we have

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_0 + b_{i|j}y^i y^j = 0 \tag{4.13}$$

From lemma (3.1) and (3.2), it is clear that the hypersuface $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus from (4.13) it is obvious that $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n defined on y^i, but $b_{ij} = \nabla_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ where $b_{ij} = \nabla_j b_i$ in the following. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by (2.10). Since b_i is a gradient vector, then from (2.9) we have

$$E_{ij} = b_{ij} \qquad F_{ij} = 0 \qquad \textit{and} \qquad F^i_j = 0.$$

Thus (2.10) reduces to

$$\begin{aligned} D^i_{jk} = \; & B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k \\ & - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} \\ & + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{aligned} \tag{4.14}$$

where

(4.15)
$$B_i = 8b_i + 4\alpha^{-1}Y_i, \quad B^i = (\frac{1}{1+b^2})b^i + \frac{1}{\alpha(1+b^2)}y^i,$$
$$\lambda^m = B^m b_{00}, \quad B_{ij} = \frac{2}{\alpha}(a_{ij} - \frac{Y_iY_j}{\alpha^2}) + \frac{12}{\alpha}b_ib_j,$$
$$B^i_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-1}y^iY_j) + \frac{5}{2\alpha(1+b^2)}b^ib_j -$$
$$\frac{(1+6b^2)}{2\alpha^2(1+b^2)}b_jY^i, \quad A^m_k = B^m_k b_{00} + B^m b_{k0}.$$

In view of (4.3) and (4.4), the relation in (2.11) becomes to by virture of (4.15) we have $B^i_0 = 0,\ B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.

Now contracting (4.14) by y^k we get
$$D^i_{j0} = B^ib_{j0} + B^i_jb_{00} - B^mC^i_{jm}b_{00}$$
Again contracting the above equation with respect to y^j we have
$$D^i_{00} = B^ib_{00} = \{(\tfrac{1}{1+b^2})b^i + \tfrac{1}{\alpha(1+b^2)}y^i\}b_{00}$$
Paying attention to (4.1), along $F^{n-1}_{(c)}$, we get

(4.16)
$$b_iD^i_{j0} = \frac{b^2}{(1+b^2)}b_{j0} + \frac{(1+6b^2)}{2\alpha(1+b^2)}b_jb_{00} + \frac{1}{(1+b^2)}b_ib^mC^i_{jm}b_{00}$$

Now we contract (4.16) by y^j we have

(4.17)
$$b_iD^i_{00} = \frac{1}{(1+b^2)}b_{00}$$

From (3.3), (4.5), (4.6), (4.9) and $M_\alpha = 0$, we have
$$b_ib^mC^i_{jm}B^j_\alpha = b^2M_\alpha = 0.$$
Thus the relation $b_{i|j} = b_{ij} - b_rD^r_{ij}$ the equation (4.16) and (4.17) gives
$$b_{i|j}y^iy^j = b_{00} - b_rD^r_{00} = \tfrac{1}{1+b^2}b_{00}.$$
Consequently (4.12) and (4.13) may be written as

(4.18)
$$\sqrt{\frac{b^2}{4(1+b^2)}}H_\alpha + \frac{1}{1+b^2}b_{i0}B^i_\alpha = 0,$$
$$\sqrt{\frac{b^2}{4(1+b^2)}}H_0 + \frac{1}{1+b^2}b_{00} = 0$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus we can write,

$$2b_{ij} = b_i c_j + b_j c_i \tag{4.19}$$

Now from (4.1) and (4.19) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

Hence from (4.18) we get $H_\alpha = 0$, again from (4.19) and (4.15) we get $b_{i0} b^i = \dfrac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \dfrac{2}{\alpha} h_{\alpha\beta}$.

Now we use equation (3.3), (4.4), (4.5), (4.6), (4.9) and (4.14) then we have

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2 (4 + 3b^2)}{16\alpha(1 + b^2)^2} h_{\alpha\beta} \tag{4.20}$$

Thus the equation (4.11) reduces to

$$\sqrt{\frac{b^2}{4(1 + b^2)}} H_{\alpha\beta} + \frac{b^2 (4 + 3b^2)}{16\alpha(1 + b^2)^2} h_{\alpha\beta} = 0 \tag{4.21}$$

Hence the hypersurface $F^{n-1}_{(c)}$ is umbilic.

Theorem 4.3. The necessary and sufficient condition for $F^{n-1}_{(c)}$ to be a hyperplane of first kind is (4.19). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.

Now from lemma (3.3), $F^{n-1}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (4.20), we get

$$c_0 = c_i(x) y^i = 0$$

Therefore there exist a function $\psi(x)$ such that

$$c_i(x) = \psi(x) b_i(x)$$

Therefore (4.19) we get

$$2b_{ij} = b_i(x)\psi(x) b_j(x) + b_j(x)\psi(x) b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x) b_i b_j$$

Theorem 4.4. The necessary and sufficient condition for a hypersurface $F^{n-1}_{(c)}$ to be a hyperplane of second kind is (4.21).

Again lemma (4.4), together with (4.9) and $M_\alpha = 0$ shows that $F^{n-1}_{(c)}$ does not become a hyperplane of third kind.

Theorem 4.5. The hypersurface $F^{n-1}_{(c)}$ is not a hyperplane of the third kind.

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